ON THE STRUCTURE OF INSTABILITY IN MODULI THEORY

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Abstract. We formulate a theory of instability for points in an algebraic stack which generalizes geometric invariant theory as well as the notion of instability in several commonly studied moduli problems. We introduce the notion of a Θ-stratification of the unstable locus, which generalizes the Kempf-Ness stratification in GIT as well as the Harder-Narasimhan filtration for objects in derived categories of coherent sheaves. The analysis of instability leads to a combinatorial object associated to a point in an algebraic stack which generalizes the fan of a toric variety and the spherical building of a semisimple group.

For a certain class of algebraic stacks, which we call (weakly) reductive, the existence of Θ-stratifications can be reduced to solving a relatively straightforward "boundedness" problem. We observe that (weakly) reductive stacks arise in nature. For instance, the moduli of families of objects in the heart of a t-structure in the derived category of coherent sheaves is weakly reductive.

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Let us recall a familiar situation in algebraic geometry: One wishes to construct a quasiprojective scheme which parameterizes the isomorphism classes of a particular algebro-geometric object. Typically one has a moduli functor which is an algebraic stack, $\mathfrak{X}$, but fails to be representable by a quasiprojective scheme in a number of ways (not quasi-compact, non-separable,...), so one passes to an open substack of "semistable objects" suitably defined,$\mathfrak{X}^{ss}$, in the hope that the new moduli problem is representable (or at least has a coarse moduli space or a good moduli space).

There are fewer instances in which the unstable locus, $\mathfrak{X}^{us} = \mathfrak{X} \backslash \mathfrak{X}^{ss}$, is understood in detail. In geometric invariant theory, where $\mathfrak{X}$ is a global quotient stack, the unstable locus has a canonical stratification by disjoint locally closed substacks, $\mathfrak{X}^{us} = \mathfrak{S}_0 \cup \cdots \cup \mathfrak{S}_N$, first studied by Kempf [K1], Ness [NM], and Hesselink [H2]. Another example arises in the theory of vector bundles on a curve, and the huge list of variations on that theme (coherent sheaves on projective varieties, moduli of principle G-bundles, moduli of Higgs bundles, moduli of objects which are semistable with respect to a Bridgeland stability condition, etc.). In these examples, every unstable object has a canonical Harder-Narasimhan filtration, and the unstable locus in the moduli stack admits a stratification $\mathfrak{X}^{us} = \bigcup \mathfrak{S}_\alpha$, where $\mathfrak{S}_\alpha$ is the locally closed substack of bundles whose Harder-Narasimhan filtration has certain numerical invariants (rank and degree) recorded by the index α [S].

The lesson from these examples is that instability is a structured phenomenon. We shall introduce a notion which generalizes these stratifications, which we refer to as a Θ-stratification. In examples where these stratifications have been discovered, they have found many uses. For example, when the notion of semistability depends on a choice of parameters, the unstable stratification controls the process by which $\mathfrak{X}^{ss}$ varies as one varies these parameters. In addition, the unstable stratification can be useful in understanding the topology of $\mathfrak{X}^{ss}$ itself [K2]. Recently, these stratifications have also proved useful in understanding the relationship between derived categories $D^b(\mathfrak{X})$ and $D^b(\mathfrak{X}^{ss})$, and determining how $D^b(\mathfrak{X}^{ss})$ varies as one varies the notion of stability [HL2, BFK].

In pursuit of a general theory of instability, we revisit the problem of constructing the stratification of the unstable locus in the examples above from an intrinsic perspective. We have three primary goals:

(1) to provide a common modular interpretation for these stratifications,
(2) to develop a combinatorial structure underlying the notion of instability, which unites the theory of toric varieties with the theory of spherical buildings of semisimple groups, and
(3) to identify a type of moduli problem for which Θ-stratifications naturally arise, providing groundwork for the discovery of new examples of Θ-stratifications.

As a demonstration, we use our methods to analyze the moduli stack of objects in the heart of a t-structure on the derived category of coherent sheaves on a projective variety. This recovers known stratifications of the moduli of flat families of coherent sheaves [S, HK, N1], but it also establishes new examples of Θ-stratifications arising from Bridgeland stability conditions. The latter provides examples which are not readily approximated by GIT problems.

This paper results from a collision of two very different mathematical stories. In several active areas of research, algebraic geometers are attempting to construct moduli spaces in a GIT-like way, as outlined in our opening paragraph, without appealing to a global quotient description. Examples include the various notions of stability on the moduli of (polarized) projective varieties. In some cases, the stability condition identifying $\mathfrak{X}^{ss}$ depends on a parameter, and the questions of interest involve how the geometry of $\mathfrak{X}^{ss}$ changes as the parameter varies. One example of this kind is the theory of stable quasi-maps to a GIT quotient, which is expected to explain the Landau-Ginzburg/Calabi-Yau correspondence relating the Gromov-Witten invariants of a CY complete intersection in a GIT quotient to enumerative invariants associated to a Landau-Ginzburg model [R]. Another example is in the log minimal model program for the moduli of curves [FS], where one hopes that different birational models of $\overline{M}_{g,n}$ can be realized as good moduli spaces for open substacks of the moduli stack of all curves (possibly subject to a mild restriction on singularity type). What is missing in all of these examples is a good understanding of the structure of the unstable locus, and we hope that our intrinsic formulation of the techniques of GIT will contribute to that understanding.

On the other hand, our original motivation came from considerations of equivariant derived categories. For a smooth (or even mildly singular) quotient stack $\mathfrak{X} = X/G$ with a stratification of the kind we are discussing, [HL2] establishes a structure theorem for the derived category of coherent sheaves $D^b(\mathfrak{X})$. The theorem amounts to a semiorthogonal decomposition of $D^b(\mathfrak{X})$ with one factor identified with $D^b(\mathfrak{X}^{ss})$ by the restriction functor, and the other factors corresponding to the various strata. In fact such a decomposition of $D^-(\mathfrak{X})$ appears to hold for any locally finite type k-stack equipped with a Θ-stratification, without restrictions on the singularities of $\mathfrak{X}$, but our modular interpretation of the stratification is essential to establishing this more general result. As the investigation of this modular interpretation unfolded, it became clear that the structure underlying the construction of these stratifications justified an independent treatment. Thus we have postponed the application to derived categories for a follow-up paper, [HL3] (a preliminary version in a more restrictive level of generality is currently available [HL1]).

