ON A RESULT OF ARTIN

AISE JOHAN DE JONG

Let k be an algebraically closed field of characteristic $p > 0$. Let t be a parameter and set $S = \mathrm{Spec}(k[[t]])$. Let $f : X \to S$ be a smooth proper morphism with generic fibre X_η and special fibre X_0. In Artin's paper on supersingular K3 surfaces we find a remarkable result on the specialization $L_\eta \to L_0$ of Néron-Serveri groups of X_η and X_0. Namely, if the formal Brauer group of the generic fibre X_η is annihilated by a power of p and a mild assumption on relative Pic holds, then the cokernel of the map $L_\eta \to L_0$ is finite, see [Art74, Theorem 1.1 a]. The condition on the formal Brauer group is equivalent to the condition that all the slopes of the Newton polygon on $H^2(X_\eta)$ are 1, see [AM77].

In this note we show (under some additional hypotheses)

(1) if the slopes on $H^2(X_\eta)$ are all 1, then any crystalline cohomology class on X_0 lifts to an element of the crystalline cohomology of X, and
(2) given an invertible sheaf $\mathcal{L}_0$ on X_0 whose crystalline chern class lifts to a crystalline cohomology class on X, there exists an invertible sheaf $\mathcal{L}$ on X whose restriction to the special fibre is a power of $\mathcal{L}_0$.

Combined these results imply Artin's result in some cases, e.g., in the case of a family of K3 surfaces.

Analyzing the proof of Theorem 1 we see that an assertion as in (1) should generalize to higher degree cohomology. Moreover, Theorem 1 may be true without any assumption on torsion in crystalline cohomology. Equally likely some of the assumptions of Theorem 2 can be weakened. In higher degrees we can ask:

> Consider an algebraic cycle α_0 in codimension c on X_0 whose crystalline cohomology class $cl(\alpha_0) \in H^{2c}_{cris}(X_0/\Sigma)$ lifts to an element of $H^{2c}_{cris}(X/\Sigma)$. Is α_0 the restriction of an algebraic cycle on X?

This question can be viewed as an equi-characteristic p analogue of the p-adic variational Hodge conjecture, see [MP11, Conjecture 9.2].

Throughout this note W is a Cohen ring for k. Set $\Sigma = \mathrm{Spec}(W)$. We denote $\sigma : W[[t]] \to W[[t]]$ the unique lift of frobenius such that $\sigma(t) = t^p$. The structure sheaf on the crystalline site $(X/\Sigma)_{cris}$ of X over Σ is denoted $\mathcal{O}_{X/\Sigma}$. We write $H^2_{cris}(X/\Sigma) = H^2((X/\Sigma)_{cris}, \mathcal{O}_{X/\Sigma})$ and similarly for other schemes over Σ.

Theorem 1. *In the situation above assume*

(1) *the crystalline cohomology of X/S is torsion free*[1], *and*
(2) *all slopes of frobenius on $H^2(X_\eta)$ are* 1.

Then for any $\xi_0 \in H^2_{cris}(X_0/\Sigma)$ there exists an $n > 0$ and an element $\xi \in H^2_{cris}(X/\Sigma)$ such that $\xi|_{X_0} = p^n\xi_0$.

[1] This means that each $R^i f_* \mathcal{O}_{X/\Sigma}$ is a crystal in finite locally free $\mathcal{O}$-modules on $(S/\Sigma)_{cris}$.

Proof. Let M be the value of $R^2 f_* \mathcal{O}_{X/\Sigma}$ on $W[[t]]$. The assumption on torsion implies that M is a finite free $W[[t]]$-module and that M/tM is equal to $H^2_{cris}(X_0/\Sigma)$. The module M comes with a connection $\nabla : M \to M\mathrm{d}t$ and a horizontal σ-linear map $F : M \to M$. The assumption on slopes of Frobenius implies that M is isogenous to another crystal (M', ∇', F) such that F' is divisible by p, see [Kat79]. Then $(M', \nabla', p^{-1}F')$ is a unit root F-crystal over S. Since k is algebraically closed we see that we can find a basis $e'_1, \ldots, e'_n$ of M' such that $F(e'_i) = pe'_i$ (since unit root crystals correspond to representations of the fundamental group – see [BM90] for example). Since M is isogenous to M' we can find $e_1, \ldots, e_n \in M$ such that $\nabla(e_i) = 0$, $F(e_i) = pe_i$, and such that $M/\sum W[[t]]e_i$ is annihilated by a power of p. In particular the elements $\bar{e}_i = e_i \bmod t$ generate a W-submodule of $H^2_{cris}(X_0/\Sigma)$ of finite colength. Thus we can write

$$p^e \xi_0 = \sum \lambda_i \bar{e}_i$$

for some $e \geq 0$ and some $\lambda_i \in W$. Take $\xi = \sum \lambda_i e_i$. Note that $\nabla(\xi) = 0$. Hence ξ determines a gobal section of $R^2 f_* \mathcal{O}_{X/\Sigma}$ over $(S/\Sigma)_{cris}$. Consider the Leray spectral sequence for crystalline cohomology associated to $f : X \to S$ with E_2 page $E_2^{p,q} = H^p_{cris}(S/\Sigma, R^q f_* \mathcal{O}_{X/\Sigma})$. The obstructions to lifting ξ to an element of $H^2(X/\Sigma)$ are: first an element $d_2(\xi)$ of $H^2_{cris}(S/\Sigma, R^1 f_* \mathcal{O}_{X/\Sigma})$ followed by another element of a subquotient of $H^3_{cris}(S/\Sigma, R^0 f_* \mathcal{O}_{X/\Sigma})$. Both groups are zero. Namely $H^i_{cris}(S/\Sigma, \mathcal{E}) = 0$ for $i > 1$ for any crystal $\mathcal{E}$ of finite locally free $\mathcal{O}_{S/\Sigma}$-modules on $(S/\Sigma)_{cris}$. This is true because if $\mathcal{E}$ corresponds to the triple (N, ∇, F) over $W[[t]]$, then crystalline cohomology of $\mathcal{E}$ is computed by the two term complex $\nabla : N \to N\mathrm{d}t$. $\square$

