

STACKS

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SECTION 1. INTRODUCTION

Stacks are defined in this document. See [\[DM69\]](#).

SECTION 2. DEFINITION

Let \mathcal{C} be a site. The 2-category of stacks over \mathcal{C} will be a full sub-2-category of the 2-category of categories over \mathcal{C} , see Categories, [Definition 3.1.5](#). Thus a stack will be given by a functor of categories $p : \mathcal{S} \rightarrow \mathcal{C}$ which satisfies the following conditions:

- (1) $p : \mathcal{S} \rightarrow \mathcal{C}$ is a category fibred in groupoids, see Categories, [Definition 3.1.1](#),
- (2) descent for morphisms holds, and
- (3) descent data for objects are effective.

Subsection 2.1. Explanation. To explain this, we choose a collection of pullback functors as in Categories, [Lemma 3.1.3](#). Another approach is to use Categories, [Lemma 3.3.2](#).

First, suppose that $x, y \in \text{Ob}(\mathcal{S}_U)$ are objects in the fibre category over U . We are going to define a contravariant functor

$$\text{Isom}(x, y) : \mathcal{C}/U \longrightarrow \text{Sets}.$$

In other words this will be a presheaf on \mathcal{C}/U , see Sites, [Definition 2.1.5](#). Namely, for $f : V \rightarrow U$ we set

$$\text{Isom}(x, y)(f : V \rightarrow U) = \text{Mor}_{\mathcal{S}_V}(f^*x, f^*y).$$

We also have to define the restriction map corresponding to a morphism $(g, \text{id}_U) : (f' : V' \rightarrow U) \rightarrow (f : V \rightarrow U)$ in \mathcal{C}/U (in other words $f' = f \circ g$). This will be a map

$$\text{Mor}_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow \text{Mor}_{\mathcal{S}_{V'}}(f'^*x, f'^*y).$$

This map will basically be g^* , except that this transforms an element ϕ of the left hand side into an element $g^*\phi$ of $\text{Mor}_{\mathcal{S}_{V'}}(g^*f^*x, g^*f^*y)$. At this point we use the transformation t of Categories, [Lemma 3.1.3](#). In a formula, the restriction map maps ϕ to

$$t_y \circ g^*\phi \circ (t_x)^{-1}.$$

Lemma 2.1.1. *This actually does give a presheaf.*

Proof. Let $(g', \text{id}_U) : (f' : V' \rightarrow U) \rightarrow (f : V \rightarrow U)$ and $(g'', \text{id}_U) : (f'' : V'' \rightarrow U) \rightarrow (f' : V' \rightarrow U)$ be morphisms in \mathcal{C}/U , and let $\phi \in \text{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$. It suffices to show that

$$[(\text{Isom}(x, y))(g' \circ g'')](\phi) = [(\text{Isom}(x, y))(g'')][(\text{Isom}(x, y))(g')](\phi)$$

By [Lemma 3.1.3](#) there are pullback functors $r : (g' \circ g'')^* f'^* \rightarrow f''^*$, $t' : g'^* f'^* \rightarrow f'^*$, $t'' : g''^* f''^* \rightarrow f''^*$ and $u : g''^* g'^* \rightarrow (g' \circ g'')^*$. It follows from the uniqueness part of the lemma that $t''_y \circ g''^* t'_y = r_y \circ u_{f^*y}$ and $g''^* t'^{-1}_x \circ t''^{-1}_x = u_{f^*x}^{-1} \circ r_x^{-1}$. We now have that

$$\begin{aligned} [(\text{Isom}(x, y))(g' \circ g'')](\phi) &= r_y \circ (g' \circ g'')^* \phi \circ r_x^{-1} \\ &= r_y \circ u_{f^*y} \circ g''^* g'^* \phi \circ u_{f^*x}^{-1} \circ r_x^{-1} \\ &= t''_y \circ g''^* t'_y \circ g''^* g'^* \phi \circ g''^* t'^{-1}_x \circ t''^{-1}_x \\ &= t''_y \circ g''^* (t'_y \circ g'^* \phi \circ t'^{-1}_x) \circ t''^{-1}_x \\ &= [(\text{Isom}(x, y))(g'')][(\text{Isom}(x, y))(g')](\phi). \end{aligned}$$