0.0.1. *The modular interpretation.* We consider the stack $\Theta := \mathbb{A}^1/\mathbb{G}_m$ and the mapping stack $\underline{\mathrm{Map}}(\Theta, \mathfrak{X})$, which we denote $[\Theta, \mathfrak{X}]$. One can restrict any map $\Theta \to \mathfrak{X}$ to the generic point $\{1\} \in \mathbb{A}^1$, and extending this to families defines a canonical morphism $ev_1 : [\Theta, \mathfrak{X}] \to \mathfrak{X}$. Our modular interpretation of the stratification of $\mathfrak{X}^{us}$ is as an open substack $\mathfrak{S} \subset [\Theta, \mathfrak{X}]$ such that $ev_1 : \mathfrak{S} \to \mathfrak{X}$ is a locally-closed immersion, satisfying some additional hypotheses. In the examples above, the strata would correspond to connected components of this $\mathfrak{S}$. In Definition 3.4 we formulate a precise notion of the properties which $\mathfrak{S}$ must satisfy in order to be considered a Θ-stratification. Basically, the connected components of $\mathfrak{S}$ must be totally ordered in such a way that the union of the strata above a fixed level is closed, and the remaining strata must define a Θ-stratification of the complementary open substack.

A large fraction of this paper is devoted to unpacking this simple idea. First of all, it is not obvious that $[\Theta, \mathfrak{X}]$, which is a stack defined by its functor of points, is algebraic. The algebraicity of $[\Theta, \mathfrak{X}]$ follows from the general results of [HP]: the stack Θ is cohomologically projective, and thus the mapping stack to any quasi-geometric stack locally of finite presentation is algebraic.

We are interested in a more concrete understanding of $[\Theta, \mathfrak{X}]$, however, so we explicitly compute $[\Theta, \mathfrak{X}]$ for a global quotient stack $\mathfrak{X} = X/G$, and provide an alternative proof that it is algebraic.[1] In this case $[\Theta, \mathfrak{X}]$ is an infinite disjoint union of quotient stacks, and its set of connected components is naturally partitioned into subsets corresponding to each conjugacy class of one parameter subgroup of G. This

[1]In fact, the results of [HP] were motivated by early versions of the results in subsection 1.3 of this paper.

illustrates a general phenomenon, whereby a map $\Theta \to \mathfrak{X}$ admits several discrete invariants which divide $[\Theta, \mathfrak{X}]$ into many more connected components than $\mathfrak{X}$ itself.

Global quotient stacks serve as toy-models for studying algebraic stacks which are described more implicitly via moduli problems. In general, if $\mathfrak{X}(T)$ is described as the groupoid of "families of objects" over a test scheme, T, then points of $[\Theta, \mathfrak{X}]$ correspond to $\mathbb{G}_m$-equivariant families over $\mathbb{A}^1$. In Section 4, we shall consider two modular examples of our general framework: the moduli of objects in the heart of a t-structure on $D^b(X)$ for some projective variety X, which includes both the classical theory of slope stability of coherent sheaves [HL5] as well as more recent examples arising in the study of Bridgeland stability conditions; and the moduli of polarized varieties, where points of $[\Theta, \mathfrak{X}]$ correspond to "test configurations" as studied in the theory of K-stability [D$^+$, DT].

Remark 0.1. As mentioned above, [HL2] presents a structure theorem for the derived category for smooth and some singular quotient stacks. The singular version is much more subtle than the smooth version. It requires the stratification to satisfy two technical hypotheses which are non-vacuous. Basically, the relative cotangent complex of each stratum must have positive weights with respect to a certain one-parameter subgroup, but one can even find examples of an affine hypersurface modulo a $\mathbb{G}_m$-action for which the single unstable stratum fails to satisfy this hypothesis. However, if we regard $\mathfrak{X}$ as a derived stack, then the stratum $\mathfrak{S}$ obtains a non-trivial derived structure from its modular interpretation as a mapping stack, and with this derived structure the relative cotangent complex for the inclusion of the stratum automatically has positive weights. Thus the modular interpretation of the stratification is crucial to the second half of this project, to appear in [HL3].

0.0.2. *Instability conditions on algebraic stacks.* The modular interpretation allows us to identify the data required to specify a notion of "instability" of points in an arbitrary stack, $\mathfrak{X}$. This raises the question of the existence of Θ-stratifications in many new contexts. Our main contribution to the pursuit of Θ-stratifications is to identify a class of stacks, which we call "reductive," for which the construction of a Θ-stratification for a given notion of instability can be reduced to verifying a single property, which we call Principle (B+) below. Furthermore, reductive stacks seem to arise in nature, as we will see in Section 4.

Recall the definition of the Kempf-Ness stratification in geometric invariant theory: for a point $p \in X$ one associates to every nontrivial one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists a (normalized) numerical invariant $\mu(x, \lambda)$ [DH]. Provided x is unstable, the stratum of $\mathfrak{X}^{us}$ on which x lies is determined by the one parameter subgroup, λ, which maximizes $\mu(x, \lambda)$ and the connected component of X^λ in which $\lim_{t\to 0} \lambda(t) \cdot x$ lies. We can interpret the pair (x, λ) as a morphism $f : \Theta \to \mathfrak{X}$ (See Theorem 1.21), and so the stratum on which x lies is determined by the f which maximizes μ subject to the existence of an isomorphism $f(1) \simeq x$.

In order to generalize the construction of the Θ-stratification in GIT, we observe a general method of defining numerical invariants. The mapping stack comes equipped with a canonical evaluation morphism $ev : \Theta \times [\Theta, \mathfrak{X}] \to \mathfrak{X}$. Using the Kunneth decomposition we have $H^*(\Theta \times [\Theta, \mathfrak{X}]) \simeq H^*([\Theta, \mathfrak{X}])[[q]]$ with q in cohomological degree 2, and one can use this to transfer cohomology classes in $H^{even}(\mathfrak{X})$, pulled back via ev^*, to cohomology classes in $H^0([\Theta, \mathfrak{X}])$. Thus cohomology classes on $\mathfrak{X}$ lead to locally constant functions on $[\Theta, \mathfrak{X}]$.

