A Remark on Isotrivial Families

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Introduction

This is a preliminary writeup; we decided to post this on the web since it is easier to understand than the more recent version [2] which proves stronger results, but is formulated in terms of stacks.

In the paper [1], the second named author proved that you can reduce a specific problem on finding rational points on Brauer-Severi varieties or more precisely on "families of Grassmanians" to the case where the discriminant locus is empty. This is explained in Section 7 of that paper, allthough what is written there is somewhat technical. The new result in this note is that this is actually a completely general phenomenon that holds for "isotrivial" families, provided that the fibre has a reductive automorphism group. The precise theorem (Theorem 1.3) is in Section 1 of this note.

The idea of proving Theorem 1.3 and its proof are completely due to the first named author.

In Section 5 of this note we briefly indicate the application to the period-index problem explained above.

1. Isotrivial families

The title of this section is a little misleading as usually one thinks of an isotrivial family as having finite monodromy. As the reader will see such families are certainly examples to which our discussion applies, but we also allow for a positive dimensional structure group. The families will be isotrivial in the sense that the fibres over a Zariski open will be all isomorphic to a fixed variety V.

So let k be an algebraically closed field of any characteristic. We assume given a variety V over k and an ample invertible sheaf $\mathcal{L}$ over V. We let $m = \dim V$. We introduce another integer $d \geq 1$ which will be an upper bound for the dimension of the base of our families. We are going to ask the following question: Is it true that for any polarized family of varieties over a $\leq d$-dimensional base whose general fibre is V there is a rational point on the generic fibre? We make this more precise as follows.

1.1. Situation. Here we are given a triple $(K/k, X \to S, \mathcal{N})$, with the following properties: (a) The field K is an algebraically closed field extension of k. (b) The map $X \to S$ is a morphism of projective varieties over K. (c) The dimension of S is at most d. (d) We are given an invertible sheaf $\mathcal{N}$ on X. (e) For a general point $s \in S(K)$ we have $(X_s, \mathcal{N}_s) \cong (V_K, \mathcal{L}_K)$.

The notation $(V_K, \mathcal{L}_K)$ refers to the base change of the pair $(V, \mathcal{L})$ to $Spec K$. Thus (e) means that there exists a Zariski open $U \subset S$ such that $(X_s, \mathcal{N}_s) \cong (V_K, \mathcal{L}_K)$ as pairs over K. Considering a suitable Hilbert scheme this then implies that all geometric fibres of $X \to S$ over U are isomorphic to a suitable base change of V.

1.2. Question. Suppose we are in Situation 1.1. Is there a rational point on the generic fibre of $X \to S$? In other words: Is $X(K(S))$ not empty?

A natural problem that arises when studying this question is the possibility of bad fibres in the family $X \to S$. Let us define the discriminant Δ of a family $(K/k, X \to S, \mathcal{N})$ as in 1.1 as the Zariski closure of the set of points $s \in S(K)$ such that $(X_s, \mathcal{L}_s)$ is not isomorphic to $(V_K, \mathcal{L}_K)$. A priori the codimension of (the closure of) Δ is assumed ≥ 1, and typically it will be 1. In this section we show that it often suffices to answer Question 1.2 in cases where the codimension of Δ is bigger, at least as long as we are answering the question for all families.

It is not surprising that the automorphism group G of the pair $(V, \mathcal{L})$ is an important invariant of the situation. The group scheme G has T-valued points which are pairs (ϕ, α), where $\phi : V_T \to V_T$ is an automorphism of schemes over T, and $\alpha : \phi^*\mathcal{L}_T \to \mathcal{L}_T$ is an isomorphism of invertible sheaves. The group law is given by $(\phi, \alpha) \cdot (\psi, \beta) = (\phi \circ \psi, \beta \circ \psi^*(\alpha))$. We leave it to the reader to show that G is an affine group scheme (since $\mathcal{L}$ is ample). In the following theorem G°_{red} denotes the reduction of the connected component of G. Note that G°_{red} is a smooth affine group scheme (since k is algebraically closed, and hence perfect).

1.3. Theorem. Fix $(V, \mathcal{L})$ and d as above. Assume that G°_{red} is reductive. Then: If the answer to Question 1.2 is yes whenever $\Delta = \emptyset$ then the answer to Question 1.2 is yes in all cases.

2. An auxillary family

In this section we begin with a pair $(V, \mathcal{L})$ as above. We will use the notation $\mathcal{L} = \mathcal{O}(1)$. Pick an integer N such that both $\mathcal{O}(N)$ and $\mathcal{O}(N+1)$ are very ample. Consider the k-vector spaces $L_1 = \Gamma(V, \mathcal{O}(N))$ and $L_2 = \Gamma(V, \mathcal{O}(N+1))$. Clearly we obtain a "diagonal" embedding

$$f : V \longrightarrow \mathbf{P}(L_1) \times \mathbf{P}(L_2).$$

The notation we use here is that $\mathbf{P}(L) = Proj(Sym^*(L))$. Since $Proj$ is a contravariant functor and since we like automorphisms of schemes to act from the left, we use the notation that $GL(L)$ and $PGL(L)$ act on the right on a vector space L: $L \times GL(L) \to L$ induces $GL(L) \times \mathbf{P}(L) \to \mathbf{P}(L)$.

