

SET THEORY

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1. Introduction

We need some set theory every now and then. We use Zermelo-Fraenkel set theory with the axiom of choice (ZFC) as described in [Kun83] and [Jec02].

2. Everything is a set

Most mathematicians think of set theory as providing the basic foundations for mathematics. So how does this really work? For example, how do we translate the sentence “ X is a scheme” into set theory? Well, we just unravel the definitions: A scheme is a locally ringed space such that every point has an open neighbourhood which is an affine scheme. A locally ringed space is a ringed space such that every stalk of the structure sheaf is a local ring. A ringed space is a pair (X, \mathcal{O}_X) consisting of a topological space X and a sheaf of rings \mathcal{O}_X on it. A topological space is a pair (X, τ) consisting of a set X and a set of subsets $\tau \subset \mathcal{P}(X)$ satisfying the axioms of a topology. And so on and so forth.

So how, given a set S would we recognize whether it is a scheme? The first thing we look for is whether the set S is an ordered pair. This is defined (see [Jec02], page 7) as saying that S has the form $(a, b) := \{\{a\}, \{a, b\}\}$ for some sets a, b . If this is the case, then we would take a look to see whether a is an ordered pair (c, d) . If so we would check whether $d \subset \mathcal{P}(c)$, and if so whether d forms the collection of sets for a topology on the set c . And so on and so forth.

So even though it would take a considerable amount of work to write a complete formula $\phi_{\text{scheme}}(x)$ with one free variable x in set theory that expresses the notion

“ x is a scheme”, it is possible to do so. The same thing should be true for any mathematical object.

3. Classes

Informally we use the notion of a *class*. Given a formula $\phi(x, p_1, \dots, p_n)$ we call

$$C = \{x : \phi(x, p_1, \dots, p_n)\}$$

a *class*. A class is easier to manipulate than the formula that defines it but it is not strictly speaking a mathematical object. For example, if R is a ring then we may consider the class of all R -modules (since after all we may translate the sentence “ M is an R -module” into a formula in set theory which then defines a class). A *proper class* is a class which is not a set.

In this way we may consider the category of R -modules which is a “big” category, in other words it has a proper class of objects. Similarly we may consider the “big” category of schemes, the “big” category of rings, etc.

4. The hierarchy of sets

A set T is *transitive* if $x \in T$ implies $x \subset T$. A set α is an *ordinal* if it is transitive and well-ordered by \in . In this case we define $\alpha + 1 = \alpha \cup \{\alpha\}$, which is another ordinal called the *successor* of α . An ordinal α is called a *successor ordinal* if there exists an ordinal β such that $\alpha = \beta + 1$. If α is not a successor ordinal, then α is called a *limit ordinal* and we have

$$\alpha = \bigcup_{\gamma \in \alpha} \gamma.$$

The smallest ordinal is \emptyset which is also denoted 0 . The first limit ordinal is ω and it is also the first infinite ordinal. The collection of all ordinals is a proper class. It is well-ordered by \in in the following sense: any nonempty set (or even class) of ordinals has a least element. Given a set A of ordinals we define the *supremum* of A to be $\sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha$. It is the least ordinal bigger or equal to all $\alpha \in A$. Given any well ordered set (S, \geq) there is a unique ordinal α such that $(S, \geq) \cong (\alpha, \in)$; this is called the *order type* of the well ordered set.

We define, by transfinite induction, $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ (power set), and for a limit ordinal α ,

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta.$$

Note that each V_α is a transitive set.

Lemma 4.1. (See [Jec02], Lemma 6.3.) *Every set is an element of V_α for some ordinal α .*

In [Kun83, Chapter III] it is explained that this lemma is equivalent to the axiom of foundation. The *rank* of a set S is the least ordinal α such that $S \in V_\alpha$.

5. Cardinality

The *cardinality* of a set A is the least ordinal α such that there exists a bijection between A and α . We say an ordinal α is a *cardinal* if and only if it occurs as the cardinality of some set, in other words, if $\alpha = |\alpha|$. We use the greek letters κ, λ for cardinals. The first infinite cardinal is ω , and in this context it is denoted \aleph_0 . A set is *countable* if its cardinality is $\leq \aleph_0$. The *addition* of cardinals κ, λ is denoted $\kappa \oplus \lambda$; it is the cardinality of $\kappa \amalg \lambda$. The *multiplication* of cardinals κ, λ is denoted $\kappa \otimes \lambda$; it is the cardinality of $\kappa \times \lambda$. The *exponentiation* of cardinals κ, λ is denoted κ^λ ; it is the cardinality of the set of (set) maps from λ to κ . Given any set K of cardinals the *supremum* of K is $\sup_{\kappa \in K} \kappa = \bigcup_{\kappa \in K} \kappa$. This is a cardinal.

