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STACKS AND GROUPOIDS

Contents

Section 1. Introduction

Dear Johan

I promised to send you a continued explanation of this homotopical version of stacks. It turns out that I did not need to say much else — my source for this perspective on stacks is, by the way, a paper of Sharon Hollander's which you probably know about (see [Hol]).

Let Gpd be the category of groupoids. For a groupoid G we will let G_0 denote the objects and G_1 denote the morphisms. There is a notion of homotopy attached to this category where the homotopies are given by natural transformations. There is a model category structure which reflects this, and for which the weak equivalences are precisely the equivalences of categories.

The homotopy category of groupoids is equivalent to the homotopy category of 1-coconnective spaces - that is - the homotopy category of spaces with non-trivial homotopy groups in degrees 0 and 1. The functors which give this equivalence on the homotopy category are the classifying space functor, and the fundamental groupoid functor.

Let $\mathrm{PreShv}_{\mathcal{C}}(\mathrm{Gpd})$ be the category of presheaves of groupoids on $\mathcal{C}$. Let F be an object of this category. The presheaf F is a stack if it satisfies homotopy descent. That is, for every cover $\{U_i\} \to U$, the natural map

$$F(U) \to \mathrm{holim}\left(\prod F(U_i) \Rightarrow \prod F(U_{ij}) \Rrightarrow \prod F(U_{ijk})\right)$$

is an equivalence of categories. Let's examine this condition closer. Let $L(\{U_i\})$ denote the homotopy limit groupoid displayed above.

What is an object (0-simplex) of $L(\{U_i\})$? It consists of

- a collection of $x_i \in F(U_i)_0$
- a collection of $f_{ij} \in F(U_{ij})_1$.

This data is required to satisfy the cosimplicial identities up to homotopy. I'll draw simplex-shaped diagrams to make this translation clear.

- The maps f_{ij} have source x_i and target x_j.

$$x_i \xrightarrow{f_{ij}} x_j$$

- The maps f_{ij} satisfy a cocycle condition.

$$\begin{tikzcd} x_i \arrow[rr, "f_{ik}"] \arrow[dr, "f_{ij}"'] & \arrow[d, phantom, "Id"] & x_k \\ & x_j \arrow[ur, "f_{jk}"'] & \end{tikzcd}$$

That mysterious Id floating in the middle of the 2-simplex above is supposed to be the identity 2-morphism. Of course, since Gpd is a category of 1-categories, there are no non-trivial 2-morphisms, but it makes the pattern clear.

The morphisms of $L(\{U_i\})$ from an object (x_i, f_{ij}) to an object $(\overline{x}_i, \overline{f}_{ij})$ consist of a collection of $g_i \in F(U_i)$ such that the following diagram commutes

$$\begin{tikzcd} x_i|_{U_{ij}} \arrow[r, "g_i|_{U_{ij}}"] \arrow[d, "f_{ij}"'] & \overline{x_i}|_{U_{ij}} \arrow[d, "\overline{f}_{ij}"] \\ x_j|_{U_{ij}} \arrow[r, "g_j|_{U_{ij}}"'] & \overline{x}_j|_{U_{ij}} \end{tikzcd}$$

Note that the g_i restrict because of the presheaf condition. The above diagram is suppose to represent a 1-simplex in the category whose objects are morphisms are whose morphisms are commuting squares.

The functoriality of presheaves of groupoids makes this definition of a stack rather compact. However, this functoriality of pullbacks is not present in either the categories fibered in groupoids approach or the lax presheaf of groupoids approach, and these do seem to be what you might be handed in practice. Some sort of rectification must take place, and I think Sharon talks about this. I imagine you guys have your own rectification techniques.

I wrote everything so that you could try to replace the role of groupoid with that of n-groupoid, or n-connective simplicial set if you like. There are probably issues that crop up in the more general approach. (like fibrancy! Every groupoid is fibrant.)

Mark

To continue reading,

(1) visit the next section: Schemes as stacks and representability, Section 1, or
(2) go back to the table of contents: index.html#contents.

References

[Hol] Sharon Hollander. A homotopy theory for stacks.