Correspondingly, we let $H \subset PGL(L_1) \times PGL(L_2)$ be the closed subgroup scheme of elements which stabilize the closed subscheme $f(V)$ in $\mathbf{P}(L_1) \times \mathbf{P}(L_2)$. In addition we let $V_i \subset \mathbf{P}(L_i)$ be the image of $f(V)$ under the ith projection morphism $\mathbf{P}(L_1) \times \mathbf{P}(L_2) \to \mathbf{P}(L_i)$.

2.1. Lemma. Under the assumptions of Theorem 1.3 the group scheme H is geometrically reductive.

Proof. There is a morphism of group schemes $G \to H$. Namely, by construction G acts on the vector spaces L_i (from the right) via its action on $\mathcal{L} = \mathcal{O}(1)$. In a formula $s \cdot (\phi, \alpha) = \alpha(\phi^*(s))$ for $s \in \Gamma(V, \mathcal{O}(a))$, $a \in \mathbf{N}$. Let us show that this morphism is surjective. Pick any T-valued point h of H. Zariski locally on T we can lift h to a point (h_1, h_2) of $GL(L_1) \times GL(L_2)$ (since $GL \to PGL$ is surjective with kernel $\mathbf{G}_m$). Consider the induced action of h on $f(V) \times T$, and denote ϕ the corresponding automorphism of V_T (so that $f \circ \phi = h|_{f(V)\times T} \circ f$). The existence of h_1 shows that $\phi^*\mathcal{O}(N) \cong \mathcal{O}(N)$. The existence of h_2 implies that $\phi^*\mathcal{O}(N+1) \cong \mathcal{O}(N+1)$. Hence there exists an $\alpha : \phi^*\mathcal{O}(1) \to \mathcal{O}(1)$ such that $g = (\phi, \alpha) \in G(T)$. The induced action of g on L_i agrees with h_i up to a scalar since $L_1 = \Gamma(V, \mathcal{O}(N))$ resp. $L_2 = \Gamma(V, \mathcal{O}(N+1))$. Thus our point h locally comes from a T-valued point of G.

Not only does this show that the morphism $G \to H$ is a surjection, but also that its kernel is $\mathbf{G}_m$. From the definition of geometric reductivity it follows that it is sufficient to prove that G is geometrically reductive. By assumption the normal closed subgroup scheme G°_{red} is geometrically reductive. The group scheme G/G°_{red} is finite and hence geometrically reductive. Any extension of geometrically reductive group schemes is geometrically reductive, and hence we're done. (References ???) QED

Below we will use the natural self duality of $\mathrm{End}(L)$ coming from the trace pairing $(A, B) = Trace(AB)$. Hence we can think of a nonzero element $A \in \mathrm{End}(L)$ up to scaling as a point of $\mathbf{P}(\mathrm{End}(L))$, by the algebra map $Sym^*(\mathrm{End}(L)) \to k[T]$ induced by $B \mapsto Trace(AB)T$ in degree 1. Using this convention we may think of $PGL(L)$ as an open subvariety of $\mathbf{P}(\mathrm{End}(L))$.

We are going to consider a geometric invariant type quotient of the projective scheme

$$\mathbf{P}(2) := \mathbf{P}(\mathrm{End}(L_1)) \times \mathbf{P}(\mathrm{End}(L_2))$$

by the group scheme H acting *from the right*. Namely, H acts on $\mathrm{End}(L_i)$ by multiplication from the left, and as $Proj$ is contravariant, this gives a right action on $\mathbf{P}(2)$. Using the remark above, let us think of a point of $\mathbf{P}(2)$ as a pair $([A_1], [A_2])$ of nonzero endomorphisms $A_i \in \mathrm{End}(L_i)$, each given up to scalars. Then the point $h = (h_1, h_2)$ of H acts as follows: $h \cdot ([A_1], [A_2]) = ([A_1 h_1], [A_2 h_2])$. This is true because $Trace((Ah)B) = Trace(A(hB))$.

