ALMOST PROPER GIT-STACKS AND DISCRIMINANT AVOIDANCE

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1. Introduction

Consider an algebraic stacks of the form $[\mathrm{Spec}(k)/G]$ where G is a geometrically reductive group scheme over the field k. It turns out that such a stack is nearly proper, see Proposition 2.4.1.

Next, consider a pair $(V, \mathcal{L})$ consisting of a projective variety V over k and an invertible sheaf. Also, fix an integer $d \geq 1$. We would like to know if every d-dimensional family of polarized varieties $X \to S$, $\mathcal{N} \in \mathrm{Pic}(X)$ all of whose fibres are isomorphic to $(V, \mathcal{L})$ has a rational section. For example this is true if V is a nodal plane cubic.

Theorem 1.0.1. *(Theorem 2.1.3.) Assume $G = Aut(V, \mathcal{L})$ is geometrically reductive. If $X \to S$ has a section whenever S is a projective variety of dimension d then there is a section no matter what $(X, \mathcal{N})/S$ you start with, provided* $\dim S \leq d$.

Loosely speaking this means that if you prove the existence of rational sections whenever the discriminant is empty then you prove it in general. For example, it implies that if you are trying to find rational sections on families of polarized homogenous varieties over surfaces then it suffices to do so in the case of families of homogenous varieties over projective nonsingular surfaces. Our proof of this theorem depends on the result on GIT-stacks mentioned above.

This effort is part of our work on using the geometry moduli spaces of rational curves on fibres of families $X \to S$ to prove the existence of rational sections of X/S. See the forthcoming papers [**?**]. In this paper we handle the special case of a family of Grassmanians over a smooth projective surface S, see Theorem 4.2.1. The method is to fibre the surface S in curves C_t. For each $C = C_t$ we study the moduli space Σ of sections of some large degree of $X|_C \to C$. We prove that the MRC-quotient of Σ equals the Jacobian of the curve C. Finally we use the Graber-Harris-Starr theorem to conclude.

In the end, putting everything together we deduce the period-index theorem for Brauer classes over function fields of surfaces, compare [**?**].

Theorem 1.0.2. *Let k be an algebraically closed field and let $k(S)$ be the function field of a surface over S. The the period equals the index for any Brauer class over $k(S)$.*

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2. Isotrivial families

The title of this section is a little misleading as usually one thinks of an isotrivial family as having finite monodromy. As the reader will see such families are certainly examples to which our discussion applies, but we also allow for a positive dimensional structure group. The families will be isotrivial in the sense that the fibres over a Zariski open will be all isomorphic to a fixed variety V.

2.1. **Statement of the result.** Let k be an algebraically closed field of any characteristic. We assume given a variety V over k and an ample invertible sheaf $\mathcal{L}$ over V. We let $m = \dim V$. We introduce another integer $d \geq 1$ which will be an upper bound for the dimension of the base of our families. We are going to ask the following question: Is it true that for any polarized family of varieties over a $\leq d$-dimensional base whose general fibre is V, there is a rational point on the generic fibre? We make this more precise as follows.

Situation 2.1.1. Here we are given a triple $(K/k, X \to S, \mathcal{N})$, with the following properties: (a) The field K is an algebraically closed field extension of k. (b) The map $X \to S$ is a morphism of projective varieties over K. (c) The dimension of S is at most d. (d) We are given an invertible sheaf $\mathcal{N}$ on X. (e) For a general point $s \in S(K)$ we have $(X_s, \mathcal{N}_s) \cong (V_K, \mathcal{L}_K)$.

The notation $(V_K, \mathcal{L}_K)$ refers to the base change of the pair $(V, \mathcal{L})$ to Spec$K$. Thus (e) means that there exists a Zariski open $U \subset S$ such that $(X_s, \mathcal{N}_s) \cong (V_K, \mathcal{L}_K)$ as pairs over K for all $s \in U$. Considering a suitable Hilbert scheme this then implies that all geometric fibres of $X \to S$ over U are isomorphic to a suitable base change of V.

Question 2.1.2. Suppose we are in Situation 2.1.1. Is there a rational point on the generic fibre of $X \to S$? In other words: Is $X(K(S))$ not empty?

A natural problem that arises when studying this question is the possibility of bad fibres in the family $X \to S$. Let us define the discriminant Δ of a family $(K/k, X \to S, \mathcal{N})$ as in Situation 2.1.1 as the Zariski closure of the set of points $s \in S(K)$ such that $(X_s, \mathcal{L}_s)$ is not isomorphic to $(V_K, \mathcal{L}_K)$. A priori the codimension of (the closure of) Δ is assumed ≥ 1, and typically it will be 1. In this section we show that it often suffices to answer Question 2.1.2 in cases where the codimension of Δ is bigger, at least as long as we are answering the question for all families.

It is not surprising that the automorphism group G of the pair $(V, \mathcal{L})$ is an important invariant of the situation. The group scheme G has T-valued points which are pairs (ϕ, α), where $\phi : V_T \to V_T$ is an automorphism of schemes over T, and $\alpha : \phi^*\mathcal{L}_T \to \mathcal{L}_T$ is an isomorphism of invertible sheaves. The group law is given by $(\phi, \alpha) \cdot (\psi, \beta) = (\phi \circ \psi, \beta \circ \psi^*(\alpha))$. We leave it to the reader to show that G is an affine group scheme (since $\mathcal{L}$ is ample). In the following theorem G°_{red} denotes the reduction of the connected component of G. Note that G°_{red} is a smooth affine group scheme (since k is algebraically closed, and hence perfect).

Theorem 2.1.3. *Fix $(V, \mathcal{L})$ and d as above. Assume that G°_{red} is reductive. If the answer to Question 2.1.2 is yes whenever $\Delta = \emptyset$, then the answer to Question 2.1.2 is yes in all cases.*

The proof has 2 parts: deformation and specialization. The deformation argument proves the following: For every triple $(K/k, X \to S, \mathcal{N})$, there is a dense open subset

$U \subset X$ and a deformation of $(X_U \to U, \mathcal{N}|_{X_U})$ to a triple $(K'/k, X' \to S', \mathcal{N}')$ with trivial discriminant. The specialization argument proves the following: Every rational point of the generic fiber of $X' \to S'$ specializes to a rational point of the generic fiber of $X \to S$. Thus Question 2.1.2 has a positive answer for $(K/k, X \to S, \mathcal{N})$ if it has a positive answer for $(K'/k, X' \to S', \mathcal{N}')$.

2.2. **A bijective correspondence.** To deform the pair $(X_U \to U, \mathcal{N}|_{X_U})$, it is convenient to first convert the pair into a G-torsor over U, deform the torsor, and then convert this back into a triple. This subsection describes how to convert between pairs and G-torsors. As in subsection 2.1, denote by G the automorphism group scheme of $(V, \mathcal{L})$.

Let U be any k-scheme. Let $(X \to U, \mathcal{N})$ be a pair where $X \to U$ is a flat proper morphism and $\mathcal{N}$ an invertible sheaf on X. We assume that every geometric fiber of $(X, \mathcal{N})$ over U is isomorphic to the base-change of $(V, \mathcal{L})$. Consider the functor that associates to a scheme $T \to U$ over U the set of pairs (ϕ, α), where $\phi : X_T \to V_T$ is an isomorphism over T and $\alpha : \phi^*\mathcal{L}_T \to \mathcal{N}_T$ is an isomorphism of invertible sheaves. This functor is representable, see [][??] notation $\mathcal{T} := \mathrm{Isom}_U((X, \mathcal{N}), (V_U, \mathcal{L}_U))$. There is a left G-action $G \times \mathcal{T} \to \mathcal{T}$ on $\mathcal{T}$ over U (by post-composing, see the definition of the group structure on G).

Lemma 2.2.1. *If U is reduced then $\mathcal{T}$ is a G-torsor over U.*

Proof. It suffices to prove that $(X, \mathcal{N})$ is locally in the fppf topology of U isomorphic to the constant family $(V, \mathcal{L}) \times U$. To prove this we need some notation.

Take N so large that $\mathcal{L}^N$ is very ample on V and has vanishing higher cohomology groups. Let $n = \dim \Gamma(V, \mathcal{L}^N)$. A choice of basis of $\Gamma(V, \mathcal{L}^N)$ determines a closed immersion $i : V \to \mathbf{P}^{n-1}$. This determines a point $[i]$ of the Hilbert scheme $\mathbf{Hilb} = \mathrm{Hilb}_{\mathbf{P}^{n-1}/k}$. The smooth algebraic group $\mathbf{PGL}_n$ acts on Hilb, and we denote Z the orbit of $[i]$. Note that Z is a smooth scheme of finite type over k, and that there is a flat surjective morphism $\mathbf{PGL}_n \to Z$. See [**?**, Generalitiesaboutorbits] By construction the pullback of the universal family over Z to $\mathbf{PGL}_n$ is canonically isomorphic to $V \times \mathbf{PGL}_n$, and the invertible sheaf $\mathcal{O}(1)$ pulls back to $\mathcal{L}^N \boxtimes \mathcal{O}$.