Alternatively we can argue with [Lemma 3.3.2](#) which says that $\mathcal{S} \rightarrow \mathcal{C}$ is equivalent to the category associated to a contravariant function $F : \mathcal{C} \rightarrow \text{Groupoids}$. Then t_y is the identity transformation for all y so the restriction map is $g \mapsto g^* \phi = F(g)\phi$ which is clearly functorial making $\text{Isom}(x, y)$ a presheaf. \square

OK, so the second condition listed in [Section 2](#) is simply the condition that $\text{Isom}(x, y)$ is a sheaf on the site \mathcal{C}/U !

In order to explain the meaning of the third condition, we must define a descent datum. First, we introduce some notation. Suppose that $\{f_i : U_i \rightarrow U\}_{i \in I}$ is a covering in the site \mathcal{C} . We will be looking at fibre products $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$. It is very important to allow $i = j$ and even $i = j = k$ in the following. The projection maps from U_{ij} to U_i , resp. U_j are denoted pr_1 , resp. pr_2 . The projection maps from U_{ijk} to the twofold fibre products are $\text{pr}_{12} : U_{ijk} \rightarrow U_{ij}$, $\text{pr}_{13} : U_{ijk} \rightarrow U_{ik}$, and $\text{pr}_{23} : U_{ijk} \rightarrow U_{jk}$. The projection maps from U_{ijk} to U_i , U_j , and U_k are pr_1 , pr_2 , and pr_3 , respectively.

This notation is potentially ambiguous, but it is standard in the literature. For an example of the ambiguity: note the relations $\text{pr}_1 = \text{pr}_1 \circ \text{pr}_{12} = \text{pr}_1 \circ \text{pr}_{13}$, $\text{pr}_2 = \text{pr}_2 \circ \text{pr}_{12} = \text{pr}_1 \circ \text{pr}_{23}$, and $\text{pr}_3 = \text{pr}_2 \circ \text{pr}_{13} = \text{pr}_3 \circ \text{pr}_{23}$.

Let $x_i \in \text{Ob}(\mathcal{S}_{U_i})$, $i \in I$ be a collection of objects, and let

$$\phi_{ij} : \text{pr}_1^* x_i \rightarrow \text{pr}_2^* x_j \quad (i, j \in I)$$

be a collection of morphisms in the fibre categories $\mathcal{S}_{U_{ij}}$. In the definition that follows we will identify, for instance, $\text{pr}_1^* x_i$ on U_{ijk} with both $\text{pr}_{12}^* \text{pr}_1^* x_i$ and $\text{pr}_{13}^* \text{pr}_1^* x_i$.

Definition 2.1.2. The collection $\{x_i, \phi_{ij}\}$ is a descent datum if the following cocycle condition is satisfied: For every triple $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} \text{pr}_1^* x_i & \xrightarrow{\text{pr}_{13}^* \phi_{ik}} & \text{pr}_3^* x_k \\ & \searrow \text{pr}_{12}^* \phi_{ij} & \nearrow \text{pr}_{23}^* \phi_{jk} \\ & \text{pr}_2^* x_j & \end{array}$$

in the fibre category $\mathcal{S}_{U_{ijk}}$ is commutative.

Remarks 2.1.3. Two remarks about this definition are in order.

(1) There is a diagonal morphism $\Delta_i : U_i \rightarrow U_{ii}$. We can pull back ϕ_{ii} via this morphism to get an automorphism $\Delta_i^* \phi_{ii} \in \text{Aut}_{U_i}(x_i)$. On pulling back the cocycle condition for the triple (i, i, i) by $\Delta_{123} : U_i \rightarrow U_{iii}$ we deduce that $\Delta_i^* \phi_{ii} \circ \Delta_i^* \phi_{ii} = \Delta_i^* \phi_{ii}$; thus $\Delta_i^* \phi_{ii} = \text{id}_{x_i}$.

(2) There is a morphism $\Delta_{13} : U_{ij} \rightarrow U_{ijj}$ and we can pull back the cocycle condition for the triple (i, j, i) to get the identity $(\sigma^* \phi_{ji}) \circ \phi_{ij} = \text{id}_{\text{pr}_1^* x_i}$, where $\sigma : U_{ij} \rightarrow U_{ji}$ is the switching morphism.