Let ω^{-1} be the inverse of the dualizing sheaf on $\mathbf{P}(2)$. This is a canonical ample invertible sheaf wich therefore is endowed with a canonical H-linearization. Thus we can talk about semi-stable and stable points for the action of H on $P(2)$ with respect to ω^{-1}. The result is an open subscheme of semi-stable points $\mathbf{P}(2)^{ss}$ and a quotient morphism

$$\pi : \mathbf{P}(2)^{ss} \longrightarrow Y := \mathbf{P}(2)^{ss} // H.$$

For a pair of matrices $(A_1, A_2) \in \mathrm{End}(L_1) \times \mathrm{End}(L_2)$ we have the expression

$$s(A_1, A_2) = \det(A_1)^{n_1} \det(A_2)^{n_2}.$$

Here $n_i = \dim L_i$. The powers are chosen so that $s(A_1, A_2)$ is a section of $\omega^{-1} \cong \mathcal{O}(n_1^2, n_2^2)$ on $\mathbf{P}(2)$. This section is $PGL(L_1) \times PGL(L_2)$-invariant and hence H-invariant. The locus where s is nonzero is exactly $PGL(L_1) \times PGL(L_2)$. The action of H is closed (and free) on this locus. Hence this gives an open subscheme of the stable locus:

$$PGL(L_1) \times PGL(L_2) \subset \mathbf{P}(2)^s \text{ and } U := PGL(L_1) \times PGL(L_2)/H \subset Y.$$

In particular this shows that $\mathbf{P}(2)^s$ is not empty.

Over U we have a natural flat family of varieties $\mathcal{V}_U$ whose fibres are all isomorphic to V. Namely, to a point $u = gH$ we can associate the closed subvariety $V_u := g(f(V))$ of $\mathbf{P}(L_1) \times \mathbf{P}(L_2)$, where $g = ([A_1], [A_2])$ acts componentwise on $\mathbf{P}(L_1) \times \mathbf{P}(L_2)$ from the left. More precisely, this family is the quotient of the variety

$$\left\{(x_1, x_2, [A_1], [A_2]) \mid ([A_1]^{-1}x_1, [A_2]^{-1}x_2) \in f(V)\right\} \subset \mathbf{P}(L_1) \times \mathbf{P}(L_2) \times PGL(L_1) \times PGL(L_2)$$

by the free closed action of H from the right: $(x_1, x_2, [A_1], [A_2]) \cdot h = (x_1, x_2, [A_1 h_1], [A_2 h_2])$. Thus there is a commutative diagram

$$\begin{array}{ccc} \mathcal{V}_U & \longrightarrow & \mathbf{P}(L_1) \times \mathbf{P}(L_2) \times U \\ \downarrow & & \downarrow \\ U & = & U \end{array}$$

where the upper horizontal arrow is a closed immersion.

Definition. The closed subscheme $\mathcal{V} \subset \mathbf{P}(L_1) \times \mathbf{P}(L_2) \times Y$ is the scheme theoretic closure of the family of projective schemes $\mathcal{V}_U/U$. The **discriminant** Δ of this family is the Zariski closure of the set

$$\{y \in Y(k) | \mathcal{V}_y \not\cong V\}$$

Note that it is not clear that the formula above defines a closed subscheme. Since U is open in Y, the fibre of $\mathcal{V}$ over $u \in U(k)$ is the variety V_u.

Recall that $V_i = \mathrm{pr}_i(f(V)) \subset \mathbf{P}(L_i)$. The *secant variety* of V_i is the union in $\mathbf{P}(L_i)$ of all lines that meet V_i in a closed subscheme of length at least 2. We will denote this closed subscheme of $\mathbf{P}(L_i)$ by $Secant(V_i)$.

Note that our conventions above force us to think of $\mathrm{End}(L)$ as acting on the vector space L from the right: $L \times \mathrm{End}(L) \to L$. Also, if $A \in \mathrm{End}(L)$ is not invertible, then the quotient map $L \to Coker(A)$ defines a closed subscheme $\mathbf{P}(Coker(A)) \subset \mathbf{P}(L)$. (In case A is invertible this also makes sense, namely in this case $\mathbf{P}(Coker(A))$ is empty.) The rational map $\mathbf{P}(A) : \mathbf{P}(L) - - \to \mathbf{P}(L)$ corresponding to A is defined on the open subscheme $\mathbf{P}(L) \setminus \mathbf{P}(Coker(A))$.

2.2. Definition. A point $([A_1], [A_2]) \in \mathbf{P}(\mathrm{End}(L_1)) \times \mathbf{P}(\mathrm{End}(L_2))$ is called **good** if both intersections $\mathbf{P}(Coker(A_i)) \cap Secant(V_i)$ are empty.

Except for the trivial case that V is a point we have $V_i \subset Secant(V_i)$, and we will use this below. Furthermore, the set of good points is clearly open, and hence the bad points (i.e., the ones which aren't good) form a closed subset.