The question is local on U so we may assume that U is affine. By our choice of N above, the invertible sheaf $\mathcal{N}^N$ is very ample on every fibre of X over U with vanishing higher cohomology groups. Hence after possibly shrinking U we can find a closed immersion $X \to \mathbf{P}^{n-1}_U$ which restricts to a the embedding given by the full linear series of $\mathcal{N}^N$ on every geometric fibre. Consider the associated moduli map $m : U \to \mathrm{Hilb}$. Since U is reduced, and since each pair $(X_s, \mathcal{N}_s)$ is isomorphic to a base change of $(V, \mathcal{L})$ we see that $m(U) \subset Z$.

This implies there is some surjective flat morphism $U' \to U$ such that there is a U' isomorphism $X' \cong V \times U'$ with the property that $\mathcal{N}^N$ pulls back to $\mathcal{L}^N$. Namely, take $U' = U \times_Z \mathbf{PGL}_n$ and unwind the definitions. To finish, do the same thing for $N+1$ to get some $U'' \to U$. Then over $U''' = U' \times_U U'$ there is an isomorphism of the pullback of $(X, \mathcal{N})$ and the base change of $(V, \mathcal{L})$. This proves the reult. $\square$

Conversely, given a left G-torsor $\mathcal{T}$ over U we will construct a flat proper family of varieties $X \to U$ and an invertible sheaf $\mathcal{N}$ on X such that $\mathrm{Isom}_U((X, \mathcal{N}), (V_U, \mathcal{L}_U))$ is isomoprhic to $\mathcal{T}$.

The structure morphism $\pi : \mathcal{T} \to U$ is a flat surjective morphism of finite type. We are going to descend the constant family $V \times \mathcal{T}$ to U using a descend datum

$$\phi : V \times \mathcal{T} \times_U \mathcal{T} \to V \times \mathcal{T} \times_U \mathcal{T}.$$

Before we describe the descent datum, we recall that the map $\psi : G \times \mathcal{T} \to \mathcal{T} \times_U \mathcal{T}$, $(g, t) \mapsto (gt, t)$ is an isomorphism. Also, let us denote $m : V \times G \to V$ the map $(v, g) \mapsto gv$, where gv denote the natural action of $g \in G$ on $v \in V$. Finally, we take

$$\phi = \mathrm{Id}_V \times \psi \circ m \times \mathrm{Id}_{\mathcal{T}} \circ (\mathrm{Id}_V \times \psi)^{-1}.$$

To verify the cocycle condition on $\mathcal{T} \times_U \mathcal{T} \times_U \mathcal{T}$, we can think of ϕ as the map $(v, gt, t) \mapsto (g^{-1}v, gt, t)$. If on $V \times \mathcal{T} \times_U \mathcal{T} \times_U \mathcal{T}$ we have a point (v, g_1g_2t, g_2t, t) then $\mathrm{pr}_{23}^*(\phi)(v, g_1g_2t, g_2t, t) = (g_2v, g_1g_2t, g_2t, t)$ and $\mathrm{pr}_{12}^*(\phi) \circ \mathrm{pr}_{23}^*(\phi)(v, g_1g_2t, g_2t, t) = (g_1g_2v, g_1g_2t, g_2t, t)$ and $\mathrm{pr}_{13}^*(\phi)(v, g_1g_2t, g_2t, t) = ((g_1g_2)v, g_1g_2t, g_2t, t)$. Thus $\mathrm{pr}_{13}^*(\phi) = \mathrm{pr}_{12}^*(\phi) \circ \mathrm{pr}_{23}^*(\phi)$ as desired.

Because all the maps in question lift canonically to the invertible ample sheaf $\mathcal{L}$ this actually defines a descent datum on the pair $(V, \mathcal{L})$ for $\mathcal{T} \to U$. As $\mathcal{L}$ is ample, this descent datum is effective, cf. [Gro62, No. 190, §B.1]. Thus there exists a pair $(X \to U, \mathcal{N})$ over U and an isomorphism $\delta : \mathcal{T} \times_U (X, \mathcal{N}) \to \mathcal{T} \times (V, \mathcal{L})$ such that ϕ equals $\mathrm{pr}_1^*\delta \circ \mathrm{pr}_2^*\delta^{-1}$.

Conclusion 2.2.2. The above constructions give a bijective correspondence between pairs $(X \to U, \mathcal{N})$ and left G-torsors over U in case U is a reduced scheme over k.

Remark 2.2.3. The construction of the family $(X, \mathcal{N})/U$ starting from the torsor $\mathcal{T}$ works more generally when k is more generally a ring as long as: (1) V is a flat projective scheme of finite presentation over k, (2) $\mathcal{L}$ is ample, and (3) the automorphism group scheme $G = \mathrm{Aut}(V, \mathcal{L})$ is flat over k.

2.3. **Deforming torsors over a Henselian DVR.** Before proving Theorem 2.1.3, it is useful to say what is known without the hypothesis that G is reductive. We thank Ofer Gabber, Jean-Louis Colliot-Thélène and Max Lieblich for explaining the following proposition.

Proposition 2.3.1. *Let R be a Henselian DVR with residue field K, and let G be a flat separated group scheme of finite type over Spec R. Every torsor for the closed fiber G_K over Spec K is the closed fiber of a torsor for G over Spec R.*

Proof. We first give a proof when G is affine which is all we will use in this paper. Choose a closed immersion $G \to \mathbf{GL}_{n,R}$, see [**?**, Generalitiesaboutgroupschemes] Let $X = \mathbf{GL}_{n,R}/G$, and note that $X \to \mathrm{Spec}\ R$ is smooth, see [**?**, Generalitiesaboutorbits] Since $H^1(K, \mathbf{GL}_{n,K}) = \{1\}$, any torsor for G_K is the fibre of the map $\mathbf{GL}_{n,K} \to X_K$ over a K-point of X. Since R is Henselian the map $X(R) \to X(K)$ is surjective, and hence every G_K-torsor lifts.

In the general case we argue as follows. By [LMB00, Prop. 10.13.1], which relies upon Artin's criterion for algebraicity of a stack, the classifying stack BG is an algebraic stack over Spec R. By [LMB00, Thm. 6.3], for each G_K-torsor there exists an affine R-scheme X, a smooth morphism $\phi : X \to BG$, and a K-point x of X such that $\phi(x)$ corresponds to the given G_K-torsor. Denote by $t : \mathrm{Spec}\ R \to BG$ the 1-morphism associated to the trivial G-torsor. Since ϕ is smooth, the base-change $\mathrm{pr}_R : \mathrm{Spec}\ R \times_{t,BG,\phi} X \to \mathrm{Spec}\ R$ is smooth. Since t is a surjective flat morphism, the base-change, $\mathrm{pr}_X : \mathrm{Spec}\ R \times_{t,BG,\phi} X \to X$ is surjective and flat. By [GD67, §6.5], it follows that X is smooth over Spec R. Since R is Henselian and X is smooth over Spec R, $X(R) \to X(K)$ is surjective; in particular there is an R-morphism Spec $R \to X$ extending the given K-point of X. The composition of this morphism with ϕ determines a G-torsor over Spec R whose closed fiber is isomorphic to the given G_K-torsor over Spec K. □

Corollary 2.3.2. *Let R be a DVR with residue field k, and let G be a separated, finite type, flat group scheme over Spec R. Let U be a finite type, integral k-scheme, and let $\mathcal{T}_U \to U$ be a G_k-torsor. There exists an integral, flat, quasi-projective R-scheme Y, with nonempty special fibre Y_k, a G-torsor $\mathcal{T} \to Y$, and an open immersion $j : Y_k \to U$ such that $j^*\mathcal{T}_U$ is isomorphic to $\mathcal{T}_k$ as G_k-torsors over Y_k.*

$$\begin{tikzcd}
\mathcal{T} \arrow[d] & \mathcal{T}_k \arrow[l] \arrow[r, "j"] \arrow[d] & \mathcal{T}_U \arrow[d] \\
Y & Y_k \arrow[l] \arrow[r, "j"] & U
\end{tikzcd}$$

Proof. First we show there exists an integral, flat, quasi-projective R-scheme Z and an open immersion $j : Z_k \to U$. Namely, we may replace U by an affine open, and hence we may assume that U is regular, see [**?**, Generalitiesabout-commutativealgebra] In particular this implies that $U \to \mathrm{Spec}\ k$ is a local complete intersection morphism, see [**?**, Generalitiesaboutcommutativealgebra] So after shrinking U some more we may assume that $U = \mathrm{Spec}\ k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a complete intersection, i.e., $\dim U = n - c$. At this point we simply put $Z = \mathrm{Spec}\ R[x_1, \ldots, x_n]/(F_1, \ldots, F_c)$, where $F_i \in R[\underline{x}]$ lifts f_i.