6. Cofinality

A *cofinal subset* S of a partially ordered set T is a subset $S \subset T$ such that $\forall t \in T \exists s \in S (t \leq s)$. Note that a subset of a well-ordered set is a well-ordered set (with induced ordering). Given an ordinal α the *cofinality* $\text{cf}(\alpha)$ of α is the least ordinal β which occurs as the order type of some cofinal subset of α . The cofinality of an ordinal is always a cardinal.

7. Reflection principle

Some of this material is in the chapter of [Kun83] called “Easy consistency proofs”.

Let $\phi(x_1, \dots, x_n)$ be a formula of set theory. Let us use the convention that this notation implies that all the free variables in ϕ occur among x_1, \dots, x_n . Let M be a set. The formula $\phi^M(x_1, \dots, x_n)$ is the formula obtained from $\phi(x_1, \dots, x_n)$ by replacing all the $\forall x$ and $\exists x$ by $\forall x \in M$ and $\exists x \in M$. So the formula $\phi(x_1, x_2) = \exists x(x \in x_1 \wedge x \in x_2)$ is turned into $\phi^M(x_1, x_2) = \exists x \in M(x \in x_1 \wedge x \in x_2)$. The formula ϕ^M is called the *relativization of ϕ to M* .

Theorem 7.1. See [Jec02, Theorem 12.14] or [Kun83, Theorem 7.4]. *Suppose given $\phi_1(x_1, \dots, x_n), \dots, \phi_m(x_1, \dots, x_n)$ a finite collection of formulas of set theory. Let M_0 be a set. There exists a set M such that $M_0 \subset M$ and $\forall x_1, \dots, x_n \in M$, we have*

$$\forall i = 1, \dots, m, \phi_i^M(x_1, \dots, x_n) \Leftrightarrow \forall i = 1, \dots, m, \phi_i(x_1, \dots, x_n).$$

In fact we may take $M = V_\alpha$ for some limit ordinal α .

We view this theorem as saying the following: Given any $x_1, \dots, x_n \in M$ the formulas hold with the bound variables ranging through all sets if and only if they hold for the bound variables ranging through elements of V_α . This theorem is a meta-theorem, since it deals with the formulas of set theory directly. It actually says that given the finite list of formulas ϕ_1, \dots, ϕ_m with at most free variables x_1, \dots, x_n the sentence

$$\forall M_0 \exists M, M_0 \subset M \forall x_1, \dots, x_n \in M \phi_1(x_1, \dots, x_n) \wedge \dots \wedge \phi_m(x_1, \dots, x_n) \Leftrightarrow \phi_1^M(x_1, \dots, x_n) \wedge \dots \wedge \phi_m^M(x_1, \dots, x_n)$$

is provable in ZFC. In other words, whenever we actually write down a finite list of formulas ϕ_i we get a theorem.

It is somewhat hard to use this theorem in “ordinary mathematics” since the meaning of the formulas $\phi_i^M(x_1, \dots, x_n)$ is not so clear! Instead we will use the idea of the proof of the reflection principle to prove the existence results we need directly.

8. Constructing categories of schemes

We will discuss how to apply this to produce, given an initial set of schemes, a “small” category of schemes closed under a list of natural operations. Before we do so we introduce the size of a scheme. Given a scheme S we define

$$\text{size}(S) = \max(\aleph_0, \kappa_1, \kappa_2)$$

where we define the cardinal numbers κ_1 and κ_2 as follows

- (1) We let κ_1 be the cardinality of the set of affine opens of S .
- (2) We let κ_2 be the supremum of all the cardinalities of all $\Gamma(U, \mathcal{O}_S)$ for all $U \subset S$ affine open.

Lemma 8.1. *For every cardinal κ there exists a set A such that every element of A is a scheme, and such that for every scheme S with $\text{Size}(S) \leq \kappa$ there is an element $X \in A$ such that $X \cong S$ (isomorphism of schemes).*

Proof. Omitted. Hint: think about how any scheme is isomorphic to a scheme obtained by glueing affines. \square

We denote $Bound$ the function which to each cardinal κ associates $Bound(\kappa) = \kappa^{\aleph_0}$. We could make this function grow much more rapidly, e.g., we could set $Bound(\kappa) = \kappa^\kappa$, and the result below would still hold. For any ordinal α we denote Sch_α the full subcategory of category of schemes whose objects are elements of V_α . Here is the result we are going to prove.