The reason for the terminology in 2.2 is the following. If $([A_1], [A_2])$ is a good point then we can make sense of image of $f(V)$ under the rational map $\mathbf{P}(A_1) \times \mathbf{P}(A_2) : \mathbf{P}(L_1) \times \mathbf{P}(L_2) - - \to \mathbf{P}(L_1) \times \mathbf{P}(L_2)$. This is true as the locus where this morphism is not defined is equal to $\mathbf{P}(Coker(A_1)) \times \mathbf{P}(L_2) \bigcup \mathbf{P}(L_1) \times \mathbf{P}(Coker(A_2))$. Let us call this image $V_{A_1 A_2}$. In addition the condition in 2.2 implies that the induced morphism $V \to V_{A_1 A_2}$ is an isomorphism.

It is tempting to think that $Y - \Delta$ is the image of the good points in $\mathbf{P}(2)^{ss}$. This is not so clear however; for example π may map a bad and a good point to the same point in Y. What is clear is that the locus of good points is open and H invariant. We will analyze this further.

To do this we recall the following result from GIT. Let $y \in Y(K)^{ss} - Y(K)^s$ be a strictly semi-stable point over an algebraically closed field K. The fibre $\pi^{-1}(y) \subset \mathbf{P}(2)^{ss}_K$ contains a unique closed orbit of smallest dimension. This orbit is affine and the stabilizer of any point in this orbit is a geometrically reductive subgroup scheme of H_K of positive dimension. See [??]. The uniqueness assertion implies that the orbit exists also when K is not algebraically closed.

Let R be the set of bad, semi-stable points. We think of R as an H-invariant reduced closed subscheme of $\mathbf{P}(2)^{ss}$. We construct an H-invariant closed subscheme $Z \subset R$ in the following manner. For every irreducible component R' of R we let ξ be the generic point of the image $\pi(R') \subset Y$. By the above, there is a unique closed H-orbit O_ξ in the fibre $R'_\xi = R' \cap \pi^{-1}(\xi)$. This is also the unique closed orbit in $\pi^{-1}(\xi)$. Next, we define Z' as the closure of O_ξ in R'. In other words, $Z' \subset R'$ is irreducible and $Z'_\xi = O_\xi$. We set Z equal to the union of the subvarieties Z'.

2.3. Lemma. The discriminant Δ is contained in $\Delta' = \pi(Z)$.

Proof. First we observe that $R \subset \pi^{-1}(\pi(Z))$. Namely, this is clear component by component. Let $W = \mathbf{P}(\mathrm{End}(L_1)) \times \mathbf{P}(\mathrm{End}(L_2)) \setminus \pi^{-1}(\pi(Z))$. This is an open H-invariant subscheme consisting entirely of good points by our first observation. It follows that the family of closed subschemes with fibres $V_{A_1A_2}$ is well defined and flat over W. In other words we obtain a morphism $W \to \mathit{Hilb}$ to the Hilbert scheme of $\mathbf{P}(L_1) \times \mathbf{P}(L_2)$. On the open dense subscheme $PGL(L_1) \times PGL(L_2) \subset W$ the morphism $W \to \mathit{Hilb}$ is constant on H-orbits. Since $PGL(L_1) \times PGL(L_2)$ is schematically dense in W we deduce that $W \to \mathit{Hilb}$ is H-invariant. Since $W \to W//H = Y - \pi(Z)$ is a categorical quotient we deduce the existence of a morphism $Y - \pi(Z) \to \mathit{Hilb}$ extending the one given by the restriction of $\mathcal{V}$ to U. The result follows because this implies that the fibre of $\mathcal{V}$ over the image of $([A_1], [A_2])$ in W is equal to $V_{A_1A_2}$. QED

Our next goal is to estimate from below the codimension of $\pi(Z)$ in Y. We will first do this for an irreducible component Z' of Z which meets the stable locus of the action of H. In this case the codimension of Z' in $\mathbf{P}(2)^s s$ is the same as the codimension of $\pi(Z')$ in Y. (Since all stable points are properly stable in our case.) So in this case it suffices to bound from below the codimension of the bad points in $\mathbf{P}(2)$.

2.4. Lemma. Let c_i be the codimension of $\mathit{Secant}(V_i)$ in $\mathbf{P}(L_i)$. Assume $c_i \geq 1$. Then the codimension of the bad points in $\mathbf{P}(2)^{ss}$ is at least $\min\{c_i + 1\}$.

