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# FLAT DESCENT FOR QUASI-COHERENT SHEAVES

Contents



## Section 1. Introduction

In this chapter we discuss the flat topology.

**Subsection** 1.1. **Descent for quasi-coherent sheaves.** Let me state our goal now so we have a clear idea of what we are trying to achieve. It is to show that the fibered category $(\mathrm{QCoh}/S)$ (of quasi-coherent sheaves over a scheme $S$) over $\mathrm{Sch}/S$ (the category of schemes over $S$) is a stack with respect to the flat topology. The idea is that we will be able to exploit the fact that there is a standard equivalence of categories between $\mathrm{QCoh}(U)$ and $\mathrm{Mod}_A$ where $U = \mathrm{Spec}(A)$, and then use Lemma 1.1.1.

**Lemma 1.1.1.** *([Vis],Lemma 4.25) Let $S$ be a scheme, $\mathcal{F}$ be a fibered category over the category $(Sch/S)$. Suppose the following conditions are satisfied.*

1. *$\mathcal{F}$ is a stack with respect to the Zariski topology.*
2. *Whenever $V \to U$ is a flat surjective morphism of affine $S$-schemes, the functor $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is an equivalence of categories.*

*Then $\mathcal{F}$ is a stack with respect to the flat (fpqc) topology.*

*Proof.* FIXME: Add proof. □

Note that the fpqc topology stands for "fidelment plat et quasi-compact" which means faithfully flat and quasi-compact. This is a finer topology than the fppf topology which is a finer topology than the etale topology.

In the case of $(\mathrm{QCoh}/S)$ over $(\mathrm{Sch}/S)$ we can easily see that the first condition is satisfied (i.e in the Zariski topology our definition for quasi-coherent sheaves is exactly the conditions needed for the descent datum to be effective).

For the second condition, it will be necessary to define certain notions and to prove certain algebraic results. We begin with some definitions.

**Definition 1.1.2.** A morphism of schemes $f : X \to Y$ is faithfully flat if it is flat and surjective. Let $B$ be an algebra over $A$, we say that $B$ is faithfully flat if the associated morphism of schemes $\mathrm{Spec}B \to \mathrm{Spec}A$ is.

**Proposition 1.1.3.** *Let $B$ be an algebra over $A$, the following are equivalent.*

1. *$B$ is faithfully flat over $A$.*

(2) *A sequence of $A$-modules $M' \to M \to M''$ is exact if and only if the induced sequence of $B$-modules $M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B$ is exact.*
(3) *A homomorphism of $A$ modules $M' \to M$ is injective if and only if the associated homomorphism of $B$-modules $M' \otimes_A B \to M \otimes_A B$ is injective.*
(4) *$B$ is flat over $A$, and if $M$ is a module over $A$ with $M \otimes_A B = 0$, we have $M = 0$.*
(5) *$B$ is flat over $A$, and $mB \neq B$ for all maximal ideals $m$ of $A$.*

**Definition 1.1.4.** Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$. Let $\mathcal{U} = \{\sigma_i : U_i \to U\}$ be a covering in $\mathcal{C}$. An object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$ is a collection of objects $\xi_i \in \mathcal{F}(U_i)$, together with isomorphisms $\phi_{ij} : pr_2^*\xi_j \simeq pr_1^*\xi_i$ in $\mathcal{F}(U_i \times_U U_j)$, such that the following cocyle condition is satisfied: for any triple of indicies $i, j, k$ we have the equality $pr_{13}^*\phi_{ik} = pr_{12}^*\phi_{ij} \circ pr_{23}^*\phi_{jk} : pr_3^*\xi_k \to pr_1^*\xi_i$. Where $pr_{ab}$ and $pr_a$ projections onto the $a^{th}$ and $b^{th}$ factor, or the the $a^{th}$ factor respectively.

An arrow between objects with descent data $\{\alpha_i\} : (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$ is a collection of arrows $\alpha_i : \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$ with the property that for each pair of indicies $i, j$ the cocycle condition is satisfied.

There is an obvious way of composing morphisms, which makes the objects with descent data the objects of a category which we will denote $\mathcal{F}(\{U_i \to U\})$

Specifically in the case of the lemma we are only interested in the case where $U = \mathrm{Spec}(A)$ is affine. So it can be covered by one affine scheme $V = \mathrm{Spec}(B)$. In this case the above definition reduces to the following: $\mathcal{F}(V \to U)$ is the category consisting of pairs $(\xi, \phi)$ where $\xi \in \mathcal{F}(V)$ and $\phi : p_1^*\xi \simeq p_2^*\xi$ is an isomorphism which satisfies the cocycle condition.

