Weil cohomology theories

Fix an algebraically closed field $\mathbf{C}$. All varieties will be varieties over $\mathbf{C}$. See the end of this note for some remarks on what to change to get a Weil cohomology theory over a nonalgebraically closed field.

A *Weil cohomology theory* H^* with coefficients in the characteristic zero field K is given by the following set of data (notation explained below):

(D1) For every nonsingular projective algebraic variety X a graded commutative algebra $H^*(X)$ over K. The grading is indexed by integers: $H^*(X) = \oplus_{n\in\mathbf{Z}} H^n(X)$ is a direct sum decomposition of K-vector spaces. The multiplication $H^*(X) \times H^*(X) \to H^*(X)$, $(\alpha, \beta) \mapsto \alpha \cup \beta$ is called the *cup product*. It is K-bilinear. Graded commutative means that $\alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha$ for homogenous elements.

(D2) For every morphism of nonsingular projective varieties $f : X \to Y$ a *pullback* map $f^* : H^*(Y) \to H^*(X)$ which is a K-algebra map preserving the grading.

(D3) A 1-dimensional K-vector space $K(1)$, which gives rise to *Tate twists* as follows. For a K-vector space V we define $V(n) = V \otimes_K K(1)^{\otimes n}$. If n is negative then $V(n) = V \otimes_K Hom(K(1)^{\otimes -n}, K)$. We will use obvious notation, e.g., given K-vector spaces U, V and W and a linear map $U \otimes_K V \to W$ we obtain a linear map $U(a) \otimes_K V(b) \to W(a + b)$ for an pair $a, b \in \mathbf{Z}$.

(D4) For every nonsingular projective variety X a *trace map* $Tr : H^{2\dim X}(X)(\dim X) \to K$.

(D5) For every nonsingular projective variety and every closed subvariety $Z \subset X$ of codimension c there is given a *cohomology class* $cl(Z) \in H^{2c}(X)(c)$.

These data should satisfy the axioms (W1)–(W10) below.

Remarks. (a) The reason to introduce Tate twists is that there is no reasonable way to canonically identify $H^{2\dim X}(X)$ with K in certain cases (especially when doing cohomology/motives over non algebraically closed fields).
(b) There are lots of relations among these data, and in fact you could express everything in terms of the data (D1)–(D4), or everything in terms of the data (D1)–(D3) & (D5). In other words, the Trace map determines the cohomology classes, and vice versa.
(c) Another equivalent piece of data could be a chern class map $c_1 : Pic(X) \to H^2(X)(1)$. See below.

(W1) Each $H^i(X)$ is a finite dimensional K-vector space.

(W2) If $H^i(X) \neq (0)$ then $i \in [0, 2\dim(X)]$.

(W3) Given morphisms $f : X \to Y$ and $g : Y \to Z$ of nonsingular projective varieties we have $(g \circ f)^* = f^* \circ g^*$ as maps $H^*(Z) \to H^*(X)$. In other words H^* is a contravariant functor.

(W4) Künneth. Given nonsingular projective varieties X, Y we have maps $pr_X^* : H^*(X) \to H^*(X \times Y)$ and $pr_Y^* : H^*(Y) \to H^*(X \times Y)$. Because of the existence of the cup product we get a morphism of K-vector spaces

$$H^*(X) \otimes_K H^*(Y) \longrightarrow H^*(X \times Y), \quad \alpha \otimes \beta \mapsto pr_X^*(\alpha) \cup pr_Y^*(\beta).$$

Axiom (W4) says that this map is an isomorphism of vector spaces.

(W5) Poincaré duality. For every nonsingular projective variety X the trace map is an isomorphism and for every $0 \leq j \leq 2\dim X$ the cupproduct

$$H^j(X) \times H^{2\dim X - j}(X)(\dim X) \longrightarrow H^{2\dim X}(X)(\dim X) - Tr \rightarrow K$$

induces a perfect duality between $H^j(X)$ and $H^{2\dim X - j}(X)(\dim X)$.

(W6) Compatibility of Trace maps and products. Given nonsingular projective varieties X, Y we require that the trace map $Tr_{X\times Y} : H^{2\dim X + 2\dim Y}(X \times Y)(\dim X + \dim Y) \to K$ satisfies

$$Tr_{X\times Y}(pr_X^*(\alpha) \cup pr_Y^*(\beta)) = Tr_X(\alpha) Tr_Y(\beta)$$

for $\alpha \in H^{2\dim X}(X)(\dim X)$ and $\beta \in H^{2\dim Y}(Y)(\dim Y)$.