Proof. To see this we stratify the bad locus by the rank of the corresponding matrices. The set of matrices A_1 whose rank is i less than maximal has codimension i^2 in $\mathbf{P}(\mathrm{End}(L_1))$. In this stratum the locus where $\mathbf{P}(\mathit{Coker}(A_1))$ meets $\mathit{Secant}(V_1)$ has codimension $\max\{0, c_1 - i + 1\}$. Thus the codimension of the bad locus in this stratum is

$$i^2 + \max\{0, c_1 - i + 1\} \geq i^2 + c_1 - i + 1.$$

The minimal value of this is attained when $i = 1$. A similar argument applies to A_2. QED

Next, we deal with those components Z' of Z whose general point corresponds to a strictly semi-stable point of $\mathbf{P}(2)$. Basically, here we will see that the strictly semi-stable locus has a suitably high codimension in $\mathbf{P}(\mathrm{End}(L_1)) \times \mathbf{P}(\mathrm{End}(L_2))$. So let $Z' \subset Z$ be such an irreducible component and let $([A_1], [A_2]) \in Z'(k)$ be a general point. By construction, $([A_1], [A_2])$ has a positive dimensional stabilizer $U \subset H$ which is geometrically reductive. Also the fibre of the morphism $\pi : Z' \to \pi(Z')$ over a general point is (set-theoretically) the H-orbit of $([A_1], [A_2])$ in $\mathbf{P}(2) = \mathbf{P}(\mathrm{End}(L_1) \times \mathbf{P}(\mathrm{End}(L_2))$. In particular this means that the image scheme $\pi(Z')$ has dimension $\dim Z' - (\dim H - \dim U)$. On the other hand, let $\tilde{U}$ be the inverse image of U in $SL(L_1) \times SL(L_2)$. The fact that the point $([A_1], [A_2])$ is fixed by U implies that there are characters $\chi_i : \tilde{U} \to \mathbf{G}_m$ such that the equation

$$A_i u = A_i \chi_i(u), \ \forall u \in \tilde{U}(k)$$

holds in $\mathrm{End}(L_i)$. The representation of $\tilde{U}$ on L_i (from the right) has a canonical largest $\tilde{U}$-invariant subspace $L_i(\chi_i)$ such that $\tilde{U}$ acts diagonally on $L_i(\chi_i)$ with character χ_i. The equation above shows that the right action $A_i : L_i \to L_i$ is a composition $L_i \to L_i(\chi_i) \to L_i$. For a fixed subgroup $U \subset H$ and a fixed pair of characters χ_i we see that the dimension of the variety of such A_i is at most $(\dim L_i)(\dim L_i(\chi_i))$. The dimension of the space of subgroup schemes $U \subset H$ occuring for points in our component Z' is at most

$\dim H - \dim U$. Namely, in an irreducible flat family of reductive subgroup schemes of H two general points correspond to conjugate subgroups. Reference ??. We conclude that the dimension of Z' is at most

$$n_1 a_1 - 1 + n_2 a_2 - 1 + \dim H - \dim U,$$

where $a_i = \dim L_i(\chi_i)$ and $n_i = \dim L_i$. Minus 1 because we are working in $\mathbf{P}(2)^{ss} = \mathbf{P}(\mathrm{End}(L_1) \times \mathbf{P}(\mathrm{End}(L_2))$ and points correspond to endomorphisms up to scalars. By an earlier remark this implies that the dimension of $\pi(Z')$ is at most

$$n_1 a_1 - 1 + n_2 a_2 - 1.$$

The dimension of Y is $n_1^2 - 1 + n_2^2 - 1 - \dim H$ and so the codimension of $\pi(Z')$ is at least

$$(n_1 - a_1) n_1 + (n_2 - a_2) n_2 - \dim H.$$

Since U and hence $\tilde{U}$ are positive dimensional and geometrically reductive we see that $\tilde{U}$ has a nontrivial 1 parameter subgroup. Since $\tilde{U} \subset SL(L_1) \times SL(L_2)$ it follows that $a_i \leq n_i - 1$. In particular we obtain that the codimension of $\pi(Z')$ is at least $n_1 + n_2 - \dim H$.

Finally, we remark that the dimension of H and the secant varieties $Secant(V_i)$ is bounded independently of the integer N we chose at the start of this section. For H this is clear from the proof of Lemma 2.1. For the secan varieties: The dimension of a secant variety of any projective variety V in any projective embedding is bounded by $\max\{2 \dim V; \dim_k m_v/m_v^2, v \in V(k)\}$.

2.5. Theorem. Pick any integer $c \geq 0$, and given c pick the integer $N >> c$. Then: In the situation as described above, the family of schemes $\mathcal{V} \to Y$ in $\mathbf{P}(L_1) \times \mathbf{P}(L_2) \times Y$ compactifiying the family $\{V_{gH} = g(f(V))\}$ over $U = PGL(L_1) \times PGL(L_2)/H$ restricts to a flat family with all geometric fibres isomorphic to V over $Y \setminus \Delta'$ where

$$\mathrm{codim}(\Delta' \subset Y) \geq c.$$

Proof. This summarizes the remarks made above. The geometrical statements are explained in the proof of Lemma 2.3. If Z' is an irreducible component of Z which meets $\mathbf{P}(2)^s$, then $\mathrm{codim}_{\pi(Z')}(Y) = \mathrm{codim}_{Z'} \mathbf{P}(s)^{ss}$ and the estimate of Lemma 2.4 applies (see the discussion just above Lemma 2.4). Since the secant varieties $Secant(V_i)$ have dimension bounded independently of N the result follows. If Z' consists of strictly semi-stable points then we win since $n_1 + n_2 - \dim H \to \infty$ as $N \to \infty$. QED