Define R' to be the local ring of Z at the generic point of Z_k. Then R' is a Noetherian 1-dimensional local ring. Denote by π a uniformizer of R. Clearly, π maps into $\mathfrak{m}_{R'}$ and $R'/\pi R'$ is the function field of Z_k, i.e., the function field of U. Because R' is R-flat, π is a nonzerodivisor. Thus R' is a DVR with residue field $K = k(U)$.

By Proposition 2.3.1, the G_k torsor over $R'/\pi R'$ lifts to a G-torsor $\mathcal{T}^h$ over the Henselization of R'. By a standard limit argument, this lift exists over an étale extension $R' \to R''$ contained in the Henselization of R'. Note that the residue field $R''/\pi R''$ of R'' is still the function field of U. By a standard limit argument, there is an étale morphism $Y \to Z$ such that $Y_k \to Z_k$ is an open immersion and such that R'' is the local ring of Y at the generic point of Y_k. After replacing Y by an open subscheme, there is a G-torsor $\mathcal{T}$ over Y that pulls back to $\mathcal{T}^h$ over R''. We leave it to the reader to see that, after possible shrinking Y again, this torsor satisfies the conditions of the corollary. □

Corollary 2.3.2 above is a weak version of the deformation principle that we will esthablish later on. The remaining issue is whether there exists a datum $(Y \to \mathrm{Spec}\ R, \mathcal{T} \to Y, j : Y_K \to U)$ such that the generic fiber of $Y \to \mathrm{Spec}\ R$ is projective. Presumably this is not always possible, but in case G is reductive we will show that it is.

Corollary 2.3.2 can be used to lift problems in char $p > 0$ to characteristic 0. Suppose that R is a complete discrete valuation ring with algebraically closed residue field k. Let Ω be an algebraic closure of the fraction field of R. We have in mind the case where $\mathrm{char}(k) = p > \mathrm{char}(L) = 0$. Suppose that V_R is a flat projective R scheme, and that $\mathcal{L}_R$ is an ample invertible sheaf over V_R. We assume that V_Ω and V_k are varieties. Let G_R denote the automorphism group scheme of $(V_R, \mathcal{L}_R)$ over R.

Corollary 2.3.3. *Notations and assumptions as above. Fix $d \in \mathbb{N}$. Assume that G_R is flat over R. If the answer to Question 2.1.2 is always "yes" for the pair $(V_\Omega, \mathcal{L}_\Omega)$ then the answer is always "yes" for the pair $(V_k, \mathcal{L}_k)$.*

Proof. Let $(K/k, X \to S, \mathcal{N})$ be a triple as in Situation 2.1.1 for the pair $(V_k, \mathcal{L}_k)$. Let U be the open subscheme of S over which all geometric fibres of $(X, \mathcal{N})$ are isomorophic to the base change of $(V_k, \mathcal{L}_k)$. The construction in Subsection 2.2 gives a corresponding G_K-torsor $\mathcal{T}_U$ over U.

There exists an extension of complete discrete valuation rings $R \subset R'$ such that the induced extension of residue fields is K/k, see[Generalitiesaboutalgebra]. We apply Corolloray 2.3.2 to obtain $Y \to \mathrm{Spec}\ R'$, $\mathcal{T} \to Y$ and $j : Y_K \to U$. According to Conclusion 2.2.2 and Remark 2.2.3 there exists a pair $(X' \to Y, \mathcal{N}')$ over Y whose restriction to Y_K is isomorphic to $(j^*X|_U, j^*\mathcal{N}|_U)$.

Let Ω' be an algebraic closure of the field of fractions $Q(R')$ of R'. Since $R \subset R'$ we may and do assume that $\Omega \subset \Omega'$. Note that we do not know that the geometric fibre $Y_{\Omega'}$ is irreducible. However, our assumptions imply that X' has a $\Omega'(Y')$-valued point for every irreducible component Y' of $Y_{\Omega'}$. To conclude we apply the lemma below. □

Lemma 2.3.4. *Suppose that R is a DVR with algebraically closed residue field K. Let Ω be an algebraic closure of $Q(R)$. Let $Y \to \mathrm{Spec}\ R$ be a flat, finite type morphism, $X \to Y$ a projective morphism and let $\xi \in Y_K$. Assume in addition that (a) ξ is the generic point of an irreducible component C of the scheme Y_K, (b) the scheme Y_K is reduced at ξ, and (c) for every irreducible component Y' of Y_Ω there exists a $\Omega(Y')$-valued point of X. Then X has a $K(C)$-valued point.*

Proof. Note that right from the start we may replace R by its completion, and hence we may assume that R is excellent. This will garantee that the integral closure of R in a finite extension of $Q(R)$ is finite over R. (Not necessary in char 0.) See [**?**, Generalitiesaboutalgebra]

By hypothesis, and a standard limit argument, there is a section of $X_\Omega \to Y\Omega$ over a dense open $V \subset Y_\Omega$, say $s : V \to X_\Omega$. By a standard limit argument, there is a finite extension $Q(R) \subset L$ such that V and s are defined over L. Let R' be the integral closure of R in L. Since R is excellent the extension $R \subset R'$ is a finite extension of DVRs. The residue field of R' is isomorphic to K as K is algebraically closed.

By construction the scheme $Y_{R'} = Y \times_R R'$ has special fibre equal to Y_K. The local ring $\mathcal{O}$ of $Y_{R'}$ at ξ is a DVR. This follows from flatness of $Y_{R'}/R'$ and property (b), see proof of 2.3.2. Thus the image of $\mathrm{Spec}\ Q(\mathcal{O}) \to Y_L$ is one of the generic points of Y_L and hence contained in V. Since $X_{R'} \to Y_{R'}$ is proper, we see that $s|_{\mathrm{Spec}\ Q(\mathcal{O})}$ extends to a $\mathcal{O}$-valued point of $X_{R'}$, and in particular we obtain a $\kappa(\xi) = K(C)$-valued point of $(X_K)_{\kappa(\xi)} = X_{K(C)}$ as desired. □

For example this corollary always applies to the case where $(V, \mathcal{L})$ is the pair consisting of a Grassmanian and its ample generator.

2.4. **Deforming torsors for a reductive group.** Under the additional hypothesis that G is a geometrically reductive linear algebraic group we can prove a stronger version of Corollary 2.3.2. First we prove that BG is proper over k in some approximate sense.

Proposition 2.4.1. *Let G be a geometrically reductive group scheme over the field k. For each integer c, there exists a smooth k-scheme X, a smooth morphism $\phi : X \to BG$, and an open immersion $j : X \to \overline{X}$ such that*

(i) *$\overline{X}$ is a projective k-scheme,*
(ii) *for every infinite field K and every morphism Spec $K \to BG$, there exists a lift Spec $K \to X$ under ϕ.*
(iii) *$\overline{X} - X$ has codimension $\geq c$,*

The proof uses geometric invariant theory to construct $X \subset \bar{X}$. With more care it may be possible to remove the assumption that K be infinite from (ii).

Proof. By definition G is a linear group scheme. Let V be a finite dimensional k-vector space, and let $\rho' : G \to \mathbf{GL}(V)$ be a closed immersion of group schemes. Consider $\rho : G \to \mathbf{SL}(V \oplus k \oplus k)$ defined by $\rho(g) = \mathrm{diag}(\rho'(g), \det(\rho'(g))^{-1}, 1)$ (diagonal blocks). Observe that the intersection of Image(ρ) and $\mathbf{G}_m\mathrm{Id}$ is the trivial group scheme. Thus, without loss of generality, assume ρ is a closed embedding of G into $\mathbf{SL}(V)$ such that Image$(\rho) \cap \mathbf{G}_m\mathrm{Id}$ is the trivial group scheme. In other words, the induced morphism of group schemes $\mathbb{P}\rho : G \to \mathbf{PGL}(V)$ is a closed immmersion.