(W7) Exterior products and cohomology classes. Given nonsingular projective varieties X, Y and closed subvarieties $Z \subset X$, $W \subset Y$ we have $cl(Z \times W) = pr_X^*(cl(Z)) \cup pr_Y^*(cl(W))$.

(W8) Cohomology classes and pushforward. Suppose that $f : X \to Y$ is a morphism of nonsingular projective varieties, and let $Z \subset X$ be a closed subvariety. Recall that $f_*[Z] = m[f(Z)]$ where m is the degree of the morphism of varieties $Z \to f(Z)$. Axiom (w7) says that

$$Tr_X(cl(Z) \cup f^*\alpha) = m\ Tr_Y(cl(f(Z)) \cup \alpha)$$

for every $\alpha \in H^{2\dim Z}(Y)(\dim Z)$. We will see later that this means the pushforward of the cohomology class is the cohomology class of the pushforward.

(W9) Cohomology classes and pullback. Suppose that $f : X \to Y$ is a morphism of nonsingular projective varieties, and let $Z \subset Y$ be a closed subvariety. Assume that $\dim f^{-1}(Z) = \dim Z + \dim X - \dim Y$. Write the cycle associated to $f^{-1}(Z)$ as follows $[f^{-1}(Z)]_k = \sum n_i Z_i$ where $k = \dim Z + \dim X - \dim Y$, see note on intersection theory. Axiom (W8) says

$$f^* cl(Z) = \sum n_i cl(Z_i).$$

In other words, the cohomology class of the pullback cycle is the pullback of the cohomology class of the cycle.

(W10) Cohomology class of a point. Let $x = Spec(\mathbf{C})$ be the one point variety. Then $Tr_x(cl(x)) = 1$.

Here is a sequence of exercises to explain additional features of a Weil cohomology theory. You can also just take these features to be additonal axioms if you like.

Exercise 1. Show that $H^0(X)$ is one dimensional and isomorphic to K as a K-algebra. Show that the element $1 \in H^0(X)$ is a unit in $H^*(X)$.

Hint: Poncaré duality shows that it is one dimensional. Choose a generator $u \in H^{2\dim X}(X)(\dim X)$ with $Tr(u) = 1$. Choose $\theta \in H^0(X)$ nonzero. Then $\theta \cup u \neq 0$ by Poincaré duality. Hence $\theta \cup u_X = cu$ for some nonzero $c \in K$. Show that $(1/c)\theta$ is a unit for the algebra $H^*(X)$.

Exercise 2. Show that for any morphism of nonsingular projective varieties $f : X \to Y$ the map $f^* : H^0(X) \to H^0(Y)$ satisfies $f^*(1) = 1$.

Hint: Use that $(id, f) : X \to X \times Y$ composed with the projection $X \times Y \to X$ is the identity morphism on X and use the Künneth formula.

Exercise 3. Show for the one point vartiety x that $cl(x) = 1$. (It is probably better to take this as given and not argue it directly.)

Hint: By normalization we have to show that $cl(x) \cup cl(x) = cl(x)$. Axiom (W7) says that $cl(x \times x) = pr_1^*(cl(x)) \cup pr_2^*(cl(x))$ in $H^0(x \times x)$. By axiom (W9) we deduce $cl(x \times x) = cl(x \times x) \cup cl(x \times x)$. Finally pullback to x via axiom (W9) by the isomorphism $x \to x \times x$.

Exercise 4. Let X be a nonsingular projective variety. Show that $cl(X) = 1$.

Hint: Note that $cl(X) = c_X$ for some $c_X \in K$. Pullback to a point and use the previous exercise.

Given a Weil cohomology theory as above we can define *pushforward* on cohomology. Namely, if $f : X \to Y$ is a morphism of nonsingular varieties, then we have $f^* : H^*(Y) \to H^*(X)$ and hence we get a dual map $(f^*)^t : Hom(H^*(Y), K) \to Hom(H^*(X), K)$. Using Poincaré duality we obtain

$$f_* : H^*(X) \longrightarrow H^{*-2r}(Y)(-r)$$

with $r = \dim X - \dim Y$. More precisely, we have the following formal definition.