3. Proof of Theorem 1.3

Suppose that $(K/k, \nu : X \to S, \mathcal{N})$ is as in Situation 1.1, and that the dimension of S is at most d. We will produce another triple $(K'/k, X' \to S', \mathcal{N}')$ with empty discriminant and with $\dim S' \leq d$, such that the existence of a rational section for X'/S' implies the existence of a rational section of X/S. Clearly this will suffice to prove Theorem 1.3.

The first step of the proof will be to find a rational map $S - - \to Y$ (defined on an open $W \subset S$) so that the pullback of the family $\mathcal{V}$ is birationally equivalent to X.

3.1. Lemma. There is a nonempty open $W \subset S$, a right G-torsor $T \to W$ an isomorphism $\phi : V_T \to X_T$ over T and an isomorphism $\alpha : \phi^* \mathcal{N} \to \mathcal{L}_T$. These isomorphisms may be chosen to be G-equivariant.

Proof. We let $Y \to S$ be the scheme representing the functor which associates to a scheme $T \to S$ the set of isomorphisms of pairs $(V_T, \mathcal{L}_T)$ with $(X_T, \mathcal{N}_T)$ as in the lemma. See [Grothendieck??]. The reader checks that G acts on Y from the right (by precomposing). By assumption (Situation 1.1) we know that a general fibre of $Y \to S$ is a G-torsor. All we have to do to show the lemma is prove that $Y \to S$ is flat over a nonempty open $W \subset S$, and set $T = Y|_W$. Flatness is automatic over a nonempty open since S is a variety (generic flatness), see [??]. QED

Pick an integer N as in Theorem 2.5 with $c = d + 1$. Choose a nonempty open subset $W \subset S$ contained in the open identified in Lemma 3.1, such that the pushforwards $\nu_* \mathcal{N}^{\otimes N}$ and $\nu_* \mathcal{N}^{\otimes N+1}$ are finite free over W. Choose isomorphisms of locally free sheaves $\gamma_1 : L_1 \otimes \mathcal{O}_W \to \nu_* \mathcal{N}^{\otimes N}$ and $\gamma_2 : L_2 \otimes \mathcal{O}_W \to \nu_* \mathcal{N}^{\otimes N+1}$. We are using the notation $L_1 = \Gamma(V, \mathcal{L}^{\otimes N})$, etc introduced in Section 2. These isomorphisms induce a closed immmersion

$$i : X_W \longrightarrow \mathbf{P}(L_1) \times \mathbf{P}(L_2) \times W.$$

Consider the torsor $T \to W$ and maps (ϕ, α) of Lemma 3.1. The composition $i_T \circ \phi$ is a composition $(g_1, g_2) \circ f_T$, where $f : V \to \mathbf{P}(L_1) \times \mathbf{P}(L_2)$ is the embedding of Section 2, $g_1 \in GL(L_1)(T)$ comes from composing

$$L_1 \otimes \mathcal{O}_T = (V_T \to T)_* \mathcal{O}_{V_T}(N) \stackrel{\gamma_{1,T}}{\longrightarrow} (X_T \to T)_* \mathcal{N}_T^{\otimes N} \stackrel{\alpha}{\longrightarrow} L_1 \otimes \mathcal{O}_T.$$

and similarly for g_2. The pair (g_1, g_2) defines a morphism $T \to \mathbf{P}(2) = \mathbf{P}(\mathrm{End}(L_1)) \times \mathbf{P}(\mathrm{End}(L_2))$. We claim this morphism is G-equivariant, where G acts on $\mathbf{P}(2)$ (from the right) via its morphism to H (see proof of Lemma 2.1 for this map). To prove this you have to trace through a commutative diagram which we leave to the reader. The image is contained in the open $PGL(L_1) \times PGL(L_2) \subset \mathbf{P}(2)$. Thus we find an induced morphism $W \to U \subset Y = \mathbf{P}(2)^{ss}//H$. By construction, the pullback of the family $\mathcal{V}|_U = \mathcal{V}_U$ is equal to $X|_W$. Clearly, if the family $\mathcal{V}$ over the image of W in U has a rational section then so does the family X/S. We have reached the following intermediate result.

3.2. Scholium. We may assume that S is a closed subvariety of dimension $\leq d$ of Y_K meeting the open subscheme U_K, and that X is the irreducible component of the fibre product $\mathcal{V} \times_Y S$ dominating S.