Lemma 8.2. *With notations $size$, $Bound$ and Sch_α as above. Let S_0 be a set of schemes. There exists a limit ordinal α with the following properties:*

- (1) We have $S_0 \subset V_\alpha$, in other words $S_0 \subset Ob(Sch_\alpha)$.
- (2) For any $S \in Ob(Sch_\alpha)$ and any scheme T with $size(T) \leq Bound(size(S))$ there exists a scheme $S' \in Ob(Sch_\alpha)$ such that $T \cong S'$.
- (3) For any countable diagram¹ category \mathcal{I} and any functor $F : \mathcal{I} \rightarrow Sch_\alpha$ the limit $\lim_{\mathcal{I}} F$ exists in Sch_α if and only if it exists in Sch and moreover in this case the natural morphism between them is an isomorphism.
- (4) For any countable diagram category \mathcal{I} and any functor $F : \mathcal{I} \rightarrow Sch_\alpha$ the colimit $\text{colim}_{\mathcal{I}} F$ exists in Sch_α if and only if it exists in Sch and moreover in this case the natural morphism between them is an isomorphism.

Proof. We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$ we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

- (1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
- (2) For any $S \in Ob(Sch_{f(\alpha)})$ and any scheme T with $size(T) \leq Bound(size(S))$ there exists a scheme $S' \in Ob(Sch_\beta)$ such that $T \cong S'$.
- (3) For any countable diagram category \mathcal{I} and any functor $F : \mathcal{I} \rightarrow Sch_{f(\alpha)}$ if the limit $\lim_{\mathcal{I}} F$ or the colimit $\text{colim}_{\mathcal{I}} F$ exists in Sch then it is isomorphic to a scheme in Sch_β .

To see β exists we argue as follows. Since $Ob(Sch_{f(\alpha)})$ is a set we see that $\kappa = \sup_{S \in Ob(Sch_{f(\alpha)})} Bound(size(S))$ exists and is a cardinal. Let A be a set of schemes obtained starting with κ as in Lemma 8.1. There is a set $CountCat$ of

¹Both the set of objects and the morphism sets are countable. In fact you can prove the lemma with \aleph_0 replaced by any cardinal whatsoever in (3) and (4).

countable categories such that any countable category is isomorphic to an element of $\mathit{CountCat}$. Hence in (3) above we may assume that \mathcal{I} is an element in $\mathit{CountCat}$. This means that the pairs (\mathcal{I}, F) in (3) range over a set. Thus there exists a set B whose elements are schemes such that for every (\mathcal{I}, F) as in (3) if the limit or colimit exists, then it is isomorphic to an element in B . Hence if we pick any β such that $A \cup B \subset V_\beta$ and $\beta > \max\{\alpha + 1, f(\alpha)\}$ then (1)-(3) hold. Since every nonempty collection of ordinals has a least element we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha' < \alpha} f(\alpha')$.

Pick β_0 such that $S_0 \subset V_{\beta_0}$. By construction $f(\beta) \geq \beta$ and we see that also $S_0 \subset V_{f(\beta_0)}$. Moreover, as f is nondecreasing this remains true for any $\beta \geq \beta_0$. Next, choose any ordinal $\beta_1 > \beta_0$ with cofinality $\text{cf}(\beta_1) > \omega = \aleph_0$. This is possible since the cofinality of ordinals gets arbitrarily large, see. For example, given a cardinal κ , the cofinality of 2^κ is bigger than κ , see [Kun83, Chapter I, Corollary 10.41]. We claim that $\alpha = f(\beta_1)$ is a solution to the problem posed in the lemma.

The first property of the lemma holds by our choice of $\beta_1 > \beta_0$ above.

Since β_1 is a limit ordinal (as its cofinality is infinite) we get $f(\beta_1) = \sup_{\beta < \beta_1} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_1\} \subset f(\beta_1)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_1)} = \bigcup_{\beta < \beta_1} V_{f(\beta)}.$$

Now, let $S \in \text{Ob}(Sch_\alpha)$. We define $\beta(S)$ to be the least ordinal β such that $S \in \text{Ob}(Sch_{f(\beta)})$. By the above we see that always $\beta(S) < \beta_1$. Since $\text{Ob}(Sch_{f(\beta+1)}) \subset \text{Ob}(Sch_\alpha)$, we see by construction of f above that the second property of the lemma is satisfied.