Definition. Let $f : X \to Y$ be a morphism of nonsingular varieties. Set $r = \dim X - \dim Y$. For $\alpha \in H^j(X)$ we define the *pushforward* of α the be the unique element $f_*(\alpha)$ of $H^{j-2r}(Y)(-r)$ characterized by the property that

$$Tr_Y(f_*(\alpha) \cup \beta) = Tr_X(\alpha \cup f^*(\beta))$$

for all $\beta \in H^{2\dim X - j}(Y)(\dim X)$.

Exercise 5. Projection formula. Show, using the axioms above, that $f_*(\alpha \cup f^*\beta) = f_*(\alpha) \cup \beta$. Here $f : X \to Y$ is as above, $\alpha \in H^*(X)$ and $\beta \in H^*(Y)$.

Exercise 6. Covariance. Suppose that $f : X \to Y$ and $g : Y \to Z$ are morphisms of nonsingular projective varieties. Show that $g_* \circ f_* = (g \circ f)_*$.

Exercise 7. Trace and degree. Let $f : X \to Y$ be a morphism of nonsingular projective varieties of the *same* dimension d. Define the degree $\deg(f)$ of f to be 0 if f is not dominant and the degree of the field extension $[\mathbf{C}(X) : \mathbf{C}(Y)]$ if f is dominant. Show that for any $\alpha \in H^{2d}(Y)(d)$ we have

$$Tr_X(f^*\alpha) = \deg(f) Tr_Y(\alpha).$$

Another way to phrase this is: $f_*(1) = \deg(f) \cdot 1$. Show that, by the projection formula, this implies $f_* f^* \alpha = \deg(f)\alpha$ for every $\alpha \in H^*(Y)$.

Hint: This is just a reformulation of a special case of (W8).

Exercise 8. Characterization of cohomology classes. Show that cohomology classes are characterized in the following manner. Let X be a nonsingular projective algebraic variety. Let $Z \subset X$ be a closed subvariety of codimension c. Let $\varphi : Z' \to Z$ be a nonsingular alteration. (See Publ. Math. I.H.E.S. 83.) Show that $cl(Z)$ is the unique element in $H^{2c}(X)(c)$ such that

$$(1/\deg(\varphi)) Tr_{Z'}(\varphi^*(\alpha)) = Tr_X(cl(Z) \cup \alpha)$$

for all $\alpha \in H^{2\dim Z}(X)(\dim Z)$. In other words, using the pushforward φ_* we have $cl(Z) = (1/\deg(\varphi))\varphi_*(1)$.

Remark. In case the ground field $\mathbf{C}$ is the complex numbers and H^* is de Rham cohomology and ω_Z is a differential form representing $cl(Z)$ this is the familiar equation $\int_Z f^*\alpha = \int_X \omega_Z \wedge \alpha$ for closed forms α.

Remark. This exercise proves (D5) is determined by the other pieces of data (D1)–(D4).

Exercise 9. Show that $(1/\deg(\varphi))\varphi_*\varphi^*(\alpha) = cl(Z) \cup \alpha$ using notation as in Exercise 8 above.

Exercise 10. Cohomology class of a point. Prove with the definitions above that for any nonsingular variety X and any closed point $x \in X$ the cohomology class of the point $cl(x) \in H^{2\dim X}(X)(\dim X)$ is the unique element such that $Tr_X(cl(x)) = 1$.

Exercise 11. Cohomology class of a graph. Let $f : X \to Y$ be a morphism of nonsingular projective varieties. Let $\Gamma_f \subset X \times Y$ denote the graph of f. This is a nonsingular closed subvariety. Its class $cl(\Gamma_f)$ is an element of

$$\begin{aligned} H^{2\dim Y}(X \times Y)(\dim Y) &= \bigoplus H^j(X) \otimes H^{2\dim Y - j}(Y)(\dim Y) \\ &= \bigoplus H^j(X) \otimes Hom(H^j(Y), K) \\ &= \bigoplus Hom(H^j(Y), H^j(X)) \end{aligned}$$

where we used Künneth and Poincaré duality. Using these identifications, show that $cl(\Gamma_f)$ corresponds to the element $\sum_j f^{*,j}$, where $f^{*,j}$ is the element of $Hom(H^j(Y), H^j(X))$ corresponding to the restriction of f^* to $H^j(Y)$. So the cohomology class of the graph *is* the action of f on cohomology.

Hint: Use the characterization in Exercise 8 of the cohomology class of Γ_f using the obvious isomorphism $\varphi : X \to \Gamma_f$.