Using this technique, we will define a numerical invariant in Construction 2.19 analogous to the Hilbert-Mumford numerical invariant, which depends on a pair of cohomology classes $l \in H^2(\mathfrak{X})$ and $b \in H^4(\mathfrak{X})$. Roughly speaking it will be a function $\mu : \pi_0 [\Theta, \mathfrak{X}] \to \mathbb{R}$, where π_0 denotes the set of connected components of the algebraic stack. The class in H^2 plays the role of (the first Chern class of) the G-linearized ample bundle in GIT, whereas the class in H^4 plays the role of an invariant positive definite inner product on the Lie algebra of the compact form of G. We can then define the unstable strata $\mathfrak{S} \subset [\Theta, \mathfrak{X}]$ to consist of points for which $\mu > 0$ and which maximize μ in a fiber $ev_1^{-1}(p)$ for some $p \in |\mathfrak{X}|$.[2]

Although any pair of classes in $H^2(\mathfrak{X})$ and $H^4(\mathfrak{X})$ leads to a numerical invariant μ, the putative definition of $\mathfrak{S}$ above does not always define a Θ-stratification. Among other things, the locus of points defining $\mathfrak{S}$ must be open, so that it canonically has the structure of an algebraic substack of $[\Theta, \mathfrak{X}]$. Furthermore, the map ev_1 must be a locally closed immersion when restricted to $\mathfrak{S}$. Also, if $\mathfrak{S}$ is to define a stratification of $\mathfrak{X}^{us}$ via $ev_1 : \mathfrak{S} \to \mathfrak{X}$, then every unstable point $p \in |\mathfrak{X}|$, which by definition is one such that the fiber $ev_1^{-1}(p)$ contains a point with $\mu > 0$, must have a unique (up to the action of $\mathbf{N}^\times$) point in the fiber $ev_1^{-1}(p)$ which maximizes μ.

0.0.3. *The degeneration space.* The existence and uniqueness of a maximizer of μ in the fiber $ev_1^{-1}(p)$ is a crucial pre-requisite for the construction of Θ-stratifications, and in Section 2 we establish a general framework for addressing this question. For any $p \in \mathfrak{X}(k)$, we will construct a large topological space $\mathcal{D}(\mathfrak{X}, p)$, which we refer to as the degeneration space of p, such that the k-points in $ev_1^{-1}(p)$ (up to the action of $\mathbf{N}^\times$) correspond to a dense set of "rational points" in $\mathcal{D}(\mathfrak{X}, p)$. Furthermore, a numerical invariant μ constructed from classes in $H^2(\mathfrak{X})$ and $H^4(\mathfrak{X})$ extends to a continuous function on an open subset $U \subset \mathcal{D}(\mathfrak{X}, p)$. Whereas finding the maximum of μ on the infinite set of connected components of $ev_1^{-1}(p)$ is a priori an intractable problem, the additional structure of the space $\mathcal{D}(\mathfrak{X}, p)$ provides a natural framework for this optimization.

Example 0.2. For the stack $\mathfrak{X} = \mathbb{A}^2/\mathbb{G}_m^2$ and the point $p = (1, 1)$, any pair of nonnegative integers, (a, b), defines a group homomorphism $\mathbb{G}_m \to \mathbb{G}_m^2$ which extends to a map of quotient stacks $f : \Theta \to \mathbb{A}^2/\mathbb{G}_m^2$ along with an isomorphism $f(1) \simeq (1, 1)$, and the morphism f corresponding to the pair (na, nb) is the pre-composition with the n-fold ramified cover $\Theta \to \Theta$. Thus the classes (up to ramified coverings) of maps $f : \Theta \to \mathbb{A}^2/\mathbb{G}_m$ with $f(1) \simeq (1, 1)$ identify with the set of rational rays in the cone $(\mathbb{R}_{\geq 0})^2$. We identify this, in turn, with the set of rational points in the unit interval, and we let $\mathcal{D}(\mathfrak{X}, p)$ be this interval. Thus the rational points in the unit interval parameterize a "family" of maps $\Theta \to \mathbb{A}^2/\mathbb{G}_m^2$ up to ramified coverings, even though each map $\Theta \to \mathbb{A}^2/\mathbb{G}_m^2$ lies on a different connected component of the algebraic stack $\left[\Theta, \mathbb{A}^2/\mathbb{G}_m\right]$.

The construction of $\mathcal{D}(\mathfrak{X}, p)$ is a generalization of this example. Roughly, to a map $f : \Theta^n \to \mathfrak{X}$ along with an isomorphism $f(1, \ldots, 1) \simeq p$, we consider a

[2] We are ignoring some subtleties which will be dealt with in Section 3, and specifically Construction 3.10. μ will only be defined on some connected components of $[\Theta, \mathfrak{X}]$. Also the monoid $\mathbf{N}^\times$ acts on $\pi_0 [\Theta, \mathfrak{X}]$ by pre-composition with the n-fold ramified covering maps $\Theta \to \Theta$, and μ is invariant under this action, so the point in the fiber $ev_1^{-1}(p)$ which maximizes μ can only be unique up to this action of $\mathbf{N}^\times$. Rather than using the $\mathfrak{S}$ as defined here, we must further select a set of connected components of $\mathfrak{S}$ which form a set of representatives for the orbits of the $\mathbf{N}^\times$-action.

copy of the standard $n-1$-simplex Δ_f^{n-1}. We then glue two simplices Δ_f^k and Δ_g^n whenever f is obtained from g by restriction along a map $\Theta^k \to \Theta^n$. More precisely, we introduce a combinatorial object, which we call a formal fan, which is analogous to a simplicial set. One can associate two topological spaces to any formal fan: a geometric realization, $|\bullet|$, and a "projective realization," $\mathbb{P}(\bullet)$. We will see that for a stack $\mathfrak{X}$ with a point $p \in \mathfrak{X}(k)$, we can assemble the set of maps $f : \Theta^n \to \mathfrak{X}$ with isomorphisms $f(1,\ldots,1) \simeq p$ into a formal fan $\mathbf{D}(\mathfrak{X},p)_\bullet$, and $\mathcal{D}(\mathfrak{X},p) = \mathbb{P}(\mathbf{D}(\mathfrak{X},p)_\bullet)$.

The concept of the formal fan $\mathbf{D}(\mathfrak{X},p)_\bullet$ is of independent mathematical interest, because it connects toric geometry with some notions in representation theory. When $\mathfrak{X} = X/T$ is a normal toric variety and $p \in X$ is a point in the open torus orbit, then $\mathbf{D}(\mathfrak{X},p)_\bullet$ encodes the data of the fan of the toric variety. The geometric realization $|\mathbf{D}(\mathfrak{X},p)_\bullet|$ is canonically homeomorphic to the support of the fan of X, and $\mathcal{D}(\mathfrak{X},p)$ is canonically homeomorphic to the polyhedron obtained by intersecting the support of the fan of X with the unit sphere (Proposition 2.13). On the other hand, when $\mathfrak{X} = BG$ for a split semisimple group G, and p is the unique point, then $\mathcal{D}(\mathfrak{X},p)$ is homeomorphic to the spherical building of G (Proposition 2.15). This suggests many questions for further inquiry as to what geometric properties of a normal variety X with a single dense open orbit under the action of a reductive group G are encoded in the degeneration fan $\mathbf{D}(\mathfrak{X},p)_\bullet$.