Denote the dimension of V by $n > 1$. Let W be a finite-dimensional k-vector space of dimension c. Denote by H the finite-dimensional k-vector space $\mathrm{Hom}(W, \mathrm{Hom}(V, V))$. There is a linear action $\sigma : \mathbf{GL}(V) \times H \to H$, where an element $g \in \mathbf{GL}(V)$ acts on a linear map $h : W \to \mathrm{Hom}(V, V)$ by $\sigma(g, h)(w) = g \circ h(w)$. This restricts to a linear action of G on H.

The linear action of G on H determines an action of G on the projective space $\mathbb{P}H$ of lines in H. It comes with a natural linearization of the invertible sheaf $L := \mathcal{O}_{\mathbb{P}H}(1)$ so that the action of G on $H^0(\mathbb{P}H, \mathcal{O}(1)) = \mathrm{Hom}(H, k)$ is the dual of ρ. Denote by $\mathbb{P}H^{\mathrm{ss}}$, resp. $\mathbb{P}H^{\mathrm{s}}_{(0)}$, the semistable, resp. properly stable, locus for the action of G on the pair $(\mathbb{P}H, L)$. Denote by $\overline{X}$ the uniform categorical quotient $\mathbb{P}H^{\mathrm{ss}} /\!\!/ G$ and denote by $p : \mathbb{P}H^{\mathrm{ss}} \to \overline{X}$ the quotient morphism. These exist by [MFK94, Thm. 1.10, App. 1.A, App. 1.C]. By the remark on [MFK94, p. 40], $\overline{X}$ is projective. Also, some power of L is the pullback under p of an ample invertible sheaf on $\overline{X}$. Thus (i) is satisfied for $\overline{X}$.

For every element $w \in W - \{0\}$, define F_w to be the homogeneous, degree n polynomial on H defined by $F_w(h) = \det(h(w))$. For every $g \in \mathbf{SL}(V)$,

$$\begin{aligned} F_w(\sigma(g, h)) &= \det(\sigma(g, h)(w)) = \det(gh(w)) \\ &= \det(g)\det(h(w)) = \det(h(w)) = F_w(h). \end{aligned}$$

Thus F_w is invariant for the action of $\mathbf{SL}(V)$. Thinking of F_w as an element of $\Gamma(\mathbb{P}H, \mathcal{O}(n))$ it is invariant for the action of G. Denote by $H_w \subset H$, resp. $\mathbb{P}H_w \subset \mathbb{P}H$, the open complement of the zero locus of F_w. By what was said above, $\mathbb{P}H_w$ is contained in $\mathbb{P}H^{\mathrm{ss}}$. The next step is to prove that $\mathbb{P}H_w$ is contained in $\mathbb{P}H^{\mathrm{s}}_{(0)}$, and, in fact, the geometric quotient $\mathbb{P}H_w \to \mathbb{P}H_w/G$ is a G-torsor.

Let W' be a subspace of W such that $W = \mathrm{span}(w) \oplus W'$. Denote by $H' \subset H$ the subspace $H' = \mathrm{Hom}(W', \mathrm{Hom}(V,V))$. There is a morphism $q_w : H_w \to \mathbf{GL}(V) \times H'$ defined by the rule $h \mapsto (h(w), h(w)^{-1}h|_{W'})$. The morphism q_w is $\mathbf{GL}(V)$-equivariant if we act on $\mathbf{GL}(V) \times H'$ on the first factor only. There is an inverse morphism $r_w : \mathbf{GL}(V) \times H' \to H_w$ sending a pair (g, h') to the unique linear map $W \to \mathrm{Hom}(V,V)$ such that $w \mapsto g$ and $w' \mapsto gh'(w')$ for every $w' \in W'$. Thus, as a scheme with a left $\mathbf{GL}(V)$-action, H_w is isomorphic to $\mathbf{GL}(V) \times H'$. For the same reason, as a scheme with a $\mathbf{PGL}(V)$-action, $\mathbb{P}H_w$ is isomorphic to $\mathbf{PGL}(V) \times H'$. Thus the categorical quotient of $\mathbb{P}H_w$ by the action of G is the induced morphism $\mathbb{P}H_w \to (\mathbf{PGL}(V)/G) \times H'$. Since the categorical quotient $\mathbf{PGL}(V) \to \mathbf{PGL}(V)/G$ is a G-torsor, see [**?**, **?**, Generalitiesaboutorbits] also the categorical quotient $\mathbb{P}H_w \to (\mathbf{PGL}(V)/G) \times H'$ is a G-torsor. In particular, the action of G on $\mathbb{P}H_w$ is proper and free so that $\mathbb{P}H_w$ is contained in $\mathbb{P}H^{\mathrm{ss}}_{(0)}$.

Denote $U = \bigcup \mathbb{P}H_w$, where the union is over all $w \in W - \{0\}$. This is a G-invariant open subscheme of $\mathbb{P}H^{\mathrm{s}}_{(0)}$. Therefore there exists a unique open subscheme $X \subset \overline{X}$ such that $p^{-1}(X) = U$. By the last paragraph, $p : \mathcal{T} \to X$ is a G-torsor. Since U is smooth and p is flat, by [GD67, §6.5] also X is smooth.

Associated to the G-torsor U over X, there is a 1-morphism $\phi : X \to BG$. There are also morphisms of stacks $[H/G] \to BG$ and $[\mathbb{P}H/G] \to BG$ (because $BG = [\mathrm{Spec}\ k/G]$. By construction, X is 2-equivalent to an open substack of $[\mathbb{P}H/G]$ as a stack over BG. The morphism $[\mathbb{P}H/G] \to BG$ is smooth, since $\mathbb{P}H$ is smooth. Hence $X \to BG$ is smooth. For every field K and 1-morphism $\mathrm{Spec}\ K \to BG$, the 2-fibered product $\mathrm{Spec}\ K \times_{BG} [H/G]$ is a K-vector space, and $\mathrm{Spec}\ K \times_{BG} [\mathbb{P}H/G]$ is the associated projective space. Thus $\mathrm{Spec}\ K \times_{BG} [\mathbb{P}H/G] \cong \mathbb{P}^{cn^2-1}$. Finally, $\mathrm{Spec}\ K \times_{BG} X$ is a nonempty open subscheme of $\mathrm{Spec}\ K \times_{BG} [\mathbb{P}H/G]$. Since K is infinite every dense open subset of $\mathbb{P}^{dn^2-1}_K$ contains a K-point. This proves (ii).

Finally, the codimension of $\overline{X} - X$ is at least as large as the codimension of $\mathbb{P}H - U$. Choosing a basis $(w_1, \dots, w_d)$ for W, $\mathbb{P}H - U$ is contained in the common zero locus of $F_{w_1}, \dots, F_{w_c}$, which clearly has codimension c. Therefore $\overline{X} - X$ has codimension at least c in $\overline{X}$. This proves (iii). ☐

Corollary 2.4.2. *Let the field k and the group scheme G be as in Proposition 2.4.1. Let R be a DVR containing k with residue field K. Let U be a finite type, integral K-scheme, and let $\mathcal{T}_U \to U$ be a G-torsor. There exists a triple $(Y \to SpecR, \mathcal{T} \to Y, j : Y_K \to U)$ as in Corollary 2.3.2 with the additional property that the generic fiber of Y is projective.*

Proof. We may assume that $\dim U > 0$. Let c be an integer larger than $\dim(U)$. Let $(\phi : X \to BG, X \subset \overline{X})$ be as in Proposition 2.4.1. The torsor $\mathcal{T}_U$ corresponds to a 1-morphism $U \to BG$. By condition (ii), the base-change morphism $\mathrm{Spec}\ K(U) \to BG$ lifts to a morphism $\mathrm{Spec}\ K(U) \to X$. (Note that $K(U)$ is infinite since $\dim U > 0$.) After replacing U by a dense open subscheme, this comes from a morphism $f : U \to X$ lifting $U \to BG$. Also, replace U by an open subscheme that is quasi-projective, say a nonempty open affine. Then for some positive integer N, there is a locally closed immersion of K-schemes, $f' : U \to (X \times \mathbb{P}^N_k)_K$ such $\mathrm{pr}_X \circ f'$ equals f. Denote by m the codimension of $f'(U)$ in $(X \times \mathbb{P}^N_k)_K$.