Exercise 12. Cohomology classes and intersection products. Let X be a nonsingular projective variety. Suppose that V and W are subvarieties of X of codimension c and c' that intersect properly. Write $V \cdot W = \sum n_i[Z_i]$ as in the note on intersection theory. Show that

$$cl(V) \cup cl(W) = \sum n_i cl(Z_i)$$

in $H^{2c+2c'}(X)(c + c')$.

Hint: Use that $\Delta^*([V \times W]) = \sum n_i[Z_i]$ in this case, see note on intersection theory. Use axiom (W9).

Definition. Cohomology classes of cycles. Given a Weil cohomology theory we define the *cycle class map* as follows. For any codimension c cycle $\alpha = \sum n_i[Z_i]$ on the nonsingular projective variety X we set $cl(\alpha) = \sum n_i cl(Z_i)$ in $H^{2c}(X)(c)$. We call $cl(\alpha)$ the *cohomology class* of the cycle α.

Definition. Algebraic equivalence, τ-equivalence and homological equivalence and numerical equivalence of cycles. Let X be a nonsingular projective variety. Let T be a nonsingular curve (curve = variety of dimension 1). Let γ be a $(k+1)$-cycle on $X \times T$. Consider a pair of closed points $t, t' \in T$ such that γ and $X \times t$ intersect properly, and γ and $X \times t'$ intersect properly. Any cycle of the form

$$pr_{X,*}(\gamma \cdot X \times t - \gamma \cdot X \times t')$$

is said to be *algebraically equivalent to zero.* Two k-cycles α and β are said to be *algebraically equivalent* if their difference is algebraically equivalent to zero. Two k-cycles α and β are said to be *τ-equivalent* if $N(\alpha - \beta)$ is algebraically equivalent to zero for some integer $N > 1$. Two k-cycles α and β are said to be *homologically equivalent* (wrt H^*) if $cl(\alpha - \beta) = 0$. Two k-cycles α and β are said to be *numerically equivalent* if $\deg(\alpha \cdot \gamma) = \deg(\beta \cdot \gamma)$ for every cycle γ of codimension k that intersects α and β properly. (The degree of of a zero cycle $\delta = \sum n_i x_i$ is just $\deg(\delta) = \sum n_i$.)

Exercise 13. Cohomology classes and τ-equivalence.
(i) Show that homological equivalence is coarser than τ-equivalence.
(ii) Show that homological equivalence is finer than numerical equivalence.

Hint: First reduce to showing $\alpha \sim_{alg} 0 \Rightarrow \alpha \sim_{hom} 0$. Next show that in the definition of algebraic equivalence we may take T to be a nonsingular projective curve. Finally, express the class of γ using Künneth on $X \times T$ to conclude.

It is a long-standing-open conjecture (which is among the consequences of the "standard conjectures") that homological equivalence agrees with numerical equivalence. As we shall see later this is a fundamental open problem in the theory of motives. Note that it is not even clear that homological equivalence is independent of the choice of the Weil cohomology theory!

By the previous exercise we deduce in particular that we get cycle class maps

$$cl : A^*(X) \longrightarrow H^{2*}(X)(*)$$

where $A^*(X)$ are the Chow groups modulo rational equivalence as defined in the note on intersection theory. Because of the moving lemma modulo rational equivalence we conclude (Exercise 12) that this is a ring map.

Exercise 14. Injectivity of dominant pullback. Suppose that $f : X \to Y$ is a dominant morphism of nonsingular projective varieties. Show that $f^* : H^*(Y) \to H^*(X)$ is injective.

Hint: By Bertini's theorem and functoriality we may replace X by a nonsingular subvariety $X' \subset X$ such that $X' \to X$ is generically finite. After this apply the Exercise on trace and degree.

Exercise 15. Euler characteristic. Let $\Delta \subset X \times X$ be the diagonal of a nonsingular projective variety X. Show that

$$Tr_{X \times X}(cl(\Delta) \cup cl(\Delta)) = \sum_{i=0}^{2\dim X} (-1)^i \dim_K H^i(X) := \chi(X).$$

Since $cl(\Delta) \cup cl(\Delta) = cl(\Delta \cdot \Delta)$ by the above and since $Tr(cl(point)) = 1$ we obtain

$$\chi(X) = \deg(\Delta \cdot \Delta).$$

Exercise 16. Cohomology of $\mathbf{P}^1$. Show that $H^1(\mathbf{P}) = (0)$.