Using this framework, we revisit Kempf's classical proof of the existence and uniqueness of optimally destabilizing one parameter subgroups for an unstable point in X/G. The existence of a maximum for the numerical invariant μ on $\mathcal{D}(\mathfrak{X},p)$ can be inferred from the fact that a continuous function on a compact space achieves a maximum (which we articulate in Principle (B)). In the case of a global quotient stack, the space $\mathcal{D}(\mathfrak{X},p)$ is actually covered by finitely many simplices, hence Principal (B) holds automatically (Proposition 2.25). For stacks which are not global quotients, such as the non finite-type stacks appearing in many moduli problems, Principle (B) is not automatic. Provided μ obtains a maximum on $\mathcal{D}(\mathfrak{X},p)$, the uniqueness of the maximizer can be deduced from the property that any two points can be connected by a path along which μ is convex upward and non-constant (which we articulate in Principle (C)).

The morphism ev_1 is actually relatively representable by locally finitely presented algebraic spaces. We say that $\mathfrak{X}$ is weakly reductive if ev_1 satisfies the valuative criterion for properness, and we say that it is reductive if it is proper on connected components (See Definition 2.27). When $\mathfrak{X}$ is weakly reductive, then any two points on $\mathcal{D}(\mathfrak{X},p)$ are connected by a unique 1-simplex (Proposition 2.29), hence $\mathcal{D}(\mathfrak{X},p)$ is convex in a suitable sense. As a consequence, any numerical invariant constructed via classes in $H^2(\mathfrak{X})$ and $H^4(\mathfrak{X})$ satisfies Principle (C), and there can be at most one maximizer for μ on $\mathcal{D}(\mathfrak{X},p)$. This re-proves Kempf's classic result in a conceptually clean way (and obtain a slight generalization) for X/G where X is affine (Corollary 2.31). For projective-over-affine X, the stack X/G is not weakly reductive, but we verify Principle (C) for certain classes in $H^2(\mathfrak{X})$ and $H^4(\mathfrak{X})$ in Proposition 2.33.

0.0.4. *Construction of Θ-stratifications and modular examples.* In Section 3 we provide a formal definition of a numerical invariant (Definition 3.1) and a Θ-stratification (Definition 3.4). For any numerical invariant, Construction 3.10 provides a potential stratification, essentially just a subset of $|[\Theta,\mathfrak{X}]|$, and so the question becomes whether or not that subset corresponds to an open substack satisfying the properties laid out in Definition 3.4. Here again the notion of a reductive stack is essential. Whereas the existence and uniqueness question has an affirmative answer on a reductive stack if and only if Principle (B) holds, we identify a slightly stronger boundedness hypothesis, Principle (B+), and show that a numerical invariant on a reductive stack defines a Θ-stratification if Principle (B+) holds (Theorem 3.16).[3]

In Section 4 we apply our methods to the moduli of families of objects in the heart of a t-structure on the derived category of coherent sheaves, $D^b(X)$, for a projective scheme X. The results in this section will not be surprising to experts, but we hope that it is instructive to connect our theory with this well-studied example. We consider a set up which we refer to as a *pre-stability condition* consisting of a full abelian subcategory, $\mathcal{A}$, of the heart of a t-structure on $D^b(X)$ along with a homomorphism (referred to as a central charge) from the numerical K-group, $Z : \mathcal{N}(X) \to \mathbb{C}$, which maps $\mathcal{A}$ to $\mathbb{H} \cup \mathbb{R}_{\leq 0}$. This generalizes both the classical notion of slope-stability for coherent sheaves as well as the notion of Bridgeland-stability.

Given this data, one can define a moduli functor of families of objects in $\mathcal{A}$ of numerical class v, $\mathcal{M}(v)$. In Proposition 4.10 we give a relatively complete description of $\mathcal{D}(\mathcal{M}(v), [E])$, whose points correspond to finite descending filtrations of E with real weights assigned to the associated graded pieces. Any pre-stability condition defines a subcategory of torsion-free objects, $\mathcal{F}$, and there is a corresponding moduli functor, $\mathcal{M}^{\mathcal{F}}(v)$. We use the central charge to define classes in $H^2(\mathcal{M}^{\mathcal{F}}(v))$ and $H^4(\mathcal{M}^{\mathcal{F}}(v))$, and we prove in Theorem 4.14 that the existence and uniqueness problem is essentially equivalent to the central charge satisfying the Harder-Narasimhan property.

We also establish circumstances under which the numerical invariant constructed from Z induces a Θ-stratification of $\mathcal{M}^{\mathcal{F}}(v)$. We observe that the stack $\mathcal{M}(v)$ is always weakly reductive (Lemma 4.22), and this is enough to establish the existence of Θ-stratifications in Proposition 4.20. When $\mathcal{A}$ is the category of coherent sheaves whose support has dimension $\leq d$, one can formulate a pre-stability condition corresponding to slope stability, and the Θ-stratification is closely related to several recent descriptions of the stratification of the moduli of coherent sheaves by Harder-Narasimhan type [GN, N1, HK]. The novel contribution of our discussion is that when applied to Bridgeland stability conditions – for instance, the family of stability conditions constructed on a K3 surface in [B+] – it provides the first construction of a Θ-stratification for a notion of instability which does not admit a known approximation by GIT models as in the case of coherent sheaves.

In Section 4 we also describe a moduli functor for flat families of projective schemes over a field of characteristic 0 along with a relatively ample line bundle, L, up to the equivalence relation $L \sim L^k$. Our moduli functor, $\mathcal{V}ar'_{\mathbb{Q}}$, parameterizes families which additionally carry a trivialization of a certain functorial line bundle $D_0(L)$ over the base of the family. The stack $\mathcal{V}ar'_{\mathbb{Q}}$ is not algebraic, but it is an fppf stack which is a colimit of local quotient stacks, and it still makes sense to consider

[3]We are ignoring some technical issues. In addition to being reductive, the stack $\mathfrak{X}$ must satisfy an additional technical hypothesis which holds for local quotient stacks and all other known examples. There is also an intrinsic subtlety involving Θ-stratifications in positive characteristic - in general, one can only hope to obtain a structure which we call a weak Θ-stratification. However. the two notions agree for local quotient stacks in characteristic 0 (Proposition 3.18).

its degeneration space. We formulate K-stability in terms of a numerical invariant on $\mathcal{V}ar'_{\mathbb{Q}}$ corresponding to the normalized Futaki invariant.