The scheme $(\overline{X} \times \mathbb{P}^N_k)_R$ is flat and projective over Spec R. Choose a closed immersion in $\mathbb{P}^M_R$ for some positive integer M. As in the proof of 2.3.2 we will use that the scheme U is a local complete intersection at a general point, and we will use that X is smooth over k. This implies that $f'(U)$ is dense in a component of a complete intersection of $(\overline{X} \times \mathbb{P}^N_k)_K$ in $\mathbb{P}^M_K$. More precisely, for some positive integer e, there exist homogoneous, degree e polynomials $F_1, \dots, F_m$ on $\mathbb{P}^M_K$ such that the scheme $\overline{Y}_K := \mathbf{V}(F_1, \dots, F_m) \cap (\overline{X} \times \mathbb{P}^N_k)_K$ has pure dimension $\dim(U)$ and contains a nonempty open subscheme U' that is an open subscheme of $f'(U)$. Let $\widetilde{F}_1, \dots, \widetilde{F}_c$ be homogeneous, degree e polynomials on $\mathbb{P}^M_R$ such that for every $i = 1, \dots, m$,

$$(*) \qquad \widetilde{F}_i \equiv F_i \pmod{\mathfrak{m}_R}.$$

Denote by $\overline{Y}$ the zero scheme $\mathbf{V}(\widetilde{F}_1, \dots, \widetilde{F}_m) \cap (\overline{X} \times \mathbb{P}^N_k)_R$. Then $\overline{Y}$ is flat over Spec R (by Grothedieck's lemma [**?**, EGA]. The closed fiber of $\overline{Y}$ equals $\overline{Y}_K$. Moreover,

$$\dim((\overline{X} - X) \times \mathbb{P}^N_k) - m \leq \dim X - c + N - m = \dim f'(U) - c < 0.$$

It is easy to see that the set of all possible choices of $\widetilde{F}_i$ satisfying $(*)$ forms a Zariski dense set of points in the relevant vector space of degree e polynomials over the field of fractions $Q(R)$ of R. Thus the dimension count shows there exists a choice of $\widetilde{F}_1, \dots, \widetilde{F}_c$ such that $\overline{Y}_{Q(R)}$ does not intersect $\big((\overline{X} - X) \times \mathbb{P}^N_k\big)_{Q(R)}$. In other words, the generic fiber of $\overline{Y} \to$ Spec R is contained in $(X \times \mathbb{P}^N_k)_{Q(R)}$.

Let η be a generic point of $\overline{Y}$ that specializes to the generic point of U'. Replace $\overline{Y}$ by the closure of η, so that now $\overline{Y}$ is integral. (Presumably, a suitable application of Bertini's theorem could be used to replace this step.) Then $\overline{Y}$ is an integral, flat, projective R-scheme, the closed fiber contains U' as an open subscheme, and the generic fiber is contained in Spec $R \times_{\text{Spec } k} (X \times \mathbb{P}^N_k)$. Define

$$Y = \overline{Y} - \big(\overline{Y} \times_{\text{Spec } R} \text{Spec } K - U'\big).$$

This is an integral, flat, quasi-projective R-scheme whose generic fiber is projective. Moreover, Y_K equals U', which admits a dense, open immersion in S. Finally, the projection $\text{pr}_X : Y \to X$, and the 1-morphism $\phi \circ \text{pr}_X : Y \to BG$ determine a G-torsor $\mathcal{T}$ over Y. By construction, the restriction of this G-torsor to U' is isomorphic to the pullback of $\mathcal{T}_U$ by the open immersion, as desired. $\square$

We remark that we did not claim that the generic fibre of $Y \to SP(R)$ is geometrically irreducible. Since X is smooth and geometrically irreducible over k, it seems that with a carefull choice of the $\widetilde{F}_i$ and some additional arguments one can obtain this property as well maving $Y_{Q(R)}$ smooth over $Q(R)$.

Next we deduce a corollary to help prove Theorem 2.1.3. Let k be an algebraically closed field, and let $(V, \mathcal{L})$ be a pair of a projective k-scheme and an ample invertible sheaf. Denote by G/k the group scheme $G = \text{Aut}(V, \mathcal{L})$. Let $(K/k, X \to S, \mathcal{N})$ be as in Situation 2.1.1. Denote by G°_{red} the reduced, connected component of the identity of G.

Corollary 2.4.3. *Notations as above. Let R be a DVR containing k and with residue field K. If G°_{red} is reductive, there exists an integral, flat, quasi-projective R-scheme Y, a projective, flat morphism $f : \widetilde{X} \to Y$, an invertible sheaf $\widetilde{\mathcal{N}}$ on X, and an open immersion $j : Y_K \to S$ such that:*

(i) *every geometric fiber of $(\widetilde{X}, \widetilde{\mathcal{N}})$ over Y equals the base-change of $(V, \mathcal{L})$,*

(ii) *the restriction of* $(\widetilde{X}, \widetilde{\mathcal{N}})$ *to* Y_K *is isomorphic to the pullback of* $j^*(X, \mathcal{N})$*, and*

(iii) *the generic fiber of* $Y \to$ *Spec* R *is projective.*

In particular, let S' *be an irreducible component of the geometric generic fibre of* $Y \to$ *Spec* R*. Then* $(\widetilde{X} \to S', \widetilde{\mathcal{N}})$ *over* R *is a triple* $(K'/k, X' \to S', \mathcal{N}')$ *with empty discriminant.*

Proof. If G°_{red} being reductive implies that it is a geometrically reductive group scheme over k by a result of Haboush, see [**?**, GIT] Note that G°_{red} is a closed normal subgroup scheme of G and that the quotient G/G°_{red} is a finite group scheme. A finite group scheme over k is geometrically reductive, and an extension of geometrically reductive group schemes is reductive, see [**?**, GeneralitiesaboutGIT] Hence G is geometrically reductive. Thus the result of the Corollora follows from Corollary 2.4.2 above by applying the bijective correspondence of Conclusion 2.2.2. □

Proof of Theorem 2.1.3. Let us start with an arbitrary triple $(K/k, X \to S, \mathcal{N})$. Let $R = K[[t]]$. So R is Henselian, contains k and has residue field K. Let $\widetilde{X} \to Y \to \mathrm{Spec}\ R$ and $\widetilde{\mathcal{N}}$ be as in Corollary 2.4.3. Denote by Ω/k an algebraic closure of the field of fractions $Q(R)$ of R. Let S' be any irreducible component of Y_Ω and let $X' = \widetilde{X}|_{S'}$, $\mathcal{N}' = \widetilde{\mathcal{N}}|_{S'}$. Thus $(\Omega/k, X' \to S', \mathcal{N}')$ is a triple as in Situation 2.1.1. By construction, this has empty discriminant. By hypothesis, the generic fiber of $X' \to S'$ has a $K'(S')$-point. At this point we apply Lemma 2.3.4 to conclude. □

3. Simple applications

We are going to apply this in a more serious way to Grassmanians, but let us indicate some simple applications of Theorem **??** this section.

3.1. **Fermat Hypersurfaces.** As a first case we take V a Fermat hypersurface of degree d in $\mathbb{P}^{d^2-1}$

$$V : T_0^d + T_1^d + \ldots T_{d^2-1}^d = 0,$$

with $\mathcal{L} = \mathcal{O}_V(1)$, say over the complex numbers $\mathbf{C}$. In this case the group scheme G is an extension of a finite group by $\mathbf{G}_m$ so certainly reductive. Consider the following family with general fibre $(V, \mathcal{L})$ over $\mathbb{P}^2$:

$$(*) \qquad \sum_{0 \le i,j \le d-1} X^i Y^j Z^{2d-2-i-j} T_{i+dj}^d = 0,$$

We learned about this family from Tom Graber. This family does not have a rational point over $k(\mathbb{P}^2)$. Reference [**?**][TomGraber]. The reader may enjoy finding an elementary proof of this by looking at what it means to have a polynomial solution to the above. We conclude from Theorem 3.1 that there is a smooth projective family over a projective surface with *every* fibre isomorphic to $(V, \mathcal{L})$, without a rational section. We like this example because it is not immediately obvious how to write one down explicitly.

There is another reason why the family given by $(*)$ is interesting. Tsen's theorem asserts that, if $n \ge d^2$ then any degree d hypersurface $X \subset \mathbb{P}^n_F$, where F is the function field of a surface has a rational point. The authors of this paper wonder what the obstruction to the existence of a rational point is in the boundary case, namely degree d in $\mathbb{P}^{d^2-1}$. One guess is that it is a Brauer class, i.e., an element α in the Brauer group of F such that for finite extensions F'/F one has: $X(F') \neq$

$\emptyset \Leftrightarrow \alpha|_{F'} = 0$. (Compare future work of the authors.) However, the example above shows that this is not the case.