Next, we will choose a 1-parameter family $\{S_t; t \in \mathbf{A}^1_K\}$ of closed subschemes $S_t \subset Y_K$ such that: (a) S is an irreducible component of (scheme theoretic) fibre S_0, and (b) $S_t \cap \Delta = \emptyset$ for t generic. Our triple $(K'/k, X'/S', \mathcal{N}')$ will be gotten by taking $K' = \overline{K(t)}$, where t is the generic point of $\mathbf{A}^1_K$.

To construct the family S_t explicitly, let us take a projective embedding $Y \to \mathbf{P}^n$. Let η be the generic point of S; we can think of this as a point of the scheme U_K. Note that U is smooth over k since it is the quotient of a smooth scheme by a free action of a linear algebraic group (reference??). In particular, Y_K is smooth at η, i.e., Y_K is smooth generically along S, which is (being a variety over $K = \bar{K}$) generically smooth. This means we can take a complete intersection $S_0 = Y_K \cap H_1 \cap \ldots \cap H_\kappa$ of hypersurfaces $H_i \subset \mathbf{P}^n_K$ with the following property: $S \subset S_0$, $\dim S = \dim S_0$, and $\mathcal{O}_{S,\eta} = \mathcal{O}_{S_0,\eta}$, where η is the generic point of S (i.e., S_0 is generically along S smooth over K). To see that we can find such a sequence H_i we can pick elements $H_i \in \Gamma(\mathbf{P}^n, I_S(\deg H_i))$ which generate I_S/I_Y at a suitable (general) point of S. Details left to the reader. Let us also take a *general* complete intersection $S_\infty = Y_K \cap F_1 \cap \ldots \cap F_\kappa$, with $\deg F_i = \deg H_i$. By our choice of N (and Theorem 2.5) we see that $S_\infty \cap \Delta = \emptyset$.

The family we take is

$$S_t = Y_K \cap (tF_1 + H_1 = 0) \cap \ldots \cap (tF_\kappa + H_\kappa = 0).$$

In this formula F_i resp. H_i stands for the equation defining F_i resp. H_i. We denote the total space of $\{S_t\}$ by $\mathcal{S}$. This satisfies the properties (a) and (b) mentioned above. The final step is to show that a rational section of

$$\mathcal{V} \times_Y \mathcal{S}_{\overline{K(t)}} \to \mathcal{S}_{\overline{K(t)}}$$

induces a rational section of $\mathcal{V} \times_Y S$. This we see as follows. Suppose σ is such a rational map. As $\overline{K(t)}$ is the limit of finte field extensions of $K(t)$, there is a finite extension $K(t) \subset L$ such that σ is defined over L. Now $L \supset K(t)$ is the function field extension induced by a finite ramified covering $C \to \mathbf{A}^1_K$ of nonsingular curves over K. Pick a point $0 \in C$ which maps to 0 in $\mathbf{A}^1_K$. We can think of σ as a rational map

$$\mathcal{S} \times_{\mathbf{A}^1_K} C \longrightarrow \mathcal{V} \times_Y \mathcal{S} \times_{\mathbf{A}^1_K} C.$$

By our choice of S_0, the scheme $\mathcal{S} \times_{\mathbf{A}^1_K} C$ is regular at the codimension 1 point $\tilde{\eta} = \eta \times_0 0$. The residue fields $\kappa(\eta)$ and $\kappa(\eta \times_0 0)$ are the same. Since $\mathcal{V} \to Y$ is projective the rational map σ extends to the point $\eta \times_0 0$ and hence we get the rational point on $\mathcal{V}_\eta = X_{K(\eta)}$ as desired.

Thus the triple $(K'/k, X' \to S', \mathcal{N}')$, with $K' = \overline{K(t)}$ and $S' = \mathcal{S}_{K'}$ has empty discriminant and the existence of a rational point for this family implies te existence of a rational point for the original family. This finishes the proof of Theorem 1.3.

4. Simple applications

We are going to apply this in a more serious way to Grassmanians, but let us indicate some simple applications of Theorem 1.3 in this section.

4.1. Fermat Hypersurfaces. As a first case we take V a Fermat hypersurface of degree d in $\mathbf{P}^{d^2-1}$

$$V : X_0^d + X_1^d + \dots X_{d^2-1}^d = 0,$$

with $\mathcal{L} = \mathcal{O}_V(1)$. In this case the group scheme G is an extension of a finite group by $\mathbf{G}_m$ so certainly reductive. Consider the following family with general fibre $(V, \mathcal{L})$ over $Spec(k[s,t])$:

$$\sum_{0 \le i,j \le d-1} s^i t^j X_{i+dj}^d = 0,$$

This family does not have a rational point over $k(s,t)$. Reference ??. There is an elementary proof of this by looking at what it means to have a polynomial solution to the above. We conclude from Theorem 3.1 that there is a family over a projective surface with every fibre isomorphic to $(V, \mathcal{L})$, without a rational section. We like this example because it is not immediately obvious how to write one down explicitly.