Suppose that $\{S_1, S_2, \dots\} \subset \text{Ob}(Sch_\alpha)$ is a countable collection. Consider the function $\omega \rightarrow \beta_1$, $n \mapsto \beta(S_n)$. Since the cofinality of β_1 is $> \omega$ the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\{S_1, S_2, \dots\} \subset \text{Ob}(Sch_{f(\beta)})$. It follows that any functor $F : \mathcal{I} \rightarrow Sch_\alpha$ factors through one of the subcategories $Sch_{f(\beta)}$. Thus, if there exists a scheme X which is the colimit or limit of the diagram F , then by construction of f we see X is isomorphic to an object of $Sch_{f(\beta+1)}$ which is a subcategory of Sch_α . This proves the last two assertions of the lemma. \square

Remark 8.3. The lemma above can also be proved using the reflection principle. However, one has to be careful. Namely, suppose the sentence $\phi_{\text{scheme}}(X)$ expresses the property “ X is a scheme”, then what does the formula $\phi_{\text{scheme}}^{V_\alpha}(X)$ mean? It is true that the reflection principle says we can find α such that for all $X \in V_\alpha$ we have $\phi_{\text{scheme}}(X) \leftrightarrow \phi_{\text{scheme}}^{V_\alpha}(X)$ but this is entirely useless. It is only by combining two such statements that something interesting happens. For example suppose $\phi_{\text{red}}(X, Y)$ expresses the property “ X, Y are schemes, and Y is the reduction of X ” (see Schemes, Definition 12.5). Suppose we apply the reflection principle to the pair of formulas $\phi_1(X, Y) = \phi_{\text{red}}(X, Y)$, $\phi_2(X) = \exists Y, \phi_1(X, Y)$. Then it is easy to see that any α produced by the reflection principle has the property that given $X \in \text{Ob}(Sch_\alpha)$ the reduction of X is also an object of Sch_α (left as an exercise).

Lemma 8.4. *Let S be an affine scheme. Let $R = \Gamma(S, \mathcal{O}_S)$. Then the size of S is equal to $\max\{\aleph_0, |R|\}$.*

Proof. There are at most $\max\{|R|, \aleph_0\}$ affine opens of $\text{Spec}(R)$. This is clear since any affine open $U \subset \text{Spec}(R)$ is a finite union of principal opens $D(f_1) \cup \dots \cup D(f_n)$ and hence the number of affine opens is at most $\sup_n |R|^n = \max\{|R|, \aleph_0\}$, see [Kun83, Ch. I, 10.13]. On the other hand, we see that $\Gamma(U, \mathcal{O}) \subset R_{f_1} \times \dots \times R_{f_n}$ and hence $|\Gamma(U, \mathcal{O})| \leq \max\{\aleph_0, |R_{f_1}|, \dots, |R_{f_n}|\}$. Thus it suffices to prove that $|R_f| \leq \max\{\aleph_0, |R|\}$ which is omitted. \square

Lemma 8.5. *Let S be a scheme. Let $S = \bigcup_{i \in I} S_i$ be an open covering. Then $\text{size}(S) \leq \max\{|I|, \sup_i \{\text{size}(S_i)\}\}$.*

Proof. Let $U \subset S$ be any affine open. Since U is quasi-compact there exist finitely many elements $i_1, \dots, i_n \in I$ and affine opens $U_i \subset U \cap S_i$ such that $U = U_1 \cup U_2 \cup \dots \cup U_n$. Thus

$$|\Gamma(U, \mathcal{O}_U)| \leq |\Gamma(U_1, \mathcal{O})| \otimes \dots \otimes |\Gamma(U_n, \mathcal{O})| \leq \sup_i \{\text{size}(S_i)\}$$

Moreover, it shows that the set of affine opens of S has cardinality less than or equal to the cardinality of the set

$$\coprod_{n \in \omega} \prod_{i_1, \dots, i_n \in I} \{\text{affine opens of } S_{i_1}\} \times \dots \times \{\text{affine opens of } S_{i_n}\}.$$

Each of the sets inside the disjoint union has cardinality at most $\sup_i \{\text{size}(S_i)\}$. The index set has cardinality at most $\max\{|I|, \aleph_0\}$, see [Kun83, Ch. I, 10.13]. Hence by [Jec02, Lemma 5.8] the cardinality of the coproduct is at most $\max\{\aleph_0, |I|\} \otimes \sup_i \{\text{size}(S_i)\}$. The lemma follows. \square

Lemma 8.6. *Let α be an ordinal as in Lemma 8.2 above. The category Sch_α satisfies the following properties:*