To continue we need to set some conventions. Let $A$ be a commutative ring, and denote $\mathrm{Mod}_A$ as the category of modules over $A$. We also have a ring homomorphism $f : A \to B$. We define a category $\mathrm{Mod}_{A \to B}$ as follows. Let the objects be pairs $(N, \phi)$ where $N$ is a $B$-module and $\phi : N \otimes_A B \simeq B \otimes_A N$ is an isomorphism of $B^{\otimes p2}$-modules such that the following cocycle condition is satisfied:

$$\begin{aligned}
\phi_1 : B \otimes_A N \otimes_A B &\to B \otimes_A B \otimes_A N, \\
\phi_2 : N \otimes_A B \otimes_A B &\to B \otimes_A B \otimes_A B, \\
\phi_3 : N \otimes_A B \otimes_A B &\to B \otimes_A N \otimes_A B,
\end{aligned}$$

where $\phi_1 = \mathrm{id}_B \otimes \phi$, $\phi_3 = \phi \otimes \mathrm{id}_B$, and $\phi_2 = \phi_1\phi_3$.

A morphism $\beta : (N, \phi) \to (N', \phi')$ is a homomorphism of $B$-modules $\beta : N \to N'$ making the following diagram commute:

$$\begin{array}{ccc}
N \otimes_A B & \xrightarrow{\phi} & B \otimes_A N \\
\beta \otimes \mathrm{id}_B \downarrow & & \downarrow \mathrm{id}_B \otimes \beta \\
N' \otimes_A B & \xrightarrow{\phi'} & B \otimes_A N'
\end{array}$$

Given a functor $F : \mathrm{Mod}_A \to \mathrm{Mod}_{A \to B}$ which takes an $A$-module $M$ to the pair $(B \otimes_A M, \phi_M)$ where $\phi_M : (B \otimes_A M) \otimes_A B \to B \otimes_A (B \otimes_A M)$ maps $b \otimes m \otimes b'$ to $b \otimes b' \otimes m$ (and satisfies the cocyle condition).

**Theorem 1.1.5.** *If $B$ is faithfully flat over $A$, the functor $F : Mod_A \to Mod_{A \to B}$ as defined above is an equivalence of categories.*

To prove this we first need the following lemma

**Lemma 1.1.6.** *Let $M$ be an $A$-module. Then the following sequence*

$$0 \to M \xrightarrow{\alpha_M} B \otimes_A M \overset{(e_1-e_2)\otimes id_M}{\longrightarrow} B^{\otimes 2} \otimes_A M \overset{(e_1-e_2+e_3)\otimes id_M}{\longrightarrow} B^{\otimes 3} \otimes_A M \to \cdots$$

*is exact. Where $e_i : B^{\otimes n} \to B^{\otimes n+1}$ is the map that puts a 1 into the $i^{th}$ place of the tensor product.*

*Proof.* It is easy to see that $\alpha_M$ is an injective map, and that images of all of the maps will be contained in the appropriate kernels. So what we need to show is that we have reverse containment (i.e. kernels contained in images) at each step of the sequence. If there existed a section $B \otimes_A M \to M$ then this will be easy (for reasons we will see soon). However, there does not always exist such a map.

Luckily we can use the fact that $B$ is flat over $A$, and that a sequence of $A$-modules is exact if and only if the sequence tensored with $B$ is exact. This is fortunate because once we tensor our sequence with $B$ we get the following sequence (with the same maps as in the statement of the lemma just with tensored with a $\mathrm{id}_B$ on the left):

$$0 \to B \otimes_A M \to B^{\otimes 2} \otimes_A M \to B^{\otimes 3} \otimes_A M \to B^{\otimes 4} \otimes_A M \to \cdots .$$

And there is a natural map $B^{\otimes 2} \otimes_A M \overset{mult}{\to} B \otimes_A M$ which just takes $b \otimes b' \otimes m$ to the element $bb' \otimes m$. In other words it is just the multiplication map $B \otimes_A B \to B$ composed with the identity on $M$.