4.2. Projective spaces. Another case is where we take the pair $(V, \mathcal{L})$ to be $(\mathbf{P}^n, \mathcal{O}(n+1))$. Note that $\mathcal{O}(n+1) = \omega_{\mathbf{P}^n}^{-1}$ so the families in question are canonically polarized, and we are just talking about the problem of having nontrivial families of Brauer-Severi varieties. In particular, our theorem reduces the problem of proving the nullity of the Brauer group of a curve to the problem of proving the nonexistence of Brauer-Severi varieties having no rational sections over projective nonsingular curves. As far as we know this is not really helpfull, since the proof of Tsen's theorem is pretty straigthforward anyway. However, it illustrates the idea!

5. Grassmanians and the period-index problem

Suppose that F is a field and that $\alpha \in \mathrm{Br}(F)$ is a Brauer class. The *period* of α is the smallest integer $m > 0$ such that $m\alpha = 0$. The *index* of α is the smallest integer $e > 0$ such that there is a central simple algebra B of degree e representing α. The period-index problem over F is the problem as to whether the period always equals the index for all Brauer classes over F.

Fix a central simple algebra A of some degree d representing the class α, i.e., $\dim_F A = d^2$. (Note that the index of α divides d.) For each integer $1 \le e \le d$ consider the variety X_e parametrizing right ideals of rank ed in A. In other words a T-valued point of X_e corresponds to a sheaf of right ideals $I \subset \mathcal{O}_T \otimes A$, locally a direct summand of rank ed as $\mathcal{O}_T$-modules.

Note that if X_e has a rational point $I \subset A$, then the commutant of A acting on I is an algebra B of degree e representing $-\alpha$. Hence the index of the Brauer class α is the smallest integer e such X_e has an F-rational point.

Geometrically the varieties X_e are isomorphic to Grassmanians. Namely, $X_{e,\bar{F}} \cong \mathrm{Grass}(e,d)$ the Grassmanian of e-dimensional subspaces of a d-dimensional vector space. Since $\mathrm{Pic}(\mathrm{Grass}(e,d)) = \mathbf{Z}$ with a canonical ample generator, we see that each X_e gives rise to a Brauer class α_e, namely the obstruction to descending this ample generator to an invertible sheaf over X_e.

5.1. Lemma. We have $\alpha_e = e\alpha$. (Up to sign.)

Proof. Omitted. See forthcoming paper [2] for more details. QED

In particular this means that if we take e equal to the period of α then the variety carries an ample invertible sheaf which is geometrically an ample generator of $\mathrm{Pic}(\mathrm{Grass}(e,d))$. We conclude that the period index-problem for the field F is equivalent the following problem for F. (It really is equivalent, allthough the above only proves one implication.)

5.2. Problem. (Equivalent to the period-index problem over F.) Consider pairs $(V, \mathcal{L})$ over F such that $V_{\bar{F}} \cong \mathrm{Grass}(e,d)$ and $\mathcal{L}_{\bar{F}}$ corresponds to an ample generator of $\mathrm{Pic}(\mathrm{Grass}(e,d))$. Is $V(F)$ not empty?

5.3. Remark. It suffices to do this in all cases where $e < d/2$ since we can also increase d by replacing A by $\mathrm{Mat}(2 \times 2, A) = \mathrm{Mat}(2 \times 2, F) \otimes_F A$.

The following theorem follows from the main theorem of these notes (Theorem 1.3).

5.4. Theorem. The period-index problem over all function fields of all surfaces $F = k(S)$ over the algebraically closed field k is equivalent to the period-index problem for unramified Brauer classes, i.e., those classes that come from $\mathrm{Br}(S)$, where S is a nonsingular *projective* surface over k. , thereby producing a new proof of the period-index problem for function fields of surfaces (the main theorem of [1]).

In other words, we have to find a rational section for every $X \to S$ that satisfies the following properties:
1. S is a smooth projective surface over k,
2. $X \to S$ is a smooth projective morphism,
3. for all $s \in S(k)$ the fibre X_s is isomorphic to the Grassmanian $\mathrm{Grass}(e, d)$, and
4. there exists an invertible sheaf $\mathcal{L}$ on X such that the restriction of $\mathcal{L}$ to every fibre is the ample generator of $\mathrm{Pic}(X_s)$.

In the forthcoming paper [2] we will apply our methods on rational 1-connectedness to prove this result, thereby producing a new proof of the period-index problem for function fields of surfaces (the main theorem of [1]).

References

[1] Aise Johan de Jong, *The period-index problem for the Brauer group of an algebraic surface*, Duke Mathematical Journal, **123**, No. 1, pp. 71-94.

[2] Jason Starr and Aise Johan de Jong, *Almost proper GIT-stacks and discriminant avoidance*, in preparation.