- (1) *If $X, Y, S \in \text{Ob}(\text{Sch}_\alpha)$, then for any morphisms $f : X \rightarrow S, g : Y \rightarrow S$ the fibre product $X \times_S Y$ in Sch_α exists and is a fibre product in the category of schemes.*
- (2) *Given any at most countable collection S_1, S_2, \dots of elements of $\text{Ob}(\text{Sch}_\alpha)$ the coproduct $\coprod_i S_i$ exists in $\text{Ob}(\text{Sch}_\alpha)$ and is a coproduct in the category of schemes.*
- (3) *For any $S \in \text{Ob}(\text{Sch}_\alpha)$ and any open immersion $U \rightarrow S$, there exists a $V \in \text{Ob}(\text{Sch}_\alpha)$ with $V \cong U$.*
- (4) *For any $S \in \text{Ob}(\text{Sch}_\alpha)$ and any closed immersion $T \rightarrow S$, there exists a $S' \in \text{Ob}(\text{Sch}_\alpha)$ with $S' \cong T$.*
- (5) *For any $S \in \text{Ob}(\text{Sch}_\alpha)$ and any finite type morphism $T \rightarrow S$, there exists a $S' \in \text{Ob}(\text{Sch}_\alpha)$ with $S' \cong T$.*
- (6) *Suppose S is a scheme which has an open covering $S = \bigcup_{i \in I} S_i$ such that there exists a $T \in \text{Ob}(\text{Sch}_\alpha)$ with (a) $\text{size}(S_i) \leq \text{size}(T)^{\aleph_0}$ for all $i \in I$, and (b) $|I| \leq |\text{size}(T)|^{\aleph_0}$. Then S is isomorphic to an object of Sch_α .*
- (7) *For any $S \in \text{Ob}(\text{Sch}_\alpha)$ and any morphism $f : T \rightarrow S$ locally of finite type such that T can be covered by at most $\text{size}(S)^{\aleph_0}$ open affines, there exists a $S' \in \text{Ob}(\text{Sch}_\alpha)$ with $S' \cong T$. For example this holds if T can be covered by at most $|\mathbf{R}| = 2^{\aleph_0} = \aleph_0^{\aleph_0}$ open affines.*
- (8) *Suppose that $T \in \text{Ob}(\text{Sch}_\alpha)$ is affine. Write $R = \Gamma(T, \mathcal{O}_T)$. Then Sch_α contains a scheme isomorphic to any of the following schemes:*
 - (a) *For any ideal $I \subset R$ with completion $R^* = \lim R/I^n$ the scheme $\text{Spec}(R^*)$.*

- (b) For any finite type R -algebra R' the scheme $\text{Spec}(R')$.
- (c) For any localization $S^{-1}R$ the scheme $\overline{\text{Spec}(S^{-1}R)}$.
- (d) For any prime $\mathfrak{p} \subset R$ the scheme $\overline{\text{Spec}(\kappa(\mathfrak{p}))}$.
- (e) For any subring $R' \subset R$ the scheme $\text{Spec}(R')$.
- (f) Any scheme of finite type over a ring of cardinality at most $|R|^{\aleph_0}$.
- (g) And so on.

Proof. Statements (1) and (2) follow directly from the definitions. Statement (3) follows as the size of an open subscheme U of S is clearly smaller or equal than the size of S . Statement (4) follows from (5). Statement (5) follows from (7). Statement (6) follows as the size of S is $\leq \max\{|I|, \sup_i \text{size}(S_i)\} \leq \text{size}(T)^{\aleph_0}$ by Lemma 8.5. Statement (7) follows from (6). Namely, for any affine open $V \subset T$ there exist finitely many affine opens $U_1, \dots, U_n \subset S$ and affine opens $V_i \subset V$ such that $f(V_i) \subset U_i$ and $\bigcup V_i = V$. Then $V_i \rightarrow U_i$ is of finite type and hence $|\Gamma(V_i, \mathcal{O}_T)| \leq \max\{\aleph_0, |\Gamma(U_i, \mathcal{O}_T)|\}$ (see below). This shows that $\Gamma(V, \mathcal{O}_T) \subset \prod_{i=1, \dots, n} \Gamma(V_i, \mathcal{O}_T)$ and hence $|\Gamma(V, \mathcal{O}_T)| \leq \max\{\aleph_0, |\Gamma(V_i, \mathcal{O}_T)|\} \leq \max\{\aleph_0, |\Gamma(U_i, \mathcal{O}_S)|\} \leq \text{size}(S)$. By Lemma 8.4 this bounds the size of V . Thus we see that (6) applies in the situation of (7).