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During the preparation of this paper, Jochen Heinloth has announced a formulation of stability for points in an algebraic stack using a similar approach of generalizing the Hilbert-Mumford criterion. The two projects were carried out independently, and to my knowledge focus on different aspects of the question. Heinloth's research focuses primarily on the properties of the semistable locus, an important aspect of the moduli story complementary to the results presented here.

Notation. All of our schemes and stacks over a ground field,k, will be considered as stacks over the big étale site of k-schemes. We will typically denote stacks with fraktur font ($\mathfrak{X}$,$\mathfrak{Y}$, etc.), and we will denote quotient stacks as X/G. For an algebraic group, G, we will sometimes denote the quotient stack $*/G$ as BG. If S and X are a schemes over k, we will use the notation X_S to denote the S scheme $S \times X$, and we use similar notation for the pullback of stacks over k.

1. The mapping stack $[\Theta, \mathfrak{X}]$

As mentioned in the introduction, the quotient stack $\Theta := \mathbb{A}^1/\mathbb{G}_m$ plays a key role in this paper. In this section we will study the mapping stack $[\Theta, \mathfrak{X}] := \underline{\mathrm{Map}}(\Theta, \mathfrak{X})$, where $\mathfrak{X}$ is an algebraic stack. By definition, as a presheaf of groupoids on the big étale site of k-schemes we have

$$[\Theta, \mathfrak{X}] : T \mapsto \mathrm{Map}(\Theta \times T, \mathfrak{X})$$

where Map denotes category of natural transformations of presheaves of groupoids, or equivalently the category of 1-morphisms between stacks.

This definition makes sense for any presheaf of groupoids, but if $\mathfrak{X}$ is a stack for the fppf topology, such as an algebraic stack, then we can describe $[\Theta, \mathfrak{X}]\,(T)$ more explicitly in terms of descent data [V]. We consider the first 3 levels of the simplicial scheme determined by the action of $\mathbb{G}_m$ on $\mathbb{A}^1 \times T$

$$\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1 \times T \overset{\mu}{\underset{a}{\overset{\sigma}{\rightrightarrows}}} \mathbb{G}_m \times \mathbb{A}^1 \times T \overset{\sigma}{\underset{a}{\rightrightarrows}} \mathbb{A}^1 \times T \tag{1}$$

Where μ denotes group multiplication, σ denotes the action of $\mathbb{G}_m$ on $\mathbb{A}^1$, and a forgets the leftmost group element. Then the category $[\Theta, \mathfrak{X}]\,(T)$ has

- *objects:* $\eta \in \mathfrak{X}(\mathbb{A}^1 \times T)$ along with a morphism $\phi : a^*\eta \to \sigma^*\eta$ satisfying the cocycle condition $\sigma^*\phi \circ a^*\phi = \mu^*\phi$
- *morphisms:* $f : \eta_1 \to \eta_2$ such that $\phi_2 \circ a^*(f) = \sigma^*(f) \circ \phi_1 : a^*\eta_1 \to \sigma^*\eta_2$

It follows from this description, for instance, that the functor $\mathrm{Hom}(\Theta \times T, \mathfrak{X}) \to \mathrm{Hom}(\mathbb{A}^1 \times T, \mathfrak{X})$ is faithful. Furthermore, it is straightforward from this perspective to show that if $\mathfrak{X}$ is a stack for the fppf topology, then so is $[\Theta, \mathfrak{X}]$.

In this section, we establish the main results regarding $[\Theta, \mathfrak{X}]$ which allow us to interpret the Kempf-Ness stratification as an open substack of $[\Theta, \mathfrak{X}]$. After some general remarks on $[\Theta, \mathfrak{X}]$, including conditions under which it is algebraic, we turn to the case where $\mathfrak{X} = X/G$ for a qc.qs. algebraic space, X, with the action of a smooth k-group G containing a split maximal torus. Just as a map $T \to X/G$ can be described as a map $T \to BG$ along with the data required to lift that map to X/G, we shall first describe $[\Theta, BG]$ in Proposition 1.12, and then extend this to a complete desciption of $[\Theta, \mathfrak{X}]$ in Theorem 1.21. $[\Theta, X/G]$ is a (typically infinite) disjoint union of quotient stacks of locally closed sub-spaces of X by subgroups of G.

1.1. **Some general properties of $[\Theta, \mathfrak{X}]$.** From the definition of the mapping stack, it is not obvious that $[\Theta, \mathfrak{X}]$ is an algebraic stack for many algebraic $\mathfrak{X}$, locally of finite type. In contrast, the stack $\underline{\mathrm{Hom}}(\mathbb{A}^1, \mathfrak{X})$ is almost never algebraic. Nevertheless we have

Proposition 1.1 ([HP], [BHL])**.** *Let R be a G-ring, and let $\mathfrak{X}$ be an algebraic stack which is locally of finite presentation over* $\operatorname{Spec} R$ *and which is quasi-geometric (for instance $\mathfrak{X}$ could be locally of the form X/G where X is a quasi-separated, quasi-compact algebraic space and G is a smooth affine group scheme). Then for any $n \geq 1$, $[\Theta^n, \mathfrak{X}]$ and $[B\mathbb{G}_m^n, \mathfrak{X}]$ are algebraic stacks, locally of finite presentation over* $\operatorname{Spec} R$*.*

Proof. The stacks $B\mathbb{G}_m^n$ and Θ^n are cohomologically projective over $\operatorname{Spec} R$, hence [HP] contains the result when $\mathfrak{X}$ is geometric. The strengthening to the situation where $\mathfrak{X}$ is quasi-geometric follows from [BHL]. □

We will primarily be interested in the case when $R = k$ is a field.

As with any mapping stack, one has a universal evaluation 1-morphism $ev : \Theta \times [\Theta, \mathfrak{X}] \to \mathfrak{X}$. Restricting this to the point[4] $1 \in \mathbb{A}^1$ gives a morphism $ev_1 : [\Theta, \mathfrak{X}] \to \mathfrak{X}$, which is actually the restriction of the evaluation morphism to the open substack $\mathfrak{X} \simeq (\mathbb{A}^1 - \{0\})/\mathbb{G}_m \times [\Theta, \mathfrak{X}] \subset \Theta \times [\Theta, \mathfrak{X}]$. Likewise, the projection $\Theta \to B\mathbb{G}_m$ and the inclusion $B\mathbb{G}_m \hookrightarrow \Theta$, corresponding to the point $\{0\} \in \mathbb{A}^1$, define the morphisms σ and ev_0 respectively:

$$[B\mathbb{G}_m, \mathfrak{X}] \underset{\sigma}{\overset{ev_0}{\rightleftarrows}} [\Theta, \mathfrak{X}] \xrightarrow{ev_1} \mathfrak{X} \tag{2}$$

The composition $B\mathbb{G}_m \to \Theta \to B\mathbb{G}_m$ is equivalent to the identity morphism, so $ev_0 \circ \sigma \simeq \mathrm{id}_{[B\mathbb{G}_m, \mathfrak{X}]}$.