Now, to prove that the sequence is exact at $B^{\otimes 2} \otimes_A M$, we pick an element $\Sigma b_i \otimes b_i' \otimes m_i$ in the kernel of the map $\mathrm{id}_B \circ (e_1 - e_2) \circ \mathrm{id}_M$. This means that in $B^{\otimes_3} \otimes_A M$ we have the following relation: $\Sigma b_i \otimes b_i' \otimes 1 \otimes m_i = \Sigma b_i \otimes 1 \otimes b_i' \otimes m_i$. We then apply the map $mult \circ \mathrm{id}_B \circ \mathrm{id}_M$ to the equality and get the $\Sigma b_i b_i' \otimes 1 \otimes m_i = b_i \otimes b_i' \otimes m_i$ in $B^{\otimes 2} \otimes_A M$. So we are getting $\mathrm{id}_B \circ \alpha_M(\Sigma b_i b_i' \otimes m_i) = \Sigma b_i \otimes b_i' \otimes m_i$ which was the element from the kernel that we chose. Thus every element in the kernel is also in the image of the appropriate map hence the sequence is exact there. The same argument can be made at each step of the sequence to show that the appropriate kernels are contained in the appropriate images. A choice for sections that work is simply multiplication composed with whatever number of identity maps are necessary. □

*Proof of the theorem.* So to prove the theorem we need to first consider the functor $F : \mathrm{Mod}_A \to \mathrm{Mod}_{A\to B}$ which takes an $A$-module $M$ to the pair $(B \otimes_A M, \phi_M)$ where $\phi_M : (B \otimes_A M) \otimes_A B \to B \otimes_A (B \otimes_A M)$ maps $b \otimes m \otimes b'$ to $b \otimes b' \otimes m$.

To show that $F$ is an equivalence of categories, we need to show that there is a functor $G : \mathrm{Mod}_{A\to B} \to \mathrm{Mod}_A$ such that $GF$ and $FG$ are isomorphic to the identity.

So let us define a functor $G$ to take pairs $(N, \phi)$ to elements $GN = \{n \in N | 1 \otimes n = \phi(n \otimes 1)\}$ and given a morphism $\beta : (N, \phi) \to (N', \phi')$ in $\mathrm{Mod}_{A\to B}$ we get a morphism $\beta_G : GN \to GN'$.

So first let us check that $GF$ is isomorphic to the identity. Notice that

$$((e_1 - e_2) \otimes \mathrm{id}_M)(b \otimes m) = b \otimes 1 \otimes m - 1 \otimes b \otimes m = \phi_M(b \otimes m \otimes 1) - 1 \otimes b \otimes m$$

for all $m$ and $b$. For simplicity we can rewrite this as $((e_1 - e_2) \otimes \mathrm{id}_M)(x) = \phi_M(x) - 1 \otimes x$ for all $x \in B \otimes_A M$. So then by our definition of the functor $G$ we get that $G(B \otimes_A M, \phi_M) = \ker((e_1 - e_2) \otimes \mathrm{id}_M)$. However, due to our lemma, we know that the sequence

$$0 \to M \to^{\alpha_M} B \otimes_A M \to B^{\otimes 2} \otimes_A M \to B^{\otimes 3} \otimes_A M \to \cdots$$

is exact, thus $\ker((e_1 - e_2) \otimes \mathrm{id}_M) = \mathrm{im}(\alpha_M) \simeq M$. So $M \simeq G(B \otimes_A M) = GF(M)$ as needed.

Now we will show that $FG$ is isomorphic to the identity. So we take $(N, \phi)$ in $\mathrm{Mod}_{A \to B}$ and we set $M = G(N, \phi) = \{n \in N | 1 \otimes n = \phi(n \otimes 1)\}$. Since $M$ is an $A$-submodule of the $B$-submodule $N$ we get a homomorphism of $B$-modules $\theta : B \otimes_A M \to N$ which takes $b \otimes m$ to $bm$. It is easy to check that this is a morphism in $\mathrm{Mod}_{A \to B}$. So notice that we can also think of $\theta$ as a map $F(M) \to N$, thus we can see that $\theta$ defines a natural transformation $id \to FG$. So to complete the proof we need to show that $\theta$ is an isomorphism.

First we will need to define some maps $i$ will just be inclusion, $\iota_M : M \otimes B \to B \otimes M$ is the map taking $m \otimes b$ to $b \otimes m$, and $\alpha, \beta : N \to B \otimes_A M$ are defined by $\alpha(n) = 1 \otimes n$ and $\beta(n) = \phi(n \otimes 1)$. So by definition, $M = \ker(\alpha - \beta)$. We have the following diagram, where the rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_A B & \stackrel{i \otimes \mathrm{id}_B}{\longrightarrow} & N \otimes_A B & \stackrel{(\alpha - \beta) \otimes \mathrm{id}_B}{\longrightarrow} & B \otimes_A N \otimes_A B \\ & & \downarrow \theta \circ \iota_M & & \downarrow \phi & & \downarrow \phi_1 \\ 0 & \longrightarrow & N & \stackrel{\alpha_N}{\longrightarrow} & B \otimes_A N & \stackrel{(e_1 - e_2) \otimes \mathrm{id}_N}{\longrightarrow} & B \otimes_A B \otimes_A N \end{array}$$