Statement (8) is translated, via Lemma 8.4, into a bound on the cardinality of the rings R^* , $S^{-1}R$, $\overline{\kappa(\mathfrak{p})}$, R' , etc. Perhaps the most interesting one is the ring R^* . As a set it is the image of a surjective map $R^{\aleph_0} \rightarrow R^*$. Since $|R^{\aleph_0}| = |R|^{\aleph_0}$ we see that it works by our choice of $\text{Bound}(\kappa) = \kappa^{\aleph_0}$. Phew! (The cardinality of the algebraic closure of a field is the same as the cardinality of the field, or it is \aleph_0 .) \square

Remark 8.7. Let R be a ring. Suppose we consider the ring $\prod_{\mathfrak{p} \in \text{Spec}(R)} \kappa(\mathfrak{p})$. The cardinality of this ring is bounded by $|R|^{|R|}$, but is not bounded by $|R|^{\aleph_0}$ in general (for example if $R = \mathbf{C}[x]$). Thus the ‘‘And so on’’ of Lemma 8.6 above should be taken with a grain of salt. Of course, if it ever becomes necessary to consider these rings in arguments pertaining to fppf/etale cohomology, then we can change the function Bound above into the function $\kappa \mapsto \kappa^\kappa$.

9. Sets with group action

Let G be a group. We denote $G\text{-Sets}$ the ‘‘big’’ category of G -sets. For any ordinal α we denote $G\text{-Sets}_\alpha$ the full subcategory of $G\text{-Sets}$ whose objects are in V_α . As a notion for size of a G -set we take $\text{size}(S) = \max\{\aleph_0, |G|, |S|\}$ (where $|G|$, and $|S|$ is the cardinality of the underlying set). As above we use the function $\text{Bound}(\kappa) = \kappa^{\aleph_0}$.

Lemma 9.1. *We notations G , $G\text{-Sets}_\alpha$, size, and Bound as above. Let S_0 be a set of G -sets. There exists a limit ordinal α with the following properties:*

- (1) We have $S_0 \cup \{G\} \subset \text{Ob}(G\text{-Sets}_\alpha)$.
- (2) For any $S \in \text{Ob}(G\text{-Sets}_\alpha)$ and any G -set T with $\text{size}(T) \leq \text{Bound}(\text{size}(S))$ there exists a $S' \in \text{Ob}(G\text{-Sets}_\alpha)$ which is isomorphic to T .
- (3) For any countable diagram category \mathcal{I} and any functor $F : \mathcal{I} \rightarrow G\text{-Sets}_\alpha$ the limit $\lim_{\mathcal{I}} F$ and colimit $\text{colim}_{\mathcal{I}} F$ exist in $G\text{-Sets}_\alpha$ and are the same as in $G\text{-Sets}$.

Proof. Omitted. Similar to but easier than the proof of Lemma 8.2 above. \square

Lemma 9.2. *Let α be an ordinal as in Lemma 9.1 above. The category $G\text{-Sets}_\alpha$ satisfies the following properties:*

- (1) *The G -set ${}_C G$ is an object of $G\text{-Sets}_\alpha$.*
- (2) *(Co)Products, fibre products and pushouts exist in $G\text{-Sets}_\alpha$ and are the same as their counterparts in $G\text{-Sets}$.*
- (3) *Given an object U of $G\text{-Sets}_\alpha$ any G -stable subset $O \subset U$ is isomorphic to an object of $G\text{-Sets}_\alpha$.*

Proof. Omitted. □

10. Coverings of a site

Suppose that \mathcal{C} is a category (as in Categories, Definition 2.1) and that $\text{Cov}(\mathcal{C})$ is a proper class of coverings satisfying properties (1), (2) and (3) of Sites, Definition 6.2. We list them here:

- (1) If $V \rightarrow U$ is an isomorphism then $\{V \rightarrow U\} \in \text{Cov}(\mathcal{C})$.
- (2) If $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \rightarrow U_i\}_{j \in J_i} \in \text{Cov}(\mathcal{C})$, then $\{V_{ij} \rightarrow U\}_{i \in I, j \in J_i} \in \text{Cov}(\mathcal{C})$.
- (3) If $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})$ and $V \rightarrow U$ is a morphism of \mathcal{C} then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \rightarrow V\}_{i \in I} \in \text{Cov}(\mathcal{C})$.