Corollary 1.2. *Under the hypotheses of Proposition 1.1, the morphism $ev_1 : [\Theta^n, \mathfrak{X}] \to \mathfrak{X}$ is relatively representable by algebraic spaces, locally of finite presentation over R.*

Proof. We suppress the base scheme $\operatorname{Spec} R$ from out notation. We know that ev_1 is relatively representable by algebraic stacks, locally of finite presentation, so we must show that $[\Theta^n, \mathfrak{X}]$ is a category fibered in sets over $\mathfrak{X}$, with respect to the morphism ev_1.

Let S be a scheme over $\operatorname{Spec} R$ and let $f : \Theta^n_S \to \mathfrak{X}$ be a morphism, an element of $[\Theta^n, \mathfrak{X}](S)$. Then $\mathrm{Aut}(f)$ is equivalent to the group of sections of $\mathfrak{Y} := \Theta^n_S \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X} \to$

[4]Note that Θ has two k-points, the generic point corresponding to $1 \in \mathbb{A}^1$ and the special point $0 \in \mathbb{A}^1$

Θ^n_S, where $\Theta^n_S \to \mathfrak{X} \times \mathfrak{X}$ in this fiber product is classified by (f, f) and $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is the diagonal. $ev_1(f) \in \mathfrak{X}(S)$ is the restriction of f to $\{1\} \times S$, and automorphisms of f which induce the identity on $ev_1(f)$ correspond to those sections of $\mathfrak{Y} \to \Theta^n_S$ which agree with identity on the open substack $(\mathbb{A}^1 - \{0\})^n \times S/\mathbb{G}_m^n \subset \Theta^n_S$. By hypothesis $\mathfrak{Y} \to \Theta^n$ is representable by separated algebraic spaces, so a section is uniquely determined by its restriction to $(\mathbb{A}^1 - \{0\})^n \times S$. Hence $\mathrm{Aut}(f) \to \mathrm{Aut}_{\mathfrak{X}}(ev_1(f))$ has trivial kernel. □

Remark 1.3. The fact that the fiber of ev_1 over an S point of $\mathfrak{X}$ is (equivalent to) a set does not depend on the representability of $[\Theta^n, \mathfrak{X}]$, only the fact that the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable by separated algebraic spaces.

Our analysis of $[\Theta, \mathfrak{X}]$ when $\mathfrak{X}$ is a quotient stack will ultimately give explicit descriptions of the morphisms σ, ev_0, and ev_1. In this case we will see that ev_1 is relatively representable by schemes, and in the case of torus actions is even a locally closed immersion (Corollary 1.17).

We can also study the morphism $[\Theta, \mathfrak{W}] \to [\Theta, \mathfrak{X}]$ induced by a morphism of stacks $\mathfrak{W} \to \mathfrak{X}$.

Proposition 1.4. *Let $f : \mathfrak{W} \to \mathfrak{X}$ be a morphism of quasi-geometric schemes which is representable by algebraic spaces. Then so is the induced morphism $\tilde{f} : [\Theta, \mathfrak{W}] \to [\Theta, \mathfrak{X}]$. Furthermore:*

(1) If f is an open immersion, then so is $\tilde{f}$, and $\tilde{f}$ identifies $[\Theta, \mathfrak{W}]$ with the preimage of $\mathfrak{W} \subset \mathfrak{X}$ under the composition $[\Theta, \mathfrak{X}] \xrightarrow{ev_0} [B\mathbb{G}_m, \mathfrak{X}] \to \mathfrak{X}$.

(2) If f is a closed immersion, then so is $\tilde{f}$, and $\tilde{f}$ identifies $[\Theta, \mathfrak{W}]$ with the closed substack $ev_1^{-1}\mathfrak{W} \subset [\Theta, \mathfrak{X}]$.

Proof. Let $S \to [\Theta, \mathfrak{X}]$ be an S-point defined by a morphism $\Theta_S \to \mathfrak{X}$. Then the fiber product $\Theta_S \times_{\mathfrak{X}} \mathfrak{W} \to \Theta_S$ is representable and is thus isomorphic to $E/\mathbb{G}_m$ for some algebraic space E with a $\mathbb{G}_m$-equivariant map $E \to \mathbb{A}^1_S$. The fiber of $[\Theta, \mathfrak{W}] \to [\Theta, \mathfrak{X}]$ over the given S-point of $[\Theta, \mathfrak{X}]$ corresponds to the groupoid of sections of $E/\mathbb{G}_m \to \mathbb{A}^1_S/\mathbb{G}_m$, which form a set. Thus $[\Theta, \mathfrak{W}]$ is equivalent to a sheaf of sets as a category fibered in groupoids over $[\Theta, \mathfrak{X}]$. Because $[\Theta, \mathfrak{W}]$ and $[\Theta, \mathfrak{X}]$ are algebraic, the morphism $[\Theta, \mathfrak{W}] \to [\Theta, \mathfrak{X}]$ is relatively representable by algebraic spaces.

Now let $f : \mathfrak{W} \subset \mathfrak{X}$ be an open substack. For any test scheme, S, a morphism $\Theta \times S \to \mathfrak{X}$ factors through $\mathfrak{W}$ if and only if the composition $B\mathbb{G}_m \times S \to \mathfrak{X}$ factors through $\mathfrak{W}$ because any open substack containing $B\mathbb{G}_m \times S$ contains all of $\Theta \times S$. Furthermore, when the factorization exists it is unique. On the other hand, the topological space of $B\mathbb{G}_m \times S$ agrees with that of S, hence the morphism $B\mathbb{G}_m \times S \to \mathfrak{X}$ factors (uniquely) through $\mathfrak{W}$ if and only if the composition $S \to \mathfrak{X}$ factors through $\mathfrak{W}$. We have thus described the functor of points of $[\Theta, \mathfrak{X}]$ and verified (1).