Namely, in our example $F = \mathbf{C}(x, y)$ where $x = X/Z$ and $y = Y/Z$. Anand Depokar pointed out that $(*)$ obtains a rational point over $F(\xi)$ where ξ is a dth root of a nonzero polynomial of the form

$$f(x, y) = - \sum_{0 \leq i,j \leq d-1, (i,j) \neq (0,0)} a_{i,j} x^i y^j.$$

(Just take $T_0 = \xi$ and $T_{i+jd} = a_{i,j}^{1/d}$.) Let $C \subset \mathbb{P}^2$ be an irreducible curve, not the line at infinity $Z = 0$. Suppose that α ramifies along C. The ramification data gives a cyclic extension $\mathbf{C}(C) \subset \mathbf{C}(C)[g^{1/d'}]$ of degree d', where $1 < d'|d$. There is a choice of of $a_{i,j}$ such that the rational function $f(x, y)$ restricts to a rational function on C such that both $f|_C$ and $g^{-1}f|_C$ are not d'th powers. (Left to the reader.) Thus the pullback of α to F' is still ramified along the pullback of C to the surface whose function field is $\mathbf{C}(x, y)(\xi)$. Contradiction. Hence C does not exist. However, the only Brauer class on $\mathbb{P}^2$ ramified along a single line is 0.

3.2. **Projective spaces.** Another case is where we take the pair $(V, \mathcal{L})$ to be $(\mathbf{P}^n, \mathcal{O}(n + 1))$. Note that $\mathcal{O}(n + 1) = \omega_{\mathbf{P}^n}^{-1}$ so the families in question are canonically polarized, and we are just talking about the problem of having nontrivial families of Brauer-Severi varieties. In particular, our theorem reduces the problem of proving the nullity of the Brauer group of a curve to the problem of proving the nonexistence of Brauer-Severi varieties having no rational sections over projective nonsingular curves. As far as we know this is not really helpfull, since the proof of Tsen's theorem is pretty straigthforward anyway. However, it illustrates the idea!

4. The period-index problem

In this section we show how to use the ideas above to find a proof of the period index problem for Brauer classes over surfaces. We refer to the paper [**?**] and the lecture [**?**] for a more thorough discussion of the problem.

4.1. **Brauer classes and Grassmanians.** Suppose that F is a field and that $\alpha \in \mathrm{Br}(F)$ is a Brauer class. The *period* of α is the smallest integer $m > 0$ such that $m\alpha = 0$. The *index* of α is the smallest integer $e > 0$ such that there is a central simple algebra B of degree e representing α.

Fix a central simple algebra of some degree d representing the class α, i.e., $\dim_F A = d^2$. Note that the index of α divides d. For each integer $1 \leq e \leq d$ consider the variety X_e parametrizing right ideals of rank ed in A. In other words a T-valued point of X_e corresponds to a sheaf of right ideals $I \subset \mathcal{O}_T \otimes A$, locally a direct summand of rank ed as $\mathcal{O}_T$-modules.

Note that if X_e has a rational point $I \subset A$, then the commutant of A acting on I is an algebra B of degree e representing $-\alpha$. Hence the index of the Brauer class α is the smallest integer e such X_e has a rational point.

Geometrically the varieties X_e are isomorphic to Grassmanians. Namely, suppose we choose an isomorphism $\psi : A \otimes_F \bar{F} \cong \mathrm{Mat}(d \times d, \bar{F})$. A right ideal in $\mathrm{Mat}(d \times d, \bar{F})$ is of the form $\{M \in \mathrm{Mat}(d \times d, \bar{F}) \mid \mathrm{Im}(M) \subset V\}$ for some subspace $V \subset \bar{F}^d$. In this way, using ψ, we obtain the isomorphism

$$X_{e,\bar{F}} \cong \mathrm{Grass}(e, d)_{\bar{F}}$$

with the Grassmanian of e-dimensional subspaces of the standard d-dimensional vector space. Since $\mathrm{Pic}(\mathrm{Grass}(e,d)) = \mathbb{Z}$ with a canonical ample generator, we see that each X_e gives rise to a Brauer class α_e, namely the obstruction to descending this ample generator to an invertible sheaf over X_e.

Lemma 4.1.1. *We have $\alpha_e = e\alpha$, at least up to a sign.*

Proof. With the notation above we can write down the Galois cocycle of α. Namely, the isomorphism class of A is given by the 1-cocycle $\partial\psi = \{\sigma \mapsto \sigma(\psi) \circ \psi^{-1}\}$ in $\mathrm{Aut}(\mathrm{Mat}(d{\times}d, \bar{F})) = \mathrm{PGL}_d(\bar{F})$. The class α is the obstruction in $H^2(\mathrm{Gal}(\bar{F}/F), \bar{F}^*)$ to lift this 1-cocycle $\partial\psi$ to a 1-cocycle for GL_d, i.e., the boundery of $\partial\psi$ for the short exact sequence $\mathbf{G}_m \to \mathrm{GL}_m \to \mathrm{PGL}_m$. The obstruction to descend the ample invertible sheaf $\mathcal{O}(1)$ on $\mathrm{Grass}(e,d)$ to F is the boundary of the 1-cocycle $\partial\psi$ for the short exact sequence $\mathbf{G}_m \to G_{e,d} \to \mathrm{PGL}_d$, see Lemma 4.1.2 for notation. By Lemma 4.1.2 below this boundary is $d-e$ times α as desired. $\square$

Lemma 4.1.2. *Let $1 \leq n < m/2$ be integers. Let $\mathcal{O}(1)$ be the canonical ample invertible sheaf on Grass(n,m).*

(1) *The automorphism group scheme of Grass(n,m) is canonically isomorphic to PGL_m.*
(2) *The automorphism group scheme $G_{n,m}$ of the pair (Grass$(n,m), \mathcal{O}(1)$) the extension of PGL_m by $\mathbf{G}_m$ fitting into the following diagram*

$$\begin{tikzcd}
\mathbf{G}_m, \arrow[r, equal] \arrow[d, "{(t^{n-m},t)}"'] & \mathbf{G}_m \arrow[d, "{(t^{n-m},t\cdot I_m)}"] & \\
\mathbf{G}_m \times \mathbf{G}_m \arrow[r] \arrow[d, "{(a,b)\mapsto ab^{m-n}}"'] & \mathbf{G}_m \times GL_m \arrow[r] \arrow[d] & PGL_m \arrow[d, equal] \\
\mathbf{G}_m \arrow[r] & G_{n,m} \arrow[r] & PGL_m
\end{tikzcd}$$

with exact rows and columns.

Proof. Clearly the group scheme $\mathbf{G}_m \times \mathrm{GL}_m$ acts on the pair $(\mathrm{Grass}(n,m), \mathcal{O}(1))$. We leave it to the reader to show that one obtains the short exact sequences of the statement. (Note that if $n = m/2$ there is an additional automorphism of $\mathrm{Grass}(n,m)$ coming from taking duals.) $\square$

Proof of Theorem 1.0.2. Let S be a surface over $k = \bar{k}$, and let $F = k(S)$. Let $\alpha \in \mathrm{Br}(F) = \mathrm{Br}(k(S))$. Lemma 4.1.1 means that if we take e equal to the period of α then the variety X_e carries an ample invertible sheaf which is geometrically an ample generator of $\mathrm{Pic}(\mathrm{Grass}(e,d))$. We conclude that the period-index problem for the field F is equivalent to the problem of finding rational points on forms of Grassmanians which are endowed with an invertible sheaf that is geometrically an ample generator for the Picard group.

It suffices to do this in all cases where $e < d/2$ since we can always replace A by $\mathrm{Mat}(2 \times 2, A)$ to artificially increase d.

Thus it suffices to prove the answer to Question 2.1.2 is always yes in the following situations. Namely, k is an algebraically closed field, the integer $d = 2$, and the pair $(V, \mathcal{L})$ is the pair $(\mathrm{Grass}(n,m), \mathcal{O}(1))$, where $1 \leq n < m/2$. Lemma 4.1.2 shows that $\mathrm{Aut}(V, \mathcal{L})$ is reductive.

By Theorem 2.1.3 it suffices to consider only those cases where the surface S is projective. By resolution of surface singularities we may also assume that the surface is smooth. This case is Theorem 4.2.1 below. $\square$

4.2. **Families of Grassmanians over smooth projective surfaces.** In this subsection let S be a smooth projective surface over the algebraically closed field k. Suppose that $X \to S$ is a projective morphism and that $\mathcal{N}$ is an invertible sheaf on X. Assume that $1 \leq n < m/2$ and assume that for every point $s \in S(k)$ the fibre X_s is isomorphic to the Grassmanian $\mathrm{Grass}(n, m)$ and that $\mathcal{N}$ restricts to the ample generator of the Picard group.