For an ordinal α we set $\text{Cov}(\mathcal{C})_\alpha = \text{Cov}(\mathcal{C}) \cap V_\alpha$. Given an ordinal α and a cardinal κ we set $\text{Cov}(\mathcal{C})_{\kappa, \alpha}$ equal to the set of elements $\mathcal{U} = \{\varphi_i : U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})_\alpha$ such that $|I| \leq \kappa$.

We recall the following notion, see Sites, Definition 8.2. Two families of morphisms $\{\varphi_i : U_i \rightarrow U\}_{i \in I}$, and $\{\psi_j : W_j \rightarrow U\}_{j \in J}$ with the same target of \mathcal{C} are called *combinatorially equivalent* if there exist maps $\alpha : I \rightarrow J$ and $\beta : J \rightarrow I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$. This defines an equivalence relation on families of morphisms having a fixed target.

Lemma 10.1. *With notations as above. Let $\text{Cov}_0 \subset \text{Cov}(\mathcal{C})$ be a set contained in $\text{Cov}(\mathcal{C})$. There is a cardinal κ and a limit ordinal α with the following properties:*

- (1) *We have $\text{Cov}_0 \subset \text{Cov}(\mathcal{C})_{\kappa, \alpha}$.*
- (2) *The set of coverings $\text{Cov}(\mathcal{C})_{\kappa, \alpha}$ satisfies (1), (2) and (3) of Sites, Definition 6.2 (see above). In other words $(\mathcal{C}, \text{Cov}(\mathcal{C})_{\kappa, \alpha})$ is a site.*
- (3) *Every covering in $\text{Cov}(\mathcal{C})$ is combinatorially equivalent to a covering in $\text{Cov}(\mathcal{C})_{\kappa, \alpha}$.*

Proof. To prove this, we first consider the set \mathcal{S} of all sets of morphisms of \mathcal{C} with fixed target. In other words, an element of \mathcal{S} is a subset T of $\text{Arrows}(\mathcal{C})$ such that all elements of T have the same target. Given a family $\mathcal{U} = \{\varphi_i : U_i \rightarrow U\}_{i \in I}$ of morphisms with fixed target, we define $\text{Supp}(\mathcal{U}) = \{\varphi \in \text{Arrows}(\mathcal{C}) \mid \exists i \in I, \varphi = \varphi_i\}$. Note that two families $\mathcal{U} = \{\varphi_i : U_i \rightarrow U\}_{i \in I}$ and $\mathcal{V} = \{\psi_j : W_j \rightarrow U\}_{j \in J}$ are combinatorially equivalent if and only if $\text{Supp}(\mathcal{U}) = \text{Supp}(\mathcal{V})$. Next, we define $\mathcal{S}_\tau \subset \mathcal{S}$ to be the subset $\mathcal{S}_\tau = \{T \in \mathcal{S} \mid \exists \mathcal{U} \in \text{Cov}(\mathcal{C}) T = \text{Supp}(\mathcal{U})\}$. For every element $T \in \mathcal{S}_\tau$ set $\beta(T)$ equal to the least ordinal β such that there exists a $\mathcal{U} \in \text{Cov}(\mathcal{C})_\beta$ such that $T = \text{Supp}(\mathcal{U})$. Finally, set $\beta_0 = \sup_{T \in \mathcal{S}_\tau} \beta(T)$. At this point it follows that every $\mathcal{U} \in \text{Cov}(\mathcal{C})$ is combinatorially equivalent to some element of $\text{Cov}(\mathcal{C})_{\beta_0}$.

Let κ be the maximum of \aleph_0 , the cardinality $|\text{Arrows}(\mathcal{C})|$,

$$\sup_{\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\beta_0}} |I| \quad \text{and} \quad \sup_{\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}_0} |I|.$$

Since κ is an infinite cardinal we have $\kappa \otimes \kappa = \kappa$. Note that obviously $\text{Cov}(\mathcal{C})_{\beta_0} = \text{Cov}(\mathcal{C})_{\kappa, \beta_0}$.

We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$ we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

- (1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
- (2) If $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$ and for each i we have $\{W_{ij} \rightarrow U_i\}_{j \in J_i} \in \text{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$, then $\{W_{ij} \rightarrow U\}_{i \in I, j \in J_i} \in \text{Cov}(\mathcal{C})_{\kappa, \beta}$.
- (3) If $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, \alpha}$ and $W \rightarrow U$ is a morphism of \mathcal{C} then $\{U_i \times_U W \rightarrow W\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, \beta}$.