The argument for closed immersions is similar – we must show that if $f : \mathfrak{W} \to \mathfrak{X}$ is a closed immersion, then $[\Theta, \mathfrak{W}] \to [\Theta, \mathfrak{X}]$ is a closed subfunctor. A morphism $\psi : \Theta \times S \to \mathfrak{X}$ factors (uniquely) through $\mathfrak{W}$ if and only if the preimage of $\mathfrak{W}$ is the entire stack $\Theta \times S$. The smallest closed substack of $\Theta \times S$ containing $(\mathbb{A}^1 - \{0\}) \times S/\mathbb{G}_m \simeq S$ is $\Theta \times S$ itself. It follows that the morphism lifts to $\mathfrak{W}$ if and only if the composition $\{1\} \times S \to \Theta \times S \to \mathfrak{X}$ lifts to $\mathfrak{W}$, hence $[\Theta, \mathfrak{W}] \simeq ev_1^{-1}(\mathfrak{W})$. □

Finally, the stack $[\Theta, \mathfrak{X}]$ carries the additional structure of an action of the monoid $\mathbf{N}^\times$ cannonically commuting with the morphism $ev_1 : [\Theta, \mathfrak{X}] \to \mathfrak{X}$. We define an action of $\mathbf{N}^\times$ on Θ canonically commuting with the inclusion of the point 1. For each $n \in \mathbf{N}$, the morphism $(\bullet)^n : \Theta \to \Theta$ is defined by the map $\mathbb{A}^1 \to \mathbb{A}^1$ given by $z \mapsto z^n$, which is equivariant with respect to the group homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given by the same formula. Given a morphism $f : \Theta \to \mathfrak{X}$, we let f^n denote the precomposition of f with $(\bullet)^n$, and we also use $(\bullet)^n$ to denote the pre-composition morphism $[\Theta, \mathfrak{X}] \to [\Theta, \mathfrak{X}]$. This action on $[\Theta, \mathfrak{X}]$ also descends to an action on the set of connected components $\pi_0 [\Theta, \mathfrak{X}]$.

Lemma 1.5. *Let $\mathfrak{X}$ be a quasi-geometric stack, locally of finite presentation. Then for any $n \in \mathbf{N}$, the morphism $(\bullet)^n : [\Theta, \mathfrak{X}] \to [\Theta, \mathfrak{X}]$ is an open immersion.*

Proof. We prove the claim using some methods from derived algebraic geometry. Let us denote the morphism $(\bullet)^n : \Theta \to \Theta$ as p. Then observe that $Rp_* \mathcal{O}_\Theta \simeq \mathcal{O}_\Theta$, as can be observed after base change to $\mathbb{A}^1$, where we can identify p with the coarse moduli space map $\mathbb{A}^1/\mu_n \to \mathbb{A}^1$. It follows from the projection formula and the adjunction between Lp^* and Rp_* that $p^* : D_{qc}(\Theta) \to D_{qc}(\Theta)$ is fully faithful. So the image of p^* is a full symmetric monoidal ∞-category, and the same can be said for $\mathrm{Perf}(\Theta)$, $\mathrm{APerf}(\Theta)$, and $D_{qc}(\Theta)^{\leq 0}$. By the Tannakian formalism [L2], a map $f : \Theta \to \mathfrak{X}$ is uniquely determined by the corresponding functor on symmetric monoidal ∞-categories f^*, hence p is an epimorphism in the category of quasi-geometric stacks, and as a result the corresponding morphism $(\bullet)^n : [\Theta, \mathfrak{X}] \to [\Theta, \mathfrak{X}]$ is a monomorphism in the category of quasi-geometric stacks.

Now if we regard $\mathfrak{X}$ as a derived stack, the mapping stack $[\Theta, \mathfrak{X}]$ is not isomorphic to the classical mapping stack, but the underlying classical stack $[\Theta, \mathfrak{X}]^{cl}$ is isomorphic to the classical mapping stack. If $f \in [\Theta, \mathfrak{X}](k')$ for some k'/k, classifying a map $f : \Theta_{k'} \to \mathfrak{X}$, then the derived tangent space at f is $T_f [\Theta, \mathfrak{X}] \simeq R\Gamma(\Theta_{k'}, f^* T\mathfrak{X})$. The derived tangent space at $f^n : \Theta_{k'} \to \mathfrak{X}$ is

$$R\Gamma(\Theta_{k'}, (f^n)^* T\mathfrak{X}) \simeq R\Gamma(\Theta_{k'}, p_* p^* f^* T\mathfrak{X}) \simeq R\Gamma(\Theta_{k'}, f^* T\mathfrak{X})$$

again by the projection formula and the fact that $Rp_* \mathcal{O}_\Theta \simeq \mathcal{O}_\Theta$. Thus the induced map on derived tangent spaces $T_f [\Theta, \mathfrak{X}] \to T_{f^n} [\Theta, \mathfrak{X}]$ is an isomorphism, and it follows that the map on classical tangent spaces is an isomorphism as well.

Returning to the setting of classical stacks, we have shown that $(\bullet)^n : [\Theta, \mathfrak{X}] \to [\Theta, \mathfrak{X}]$ is a representable monomorphism of locally finitely presented stacks, hence it is radicial and formally unramified. We have also shown that it is formally smooth, hence it must be an isomorphism. $\square$

As a consequence we shall consider the following construction: Let $\mathfrak{Y} \subset [\Theta, \mathfrak{X}]$ be an open substack which is closed under the action of $\mathbf{N}^\times$ and such that $\mathbf{N}^\times$ acts freely on $\pi_0 \mathfrak{Y}$. In this case $\pi_0 \mathfrak{Y}$ can be canonically partially ordered by saying that $\alpha \leq \beta$ if $(\bullet)^n : \mathfrak{Y}_\alpha \to \mathfrak{Y}_\beta$ for some n. Furthermore, this is a disjoint union of partially ordered sets – one for each $\mathbf{N}^\times$-orbit on $\pi_0 \mathfrak{Y}$ – each of which is a directed set. We can thus canonically regard $\mathfrak{Y}$ as a system of stacks and open immersions over the partially ordered set $\pi_0 \mathfrak{Y}$. Because the morphisms are open immersions, we can form the union of these algebraic stacks along these directed systems. We denote the resulting algebraic stack $\operatorname{colim}_{\mathbf{N}} \mathfrak{Y}$. It is locally of finite presentation over k. We can repeat the construction for $[B\mathbb{G}_m, \mathfrak{X}]$ as well, and we observe that we still have all of the canonical morphisms of Equation 2 even though $\operatorname{colim}_{\mathbf{N}} \mathfrak{Y}$ is no longer a substack of $[\Theta, \mathfrak{X}]$.