Theorem 2. *In the situation above assume*

(1) *the crystalline cohomology of X/S is torsion free,*
(2) $H^1(X_0, \mathcal{O}_{X_0}) = H^0(X_0, \Omega^1_{X_0/k}) = 0$,
(3) *$\mathcal{L}_0$ is an invertible sheaf on X_0, and*
(4) *$\xi \in H^2_{cris}(X/\Sigma)$ is an element such that $\xi|_{X_0}$ is $c_1(\mathcal{L}_0)$.*

Then $\mathcal{L}_0^{\otimes p^2}$ is isomorphic to the restriction of an invertible sheaf $\mathcal{L}$ on X.

Proof. By Grothendieck's formal existence theorem it suffices to lift $\mathcal{L}_0^{\otimes p^2}$ to the schemes $X_n = X \times_S \mathrm{Spec}(k[t]/(t^{n+1})$. Recall that the crystalline chern class of an invertible sheaf $\mathcal{L}_n$ on X_n is computed using the short exact sequence

$$0 \to (1 + \mathcal{J}_{X_n/\Sigma})^* \to \mathcal{O}^*_{X_n/\Sigma} \to \underline{\mathcal{O}^*_{X_n}} \to 0$$

on the crystalline site as well as the maps

$$(1 + \mathcal{J}_{X_n/\Sigma})^* \xrightarrow{\log} \mathcal{J}_{X_n/\Sigma} \to \mathcal{O}_{X_n/\Sigma}$$

In this proof we will work with the maps

$$c'_1 : \mathrm{Pic}(X_n) \to H^2_{cris}(X_n/\Sigma, \mathcal{J}_{X_n/\Sigma})$$

instead of the usual crystalline chern class. Note that $c'_1(\mathcal{L}_0^{\otimes p^2}) = p^2 c'_1(\mathcal{L}_0)$ is equal to the restriction of $p\xi'$ to X_0 where $\xi' \in H^2_{cris}(X/\Sigma, \mathcal{J}_{X/\Sigma})$ is the image of ξ via the map $p : \mathcal{O}_{X/\Sigma} \to \mathcal{J}_{X/\Sigma}$. The equality holds because the kernel of the map $H^2_{cris}(X/\Sigma, \mathcal{J}_{X/\Sigma}) \to H^2_{cris}(X/\Sigma, \mathcal{O}_{X/\Sigma})$ is zero by the vanishing assumptions in the theorem.

Suppose we have found an invertible sheaf $\mathcal{L}_n$ on X_n such that $c_1'(\mathcal{L}_n)$ is equal to the restriction of $p\xi'$ in $H^2(\mathcal{J}_{X_n/\Sigma})$. We have a first order thickening $X_n \to X_{n+1}$ whose ideal sheaf I can be endowed with a nilpotent divided power structure (by setting all γ_n, $n \geq 2$ equal to zero). There is a short exact sequence

$$0 \to (1+I)^* \to \mathcal{O}^*_{X_{n+1}} \to \mathcal{O}^*_{X_n} \to 0.$$

Moreover, the obstruction to lifting $\mathcal{L}_n$ to an invertible sheaf on X_{n+1} is the image *ob* in $H^2(X_{n+1}, I)$ of the class of $\mathcal{L}_n$ in $H^1(X_n, \mathcal{O}^*_{X_n})$. A computation shows that *ob* is the image of $p\xi'$ under the canonical map

$$H^2_{cris}(X/\Sigma, \mathcal{J}_{X/\Sigma}) \to H^2_{cris}(X_n/\Sigma, \mathcal{J}_{X_n/\Sigma}) \to H^2(X_{n+1}, I).$$

The arrow on the right uses that $(X_n \to X_{n+1}, \gamma)$ is an object of the crystalline site of X_n and that $\mathcal{J}_{X_n/\Sigma}$ restricts to I on this object. But $p\xi'$ maps to p times the image of ξ' in $H^2(X_n, I)$ which is zero. Hence we can lift $\mathcal{L}_n$ to some invertible sheaf $\mathcal{L}_{n+1}$ on X_{n+1}.

Let (M, ∇, F) be as in the proof of Theorem 1. By our assumption that the degree 1 Hodge cohomology groups of X_0/k are zero and the assumption on torsion we conclude that $R^1 f_* \mathcal{O}_{X/\Sigma} = 0$ and that $H^2_{cris}(X_n/\Sigma, \mathcal{J}_{X_n/\Sigma}) \to H^2_{cris}(X_n/\Sigma, \mathcal{O}_{X_n/\Sigma})$ is injective. Hence, using the Leray spectral sequence and the base change theorem in crystalline cohomology we have

$$H^2_{cris}(X_n/\Sigma, \mathcal{J}_{X_n/\Sigma}) \subset H^2_{cris}(X_n/\Sigma, \mathcal{O}_{X_n/\Sigma}) = (M \otimes_{W[[t]]} D_n)^{\nabla}$$

where D_n is the divided power envelope of the ideal (t^n) in the ring $W[[t]]$. Now D_n is p-torsion free and in fact injects into $K[[t]]$ where $K = f.f.(W)$. Since both $p\xi'$ and $c_1'(\mathcal{L}_{n+1})$ have the same value modulo t and are horizontal they have to agree. This proves the desired induction step. $\square$

References

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