To see β exists we note that clearly the collection of all coverings $\{W_{ij} \rightarrow U\}$ and $\{U_i \times_U W \rightarrow W\}$ that occur in (2) and (3) form a set. Hence there is some ordinal β such that V_β contains all of these coverings. Moreover, the index set of the covering $\{W_{ij} \rightarrow U\}$ has cardinality $\sum_{i \in I} |J_i| \leq \kappa \otimes \kappa = \kappa$, and hence these coverings are contained in $\text{Cov}(\mathcal{C})_{\kappa, \beta}$. Since every nonempty collection of ordinals has a least element we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha' < \alpha} f(\alpha')$.

Pick an ordinal β_1 such that $\text{Arrows}(\mathcal{C}) \subset V_{\beta_1}$, $\text{Cov}_0 \subset V_{\beta_0}$, and $\beta_1 \geq \beta_0$. By construction $f(\beta_1) \geq \beta_1$ and we see that the same properties hold for $V_{f(\beta_1)}$. Moreover, as f is nondecreasing this remains true for any $\beta \geq \beta_1$. Next, choose any ordinal $\beta_2 > \beta_1$ with cofinality $\text{cf}(\beta_2) > \kappa$. This is possible since the cofinality of ordinals gets arbitrarily large. For example, given a cardinal λ , the cofinality of 2^λ is bigger than λ , see [Kun83, Chapter I, Corollary 10.41]. We claim that the pair $\kappa, \alpha = f(\beta_2)$ is a solution to the problem posed in the lemma.

The first and third property of the lemma holds by our choices of $\kappa, \beta_2 > \beta_1 > \beta_0$ above.

Since β_2 is a limit ordinal (as its cofinality is infinite) we get $f(\beta_2) = \sup_{\beta < \beta_2} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_2\} \subset f(\beta_2)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_2)} = \bigcup_{\beta < \beta_2} V_{f(\beta)}.$$

Now, let $\mathcal{U} \in \text{Cov}_{\kappa, \alpha}$. We define $\beta(\mathcal{U})$ to be the least ordinal β such that $\mathcal{U} \in \text{Cov}_{\kappa, f(\beta)}$. By the above we see that always $\beta(\mathcal{U}) < \beta_2$.

We have to show properties (1), (2) and (3) defining a site hold for the pair $(\mathcal{C}, \text{Cov}_{\kappa, \alpha})$. The first holds because by our choice of β_2 all arrows of \mathcal{C} are contained in $V_{f(\beta_2)}$. For the third, we use that given a covering $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, \alpha}$ we have $\beta(\mathcal{U}) < \beta_2$ and hence any base change of \mathcal{U} is by construction of f contained in $\text{Cov}(\mathcal{C})_{\kappa, f(\beta+1)}$ and hence in $\text{Cov}(\mathcal{C})_{\kappa, \alpha}$.

Finally, for the second condition, suppose that $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$ and for each i we have $\mathcal{W}_i = \{W_{ij} \rightarrow U_i\}_{j \in J_i} \in \text{Cov}(\mathcal{C})_{\kappa, f(\alpha)}$. Consider the function $I \rightarrow \beta_2, i \mapsto \beta(\mathcal{W}_i)$. Since the cofinality of β_2 is $> \kappa \geq |I|$ the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\mathcal{W}_i \in \text{Cov}_{\kappa, f(\beta)}$ for all $i \in I$. It follows that the covering $\{W_{ij} \rightarrow U\}_{i \in I, j \in J_i}$ is an element of $\text{Cov}(\mathcal{C})_{\kappa, f(\beta+1)} \subset \text{Cov}(\mathcal{C})_{\kappa, \alpha}$ as desired. \square

Remark 10.2. It is likely the case that for some limit ordinal the set of coverings $\text{Cov}(\mathcal{C})_\alpha$ satisfies the conditions of the lemma. This is after all what an application of the reflection principle would appear to give (modulo caveats as described at the end of Section 7 and in Remark 8.3).

11. Abelian categories and injectives

The following lemma applies to the category of modules over a sheaf of rings on a site.

Lemma 11.1. *Suppose given a big category \mathcal{A} (see Categories, Remark 2.2). Assume \mathcal{A} is abelian and has enough injectives. See Homology, Definitions 3.12 and 19.4. Then for any given set of objects $\{A_s\}_{s \in S}$ of \mathcal{A} there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ with the following properties*

- (1) $\text{Ob}(\mathcal{A}')$ is a set,
- (2) $\text{Ob}(\mathcal{A}')$ contains A_s for each $s \in S$,
- (3) \mathcal{A}' has enough injectives,
- (4) an object of \mathcal{A}' is injective if and only if it is an injective object of \mathcal{A} .

Proof. Omitted. □

12. Other chapters

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