Theorem 4.2.1. *With notations as above. The morphism $X \to S$ has a rational section.*

The rest of this subsection is devoted to the proof of this theorem. Some of the discussion that follows applies more generally to the problem of finding a rational section of any family of varieties over a surface. Note that we do not assume that the characteristic of k is 0 so we have to be a little carefull.

Let $\{C_t, t \in \mathbb{P}^1\}$ be a Lefschetz pencil on S. We replace S by the blow up in $C_0 \cap C_\infty$ and we replace X by the pullback to this blow up. Thus we obtain

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend right, "h"'] & S \arrow[r, "f"] & \mathbb{P}^1_k \end{tikzcd}$$

so that $C_t = f^{-1}(t)$ is the fibre over $t \in \mathbb{P}^1$. We denote $\Delta \subset \mathbb{P}^1$ the discriminant locus of f (a finite closed subscheme). We write $X_t = h^{-1}(t)$ for the fibre of $X \to \mathbb{P}^1_k$ over $t \in \mathbb{P}^1_k$. From now on we think of S as a family of curves over $\mathbb{P}^1$ and of X as a family of families of Grassmanians over curves.

After replacing $\mathcal{N}$ by $\mathcal{N} \otimes g^*(\mathcal{M})$ for some very ample invertible sheaf on S we may assume that $\mathcal{N}$ is ample on X. Let $p_a = p_a(C_t)$ be the common value of the arithmetic genus of the curves C_t (of course this is just the genus of C_t for t not in Δ). For $d \in \mathbf{N}$ set $P_d(X) = dX + 1 - p_a \in \mathbf{Z}[X]$. and set

$$\mathrm{Hilb}_d = \mathrm{Hilb}^{P_d}_{X/\mathbb{P}^1, \mathcal{N}} \longrightarrow \mathbb{P}^1.$$

In other words, this is the projective scheme over $\mathbb{P}^1$ whose fibres $\mathrm{Hilb}_{d,t}$ parametrize 1-dimensional closed subschemes $Z \subset X_t$ whose arithmetic genus is p_a and whose $\mathcal{N}$-degree is d. Let us denote

$$\iota : Z_d \longrightarrow X \times_{\mathbb{P}^1} \mathrm{Hilb}_d$$

the universal family of closed subschemes. Note that g induces a morphism between flat projective families of 1-dimensional schemes over Hilb_d as follows

$$\begin{tikzcd} Z_d \arrow[rr] \arrow[dr] & & S \times_{\mathbb{P}^1} \mathrm{Hilb}_d \arrow[dl] \\ & \mathrm{Hilb}_d & \end{tikzcd}$$

We are only going to use a small piece of this Hilbert scheme.

Let us denote $U_d \subset \mathrm{Hilb}_d$ the open subscheme whose geometric points correspond to closed subschemes $Z \subset X_t$ such that

(1) Z is a nonsingular irreducible curve,

(2) the fibre C_t is nonsingular, i.e., $t \notin \Delta$, and
(3) the morphism $Z \to S_t$ is an isomorphism.

This could be empty, but later we will show it is not empty for large d.

Lemma 4.2.2. *The restriction $Z_d|_{U_d}$ is isomorphic to the pull back $S \times_{\mathbb{P}^1} U_d$.*

Proof. Critère de platitude par fibre. □

The lemma implies that the pullback $\iota^*(\mathrm{pr}_1^*(\mathcal{N}))$ determines a section of the relative Picard functor of $S/\mathbb{P}^1$ over U_d. In other words, we obtain a morphism

$$m_{\mathcal{N}} : U_d \longrightarrow \underline{\mathrm{Pic}}^d_{S/\mathbb{P}^1}$$

of schemes over $\mathbb{P}^1$.

There may still be some irreducible components of U_d that we want to throw out as follows. Recall that $X \to S$ is given by a PGL_m torsor over S. Now consider the morphism $\mathrm{Flag}(1, n; m) \to \mathrm{Grass}(n, m)$. This is a PGL_m-equivariant morphism. We conclude that there exists a morphism $P \to X$ such that for every point $s \in S(k)$ the fibre $P_s \to X_s$ is isomorphic to the morphism $\mathrm{Flag}(1, n; m) \to \mathrm{Grass}(n, m)$. Thus $P \to X$ is a P^{n-1}-bundle over X, in other words, P is a Severi-Brauer scheme over X.

Consider a geometric point of U_d corresponding to a closed subscheme $Z \subset X_t$ for some geometric point t of $\mathbb{P}^1$. Since the Brauer group of Z is trivial we can write $P = \mathrm{Proj}(\mathcal{W})$ for some locally free sheaf $\mathcal{W}$ of rank n over Z. The sheaf $\mathcal{W}$ is well defined up to tensoring with an invertible sheaf, and hence whether or not $\mathcal{W}$ is a stable sheaf is independent of the particular choice of $\mathcal{W}$.

Lemma 4.2.3. *There exists an open subscheme $U'_d \subset U_d$ whose geometric points corresponds to those $Z \subset X_t$ such that the locally free sheaf $\mathcal{W}$ is a stable locally free sheaf on the nonsingular projective curve $Z = C_t$.*

Proof. Stability is an open condition. □

Theorem 4.2.4. *Fix n, m, $X \to S \to \mathbb{P}^1$, and $\mathcal{N}$ ample as above. Assume $p_a \geq 2$. For all $d >> 0$ we have the following:*

(1) *The morphism*

$$m_{\mathcal{N}} : U'_d \longrightarrow \underline{Pic}^d_{S/\mathbb{P}^1} \times_{\mathbb{P}^1} (\mathbb{P}^1 - \Delta)$$

is surjective and smooth with geometrically irreducible fibres. We denote F_θ the fibre of this morphism over a geometric point θ of the right hand side.

(2) *For any θ as above there exists a rational curve $\phi : \mathbb{P}^1 \to F_\theta$ so that the pullback of the tangent bundle of F_θ is ample.*

(3) *For any geometric point t of $\mathbb{P}^1 - \Delta$, there exists and integer $h(t)$ such that for any θ as above lying over t, if $\gcd(d + h(t), n) = 1$, then there exists a compactification $F_\theta \subset \overline{F}_\theta$ into a normal projective variety, so that the boundary has codimension ≥ 2.*

Proof of 4.2.1 assuming 4.2.4. First we may choose our Lefschetz pencil so that $p_a \geq 2$. Pick d as in Theorem 4.2.4. In characteristic $p > 0$ we choose it such that $\gcd(d + h(t), n) = 1$, where t is a geometric generic point of $\mathbb{P}^1$. Choose an invertible sheaf $\mathcal{L}$ such that the restriction of $\mathcal{L}$ to C_t has degree d. This is possible since the morphism $S \to \mathbb{P}^1$ has sections as it is a Lefschetz fibration. The choice of $\mathcal{L}$ determines a section $\tau : \mathbb{P}^1 \to \mathrm{Pic}^d_{S/\mathbb{P}^1}$. We consider the morphism m_τ :

$m_{\mathcal{N}}^{-1}(\tau(\mathbb{P}^1)) \to \mathbb{P}^1$. Theorem 4.2.4 gives information on the geometric generic fibre of m_τ, namely it implies that this is a rationally connected variety. In characteristic zero this suffices, using the theorem of Graber-Harris-Starr [GHS03], to conclude there exists a lift of τ to a rational section $\rho : \mathbb{P}^1 \to U'$ of $U' \to \mathbb{P}^1$. In characteristic $p > 0$ we can use Lemma 4.2.5 below to see that we may compactify the generic fibre F_η of m_τ so that all the conditions of the main theorem of [dJS03] are satisfied. Hence in this case we obtain ρ as well.

By Lemma 4.2.2 the pullback $\rho^* Z$ is birational to S. Thus

$$S \dashrightarrow \rho^* Z \dashrightarrow X$$

is the desired rational section of $X \to S$. $\square$

Lemma 4.2.5. *Let K be a field and let V be a variety over K. Suppose that $V_{\bar{K}}$ has a compactification $V_{\bar{K}} \subset \bar{V}_{\bar{K}}$ which is a projective normal variety with boundary of codimension ≥ 2. Then there is a projective normal compactification $V \subset \bar{V}$ with boundary having codimension ≥ 2.*

Proof. Let $K \subset L \subset \bar{K}$ be a finite extension of K over which the compactification is defined. Let $\mathcal{L}_L$ be an ample invertible sheaf over V_L which is the restriction of an ample invertible sheaf on $\bar{V}_L$. If we can pick $\mathcal{L}$ such that it is the pullback of an invertible sheaf on V then we are done. (Left to the reader.)