A morphism of descent data $\alpha : \{x_i, \phi_{ij}\} \rightarrow \{y_i, \psi_{ij}\}$ is given by a collection of morphisms $\alpha_i : x_i \rightarrow y_i$ in the fibre category over U_i such that the following diagrams

$$\begin{array}{ccc} \text{pr}_1^* x_i & \xrightarrow{\text{pr}_1^* \alpha_i} & \text{pr}_1^* y_i \\ \phi_{ij} \downarrow & & \downarrow \psi_{ij} \\ \text{pr}_2^* x_j & \xrightarrow{\text{pr}_2^* \alpha_j} & \text{pr}_2^* y_j \end{array}$$

commute. Note that every morphism of descent data is an isomorphism.

An object $x \in \text{Ob}(\mathcal{S}_U)$ gives rise to a descent datum in the following manner. First we set $x_i = f_i^* x$. Then we set $\phi_{ij} = t_j^{-1} \circ t_i$, where $t_i : \text{pr}_1^* f_i^* x \rightarrow (f_i \circ \text{pr}_1)^* x$ and $t_j : \text{pr}_2^* f_j^* x \rightarrow (f_j \circ \text{pr}_2)^* x$ are the canonical isomorphisms guaranteed by Categories, [Lemma 3.1.3](#). The lemma below shows this is a descent datum; we will call this the *canonical descent datum* associated to x .

Lemma 2.1.4. *The cocycle condition holds for the datum described above.*

Proof. First, note that $f_i \circ \text{pr}_1 = f_j \circ \text{pr}_2 = f_k \circ \text{pr}_3$. Then note that $\text{pr}_{13}^* \phi_{ik}$, $\text{pr}_{23}^* \phi_{jk}$ and $\text{pr}_{12}^* \phi_{ij}$ factor uniquely through $(f_i \circ \text{pr}_1)^* x = (f_j \circ \text{pr}_2)^* x = (f_k \circ \text{pr}_3)^* x$ by [Lemma 3.1.3](#). \square

Definition 2.1.5. A descent datum $\{x_i, \phi_{ij}\}$ is said to be effective when there exists an $x \in \text{Ob}(\mathcal{S}_U)$ such that $\{x_i, \phi_{ij}\}$ is isomorphic to the canonical descent datum associated to x .

At this point we are ready to give the definition of a stack.

Definition 2.1.6. A stack (in groupoids) over a site \mathcal{C} is a category $p : \mathcal{S} \rightarrow \mathcal{C}$ over \mathcal{C} such that

- (1) $p : \mathcal{S} \rightarrow \mathcal{C}$ is a category fibred in groupoids over \mathcal{C} ,
- (2) for all $U \in \text{Ob}(\mathcal{C})$ and all $x, y \in \text{Ob}(\mathcal{S}_U)$ the presheaf $\text{Isom}(x, y)$ is a sheaf on \mathcal{C}/U , and
- (3) for all coverings $\mathcal{U} = \{U_i \rightarrow U\}$ in \mathcal{C} , all descent data $\{x_i, \phi_{ij}\}$ for \mathcal{U} are effective.

Usually the hardest part to check is the third condition.

Lemma 2.1.7. *Suppose that $f : \mathcal{X} \rightarrow \mathcal{S}$ and $g : \mathcal{Y} \rightarrow \mathcal{S}$ are morphisms of stacks over \mathcal{C} . Let $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$, p , q , ψ be the explicit 2-fibre product of f and g in the 2-category of categories over \mathcal{C} described in [Lemma 3.1.7](#). Then $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a stack. In particular the 2-category of stacks over \mathcal{C} has 2-fibre products (and they are as described in [Lemma 3.1.7](#)).*

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Proof. FIXME. □

Subsection 2.2. Examples. FIXME: Here we need lots of examples.

FIXME: To be continued.

To continue reading,

- (1) visit the next section: Stacks and Groupoids, [Section 1](#), or
- (2) go back to the table of contents: [index.html#contents](#).

REFERENCES

- [DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Publ. Math. IHES*, 36:75–110, 1969.