Hint: Compute the selfintersection of $\Delta \subset \mathbf{P}^1 \times \mathbf{P}^1$!

Exercise 17. Cohomology of $\mathbf{P}^n$. Show that the odd cohomology of $\mathbf{P}^n$ is zero and that

$$\bigoplus H^{2i}(\mathbf{P}^n)(i) = K[h]/(h^{n+1})$$

with $Tr(h^n) = 1$.
Hint: First, note that the result is true for $\mathbf{P}^1$. Consider the morphism

$$\mathbf{P}^1 \times \ldots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^n,$$

which maps $(x_1, \ldots, x_n)$ to $(x_1 + \ldots + x_n, \ldots, x_1 x_2 \ldots x_n)$. It is the map that identifies $\mathbf{P}^n$ with the quotient $(\mathbf{P}^1 \times \ldots \times \mathbf{P}^1)/S_n$. By the previous results we deduce that

$$H^*(\mathbf{P}^n) \subset (H^*(\mathbf{P}^1 \times \ldots \times \mathbf{P}^1))^{S_n}.$$

This implies the vanishing of the odd cohomology groups. Let $h \in H^2(\mathbf{P}^n)(1)$ be the cohomology class of a hyperplane. Show that the pullback of h to each $\mathbf{P}^1$ is the cohomology class of a point. Show this h satisfies $Tr(h^n) = 1$ and show that the K-algebra generated by h in the cohomology of $\mathbf{P}^1 \times \ldots \times \mathbf{P}^1$ exhausts the S_n invariant classes.

Exercise 18. Linear maps and Segre maps.
(i) Show that a linear map $\mathbf{P}^n \to \mathbf{P}^m$ pulls the class h back to h.
(ii) Show that the Segre map

$$\mathbf{P}^n \times \mathbf{P}^m \longrightarrow \mathbf{P}^{nm+n+m}$$

pulls h back to $1 \otimes h + h \otimes 1$.

Geometrically, this says that the cohomology class of a hyperplane section in $\mathbf{P}^{nm+n+m}$ pulls back to the sum of the classes of the hyperplane sections of $\mathbf{P}^n$ and $\mathbf{P}^m$.

Exercise 19. Chern classes of invertible sheaves. Show that the following procedure gives a well-defined functorial homomorphism

$$c_1 : Pic(X) \longrightarrow H^2(X)(1).$$

Take an invertible sheaf $\mathcal{L}$ on X. If $\mathcal{L}$ has a nonzero section s, so $\mathcal{L} \cong \mathcal{O}(D)$ with $D = div(s)$, then set $c_1(\mathcal{L}) = cl(D)$. In general, write $\mathcal{L} = \mathcal{O}(D_1 - D_2)$ with D_i effective and set $c_1(\mathcal{L}) = cl(D_1) - cl(D_2)$ Functoriality means $f^* c_1(\mathcal{L}) = c_1(f^*\mathcal{L})$.

Exercise 20. Take your favorite smooth projective variety and try to "compute" its Weil cohomology. For example: Grassmanians, elliptic curves, K3-surfaces, etc.

Exercise 21. Chern classes. Show there is a theory of *chern classes* that assigns to every locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules an element

$$c(\mathcal{E}) \in H^{2*}(X)$$

called the total chern class with the following properties
(i) If the rank of $\mathcal{E}$ is 1 then $c(\mathcal{E}) \in H^*(X)$ equals $1 + c_1(\mathcal{E})$, where c_1 is as above.
(ii) Chern classes are functorial with respect to pullbacks.
(iii) For any short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

we have $c(\mathcal{E}_2) = c(\mathcal{E}_1) \cup c(\mathcal{E}_3)$.

In fact, these properties characterize a unique such map. It takes some work to show such a map exists, but it is not too bad. However, a better approach is to define intersection theoretic chern classes $c(\mathcal{E}) \in A^*(X)$ and obtain the cohomological from them by means of the cycle class map. It turns out that this process can also be reversed. In other words, one can define a Weil cohomology theory in terms of data (D1)–(D4) together with a suitable theory of chern classes and show that this leads to a Weil cohomology theory as above.