So by showing that the diagram commutes, using the fact that both $\phi$ and $\phi_1$ are isomorphisms we are able to get that $\theta$ is an isomorphism. To show the diagram commutes let us focus on one square at a time. For the first square we want $\phi(i \otimes \mathrm{id}_B)(m \otimes b) = \alpha_M \theta \iota_M (m \otimes b)$. We know that $\alpha_M \theta \iota_M (m \otimes b) = 1 \otimes bm$. So we just need to show that we get the same thing for $(\phi(i \otimes \mathrm{id}_B))(m \otimes b)$. We have

$$\begin{aligned} (\phi(i \otimes \mathrm{id}_B))(b \otimes m) &= \phi(m \otimes b) \\ &= \phi((1 \otimes b)(m \otimes 1)) \\ &= (1 \otimes b)\phi(m \otimes 1) \\ &= (1 \otimes b)(1 \otimes m) \\ &= 1 \otimes bm \end{aligned}$$

as needed. Now we just need to show the second square commutes. For the second square it should be clear that $\phi_1(\alpha \otimes \mathrm{id}_B) = (e_2 \otimes \mathrm{id}_N) \circ \phi$. So we just need to check that $\phi_1(\beta \otimes \mathrm{id}_B) = (e_1 \otimes \mathrm{id}_N) \circ \phi$. We have:

$$\begin{aligned} \phi_1(\beta \otimes \mathrm{id}_B)(n \otimes b) &= \phi_1(\phi(n \otimes 1) \otimes b) \\ &= \phi_1 \phi_3 (n \otimes 1 \otimes b) \\ &= \phi_2(n \otimes 1 \otimes b) \\ &= (e_1 \otimes \mathrm{id}_N)\phi(n \otimes b) \end{aligned}$$

So, by the argument above $\theta$ is an isomorphism and thus $FG(N, \phi) \simeq (N, \phi)$. □

And now we can restate and sketch the proof of our desired result

**Theorem 1.1.7.** *Let $S$ be a scheme. The fibered category $(QCoh/S)$ over $(Sch/S)$ is stack with respect to the flat (fpqc) topology.*

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*Proof.* FIXME: Sketch of proof.

Let me remind you that we just need to check the second condition of the lemma. So for a flat and surjective morphism $V \to U$ (corresponding to a faithfully flat ring homomorphism $f : A \to B$). We need to show that there is an equivalence of categories between $\mathrm{QCoh}(U)$ and $\mathrm{QCoh}(V \to U)$. We will do this using the previous theorem which states that there is an equivalence of categories between $\mathrm{Mod}_A$ and $\mathrm{Mod}_{A\to B}$.

There is a standard equivalence of categories between $\mathrm{QCoh}(U)$ and $\mathrm{Mod}_A$. So we just need to show that there is an equivalence of categories between $\mathrm{QCoh}(V \to U)$ and $\mathrm{Mod}_{A\to B}$. To do this let us look at $\mathcal{N}$ an object in $\mathrm{QCoh}(V)$ which corresponds to an $B$-module $N$. Looking at $p_1^*\mathcal{N}$ and $p_2^*\mathcal{N}$ in $V \times_U V = \mathrm{Spec}(B \otimes_A B)$ we get $N \otimes_A B$ and $B \otimes_A N$ respectively. So the descent datum $\psi : p_1^*\mathcal{N} \simeq p_2^*\mathcal{N}$ will correspond to the descent data $\phi : N \otimes_A B \simeq B \otimes_A N$ in $\mathrm{Mod}_{A\to B}$. So $(\mathcal{N}, \psi)$ is an object of $\mathrm{QCoh}(V \to U)$ if and only if $\phi$ satisfies the cocycle condition, thus giving us an equivalence of categories between $\mathrm{QCoh}(V \to U)$ and $\mathrm{Mod}_{A\to B}$. Thus the functor $\mathrm{QCoh}(U) \to \mathrm{QCoh}(V \to U)$ corresponds to the functor $\mathrm{Mod}_A \to \mathrm{Mod}_{A\to B}$! So since, the later is an equivalence we get the equivalence of categories that we need. Thus finishing the proof. □

FIXME: put in example with descent of projective schemes and put in Galois descent example.

To continue reading,

(1) visit the next section: The étale topology on schemes, Section 1, or
(2) go back to the table of contents: `index.html#contents`.


References

[Vis] Angelo Vistoli. Notes on grothendieck topologies, fibered categories and descent theory.