In characteristic zero this does not pose a problem since we may assume that L/K is Galois with group G and then we can simply replace $\mathcal{L}$ by $\otimes_{\sigma \in G} \sigma^* \mathcal{L}$ which descends. In characteristic $p > 0$ this reduces the question to the case where L/K is purely inseparable. However in this case the invertible sheaf $\mathcal{L}^n$ with $n = [L : K]$ will descend since the n-th power of an element of the structure sheaf of V_L is an element of the structure sheaf of V. $\square$

Proof of 4.2.4. Let t be a geometric point of $\mathbb{P}^1$ not in the discriminant locus Δ of $S \to \mathbb{P}^1$. Thus $C = C_t = f^{-1}(t)$ is a nonsingular geometric fibre of $S \to \mathbb{P}^1$. Let $Y = X_t \to C$ be the family of Grassmanians over C. Then $U'_{d,t}$ is an open subscheme of the Hilbert scheme of Y, consisting of points that correspond to sections of the morphism $Y \to C$, with a suitable stability condition as in Lemma 4.2.3. We claim that to prove Theorem 4.2.4 it suffices to prove the corresponding statements for $U'_{d,t} \to \underline{\mathrm{Pic}}^d_C$ for all geometric points t of $\mathbb{P}^1 - \Delta$. The only somewhat subtle point is that the bounds implicit in the statement $d >> 0$ may depend on t. There are two ways to deal with this issue. The first is to notice that in the proof of Theorem 4.2.1 we only every used the case where t is a geometric generic point of $\mathbb{P}^1$. The second way to deal with this is to notice that once the assertions of Theorem 4.2.4 have been proven for all θ lying over a geometric generic point of $\mathbb{P}^1$, then the assertions hold for all geometric points θ lying over a fixed Zariski open of $\mathbb{P}^1 - \Delta$. The additional finite list of points can then be dealt with separatedly.

From now on we work with a fixed family of Grassmanians $Y \to C$, and ample invertible sheaf $\mathcal{N} \in \mathrm{Pic}(Y)$ over a fixed algebraically closed field as above. As before the ample sheaf restricts to an ample generator on the fibres. The corresponding Hilbert scheme is

$$\mathrm{Hilb}^d = \mathrm{Hilb}^{P_d}_{Y,\mathcal{N}},$$

which is now a scheme over the ground field, and we have open subschemes $U'_d \subset U_d \subset \mathrm{Hilb}^d$ as above. (In other words we have dropped the subscript t from the notation.)

The Brauer group of C is trivial, hence we can write $Y = \mathrm{Grass}(n, \mathcal{V})$ for some locally free sheaf $\mathcal{V}$ of rank m. Let $\mathcal{W}_{\mathrm{univ}} \subset \mathcal{V} \otimes \mathcal{O}_Y$ denote the universal locally direct summand. Comparing fibre by fibre we observe that

$$\mathcal{N}|_Y \otimes \wedge^n \mathcal{W}_{\mathrm{univ}} \cong (Y \to C)^* \mathcal{H}$$

for some invertible sheaf H on C.

Thus a section $Z = \sigma(C)$ of $Y \to C$ corresponds to a locally direct summand $\mathcal{W} \subset \mathcal{V}$ of rank n. The stability condition of Lemma 4.2.3 corresponds to the stability condition for this same $\mathcal{W}$. The corresponding point in $\underline{\mathrm{Pic}}^d_C$ is the point corresponding to the invertible sheaf $(\wedge^n \mathcal{W})^{-1} \otimes \mathcal{H}$. Hence the degree of Z with respect to $\mathcal{N}$ is $d = -\deg(\mathcal{W}) + \deg(\mathcal{H})$. Thus increasing d corresponds to decreasing $d' = \deg \mathcal{W}$. We will see below that the integer $h(t)$ of the statement of Theorem 4.2.4 can be taken to be $h(t) = -\deg(\mathcal{H})$.

Let $M^s(C, n, d')$ denote the coarse moduli scheme of rank n degree d' stable locally free sheaves on C. The arguments above show that there exists a sequence of morphisms

$$U'_d \to M^s(C, n, d') \to \underline{\mathrm{Pic}}^{d'}_C \to \underline{\mathrm{Pic}}^d_C$$

factoring the morphism of the Theorem. The last of these morphisms is an isomorphism.

The first morphism is identified with an open subscheme of a projective bundle. Namely, the fibre over the point $[\mathcal{W}]$ is the open subschemes inside $\mathbb{P}(\mathrm{Hom}(\mathcal{W}, \mathcal{V}))$ corresponding to maps $\mathcal{W} \to \mathcal{V}$ having maximal rank in all points of C. Since $\mathcal{W}$ is stable, if $d >> 0$, so $d' = \deg(\mathcal{W}) << 0$ then the codimension of the complement is $m - n - 1$.

For later use we describe this projective bundle a little better. There is a surjective étale morphism $T \to M^s(C, n, d')$ such that a tautological locally free sheaf $\mathcal{W}_T$ over $T \times C$ exists. After refining $T \to M^s(C, n, d')$ we may assume that there exists an isomorphism

$$\gamma : \mathrm{pr}_1^* \mathcal{W}_T \longrightarrow \mathrm{pr}_2^* \mathcal{W}_T$$

over $T \times_{M^s(C,n,d')} T \times C$. Over the triple fibre product $T \times_{M^s} T \times_{M^s} T \times C$ the combination $\mathrm{pr}_{23}^* \gamma \circ \mathrm{pr}_{12}^* \gamma$ differs from $\mathrm{pr}_{13}^* \gamma$ by an invertible function on $T \times_{M^s} T \times_{M^s} T$. (This invertible function defines a Brauer class on M^s which is the obstruction for the existence of a tautological bundle.) All of this means that we can consider the projective bundle

$$\mathbb{P}(\mathrm{pr}_{1,*}(\mathcal{H}om(\mathcal{W}_T, \mathrm{pr}_2^* \mathcal{V})))$$

over T. The isomorphism γ induces an isomorphism of projective bundles and the cocylce condition for γ over the triple fibre product implies that this is a descent datum. The conclusion is that we get a projective bundle $\mathbb{P} \to M^s(C, n, d')$ and an open immersion

$$\begin{tikzcd} U'_d \arrow[rr] \arrow[dr] & & \mathbb{P} \arrow[dl] \\ & M^s(C, n, d') & \end{tikzcd}$$

The morphism $\det : M^s(C, n, d') \to \underline{\mathrm{Pic}}_C^{d'}$ is smooth. This fact corresponds, via deformation theory, to the fact that $H^1(\mathrm{Tr}) : H^1(C, \mathrm{End}(\mathcal{W})) \to H^1(C, \mathcal{O}_C)$ is surjective for every locally free sheaf $\mathcal{W}$.

There is a morphism $a : \mathbb{P}^1 \to M^s(C, n, d')$ so that the pull back of the tangent sheaf

$$\mathcal{T}_{M^s(C,n,d')/\underline{\mathrm{Pic}}_C^{d'}}$$

is ample. One way to see this is to consider the Ext group

$$E = \mathrm{Ext}_C(\mathcal{B} \otimes \mathcal{A}^{\otimes -n+1}, \mathcal{A}^{\oplus n-1})$$

where $\mathcal{B} \in \mathrm{Pic}^{d'}(C)$ and where $\mathcal{A}$ is an invertibe sheaf of very negative degree. A general element $e \in E$ corresponds to a stable rank n sheaf, and we obtain a rational map from the projective space of lines in E to the fibre of det over $[\mathcal{B}]$. One checks that this morphism is defined away from codimension 2 and that it is generically smooth. A general high degree rational curve in $\mathbb{P}(E)$ maps to the desired curve a.

The description of the morphism $U'_d \to M^s(C, n, d')$ above implies that we can lift the morphism a to a morphism $b : \mathbb{P}^1 \to U'_d$. By adding lines in fibres of $U'_d \to M^s(C, n, d')$ and smoothing we can make sure that the restriction of the tangent sheaf

$$\mathcal{T}_{U'_d/\underline{\mathrm{Pic}}_C^d}$$

is ample.

Finally, suppose that $\gcd(d + h(t), n) = 1$. By our choice of $h(t) = -\deg(\mathcal{H})$ above we have $d + h(t) = -d'$ and hence $\gcd(d', n) = 1$. This implies that $M^s(C, n, d')$ is a projective variety. Thus in this case the projective bundle $\mathbb{P} \to M^s(C, n, d')$ is itself a projective variety, and the fibres of the induced morphism $\mathbb{P} \to \underline{\mathrm{Pic}}_C^d$ are the desired compactifications. $\square$

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