Weil cohomology theories over non-algebraically closed fields

Let k be a nonalgebraically closed field and let $k \subset \mathbf{C}$ be an algebraic closure. A *variety* over k is an integral scheme separated and of finite type over k. Note that, with this definition, if X is a variety over k then the base change

$$X_{\mathbf{C}} := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\mathbf{C})$$

is not necessarily a variety over $\mathbf{C}$. Namely, $X_{\mathbf{C}}$ may be nonreduced and/or reducible. However, if X is smooth and projective over k then $X_{\mathbf{C}}$ is a finite disjoint union of projective nonsingular varieties over $\mathbf{C}$. An important example of a smooth projective variety over k to keep in mind is $X = \mathrm{Spec}(k')$ where $k \subset k'$ is a finite separable field extension.

A *Weil cohomology theory* H^* for smooth projective varieties over k with coefficients in the characteristic zero field K is given by the following set of data (notation explained below):

- (D1) For every smooth projective algebraic variety X over k a graded commutative algebra $H^*(X)$ over K. The grading is indexed by integers: $H^*(X) = \oplus_{n \in \mathbf{Z}} H^n(X)$ is a direct sum decomposition of K-vector spaces. The multiplication $H^*(X) \times H^*(X) \to H^*(X)$, $(\alpha, \beta) \mapsto \alpha \cup \beta$ is called the *cup product*. It is K-bilinear. Graded commutative means that $\alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha$ for homogenous elements.
- (D2) For every morphism $f : X \to Y$ of smooth projective varieties over k a *pullback* map $f^* : H^*(Y) \to H^*(X)$ which is a K-algebra map preserving the grading.
- (D3) A 1-dimensional K-vector space $K(1)$, which gives rise to *Tate twists* as follows. For a K-vector space V we define $V(n) = V \otimes_K K(1)^{\otimes n}$. If n is negative then $V(n) = V \otimes_K Hom(K(1)^{\otimes -n}, K)$. We will use obvious notation, e.g., given K-vector spaces U, V and W and a linear map $U \otimes_K V \to W$ we obtain a linear map $U(a) \otimes_K V(b) \to W(a+b)$ for an pair $a, b \in \mathbf{Z}$.
- (D4) For every smooth projective variety X over k a *trace map* $Tr : H^{2\dim X}(X)(\dim X) \to K$.
- (D5) For every smooth projective variety X over k and every closed subvariety $Z \subset X$ of codimension c there is given a *cohomology class* $cl(Z) \in H^{2c}(X)(c)$.

These data should satisfy axioms (W1)–(W10) mentioned above, with the exception that in (W5) we no longer require the trace map to be an isomorphism. In the Künneth formula, pay attention that $X \times Y$ need no longer be a smooth variety, but it is a disjoint union of smooth varities and its cohomology is defined to be the direct sum of the cohomologies of its irreducible components. Of course, in axiom (W10) we replace $Spec(\mathbf{C})$ by $Spec(k)$.

Using these axioms you can then show $H^*(Spec(k)) = H^0(Spec(k)) = K$ as an algebra with Tr equal to the identity. In general, for $X = Spec(k')$ with $k \subset k'$ finite separable you can show that $H^*(Spec(k')) = H^0(Spec(k'))$ is a finite separable extension of K. However, for a given coefficient field K and ground field k there can exists Weil cohomology theories H^* and H'^* such that $H^0(Spec(k'))$ and $H'^0(Spec(k'))$ are nonisomorphic as K-algebras. Here is an example.

Example. Suppose that $k = \mathbf{Q}_p = K$. There are two Weil cohomology theories over k with coefficients in K given by

- (i) p-adic Étale cohomology $H^*_{et}(X) := H^*_{et}(X \times_{Speck} Spec(\mathbf{C}), \mathbf{Q}_p)$
- (ii) Algebraic de Rham cohomology $H^*_{dR}(X) := \mathbf{H}^*(X, \Omega^*_{X/k})$

Let $X = Spec(\mathbf{Q}_p[\sqrt{p}])$, then as algebras we have $H^0_{et}(X) = \mathbf{Q}_p \times \mathbf{Q}_p$ and $H^0_{dR}(X) = \mathbf{Q}_p[\sqrt{p}]$.

On the other hand, if the field K is separably closed, then there is no choice for the restriction of the Weil cohomology theory H^* to the 0-dimensional smooth k-varieties. And indeed, in this perhaps less interesting case, you can prove

$$H^*(X) = H^*_{\mathbf{C}}(X \times_{Spec(k)} \mathbf{C})$$

as a functor on all smooth k-varieties for some Weil cohomology theory $H^*_{\mathbf{C}}$ for nonsingular projective varieties over $\mathbf{C}$ (one that can be built out of H^* using "descent").