1.2. **Principal G-bundles on Θ.** As a warmup, consider the case of GL_N, in which morphisms $S \times \Theta \to B\,\mathrm{GL}_N$ can be regarded as locally free sheaves on $S \times \Theta$. Recall the Rees construction:

Proposition 1.6. *Let S be a k-scheme. The category of quasicoherent sheaves on $S \times \Theta$ is equivalent to the category of diagrams of quasicoherent sheaves on S of the form $\cdots \to F_i \to F_{i-1} \to \cdots$.*

Proof. Using descent one sees that quasicoherent sheaves on $S \times \Theta$ are the same as graded $\mathcal{O}_S[t]$ modules, where t has degree -1. The equivalence assigns a diagram $\cdots \to F_i \to F_{i-1} \to \cdots$ to the module $\bigoplus F_i$ with F_i in degree i with the maps $F_i \to F_{i-1}$ corresponding to multiplication by t. □

Under this equivalence, locally free sheaves on $S \times \Theta$ correspond to diagrams such that each F_i is locally free on S, $F_i \to F_{i-1}$ is injective and $\mathrm{gr}_i(F_\bullet) = F_i/F_{i+1}$ is locally free for each i, F_i stabilizes for $i \ll 0$, and $F_i = 0$ for $i \gg 0$. In other words GL_N bundles on $S \times \Theta$ correspond to locally free sheaves with decreasing, weighted filtrations. The ranks of the locally free sheaves $\mathrm{gr}_i(F_\bullet) \neq 0$ determine a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_N$, and the discussion above shows that GL_N bundles on $S \times \Theta$ are equivalent to P_λ-bundles on S. This is the characterization that generalizes to arbitrary linear algebraic groups G. Rather than reducing general G to the case of GL_N, we will take a more intrinsic approach to G-bundles on $S \times \Theta$ in the next section.

Let G be a smooth affine group scheme over k. For a k-scheme S, we use the phrase G-bundle over S, principal G-bundle over S, and G-torsor over S interchangeably to refer to a scheme $E \to S$ along with a right action of G_S (left action of G_S^{op}) such that $E \times_S G_S \to E \times_S E$ is an isomorphism and $E \to S$ admits a section étale locally. We can equivalently think of E as the sheaf of sets which it represents over the étale site of S.[5] By definition a morphism $S \to */G$ is a principal G-bundle, thus one can define a principal G-bundle over a stack to be a map $\mathfrak{X} \to */G$.

Lemma 1.7. *Let S be a k scheme. A principal G-bundle over $\Theta \times S$ is a principal bundle $E \to \mathbb{A}^1 \times S$ with a $\mathbb{G}_m$ action on E which is compatible with the action on $\mathbb{A}^1$ under projection and which commutes with the right action of G on E.*

Proof. This is an straightforward interpretation of the descent property of the stack $*/G$ and will be discussed in general in section 1.3 below (See Diagram 1). □

If $E \to \mathbb{A}^1$ is a principal G-bundle with compatible $\mathbb{G}_m$ action, we will often say "E is a G-bundle over Θ" even though more accurately, $E/\mathbb{G}_m$ is a G-bundle over Θ. Given a one parameter subgroup $\lambda : \mathbb{G}_m \to G$, we define the standard G-bundle $E_\lambda := \mathbb{A}^1 \times G$, where G acts by right multiplication and $t \cdot (z, g) = (tz, \lambda(t)g)$. We will show that over an algebraically closed field k, every G-bundle over Θ is isomorphic to E_λ for some one parameter subgroup. In fact we will obtain a complete description of the stack of principal G-bundles over Θ.

[5]For general G a sheaf torsor may only be represented by an algebraic space $E \to S$, but E is always a scheme when G is affine, see [M, III-Theorem 4.3].

Let us recall some standard subgroups of G associated to a one parameter subgroup, λ:

$$L_\lambda = \text{ the centralizer of } \lambda$$
$$P_\lambda = \{p \in G | \lim_{t\to 0} \lambda(t) p \lambda(t)^{-1} \text{ exists}\}$$
$$U_\lambda = \{u \in G | \lim_{t\to 0} \lambda(t) p \lambda(t)^{-1} = 1\}$$

In addition to the inclusion $L_\lambda \subset P_\lambda$, we have a surjective homomorphism $\pi : P_\lambda \to L_\lambda$ mapping $p \mapsto \lim_{t\to 0} \lambda(t) p \lambda(t)^{-1}$, and P_λ is a semidirect product $P_\lambda = U_\lambda \rtimes L_\lambda$.

Proposition 1.8. *Let k be a perfect field, let S be a connected finite type k-scheme, and let E be a G-bundle over $\Theta_S := \Theta \times S$. Let $s \in S(k)$, thought of as the point $(0, s) \in \mathbb{A}^1_S$, and assume that $\mathrm{Aut}(E_s) \simeq G$. Let $\lambda : \mathbb{G}_m \to G$ be a 1PS conjugate to the one parameter subgroup $\mathbb{G}_m \to \mathrm{Aut}(E_s)$. Then*

1. *There is a unique reduction of structure group $E' \subset E$ to a P_λ-torsor such that $\mathbb{G}_m \to \mathrm{Aut}(E'_s) \simeq P_\lambda$ is conjugate in P_λ to λ, and*
2. *the restriction of E' to $\{1\} \times S$ is canonically isomorphic to the sheaf on the étale site of S mapping $T/S \mapsto \mathrm{Iso}((E_\lambda)_{\Theta_T}, E|_{\Theta_T})$.*

Proof. $(E_\lambda)_{\Theta_S} = E_\lambda \times S/\mathbb{G}_m$ is a G-bundle over Θ_S, and $\underline{\mathrm{Iso}}((E_\lambda)_{\Theta_S}, E)$ is a sheaf over Θ_S representable by a (relative) scheme over Θ_S. In fact, if we define a twisted action of $\mathbb{G}_m$ on E given by $t \star e := t \cdot e \cdot \lambda(t)^{-1}$, then

$$\underline{\mathrm{Iso}}((E_\lambda)_{\Theta_S}, E) \simeq E/\mathbb{G}_m \text{ w.r.t. the } \star\text{-action} \tag{3}$$

as sheaves over Θ_S.[6]

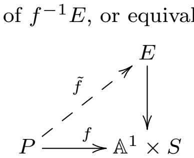
The twisted $\mathbb{G}_m$ action on E is compatible with base change. Let $T \to S$ be an S-scheme. From the isomorphism of sheaves (3), there is a natural bijection between the set of isomorphisms $(E_\lambda)_{\mathbb{A}^1_T} \xrightarrow{\simeq} E|_{\mathbb{A}^1_T}$ as $\mathbb{G}_m$-equivariant G-bundles and the set of $\mathbb{G}_m$-equivariant sections of $E|_{\mathbb{A}^1_T} \to \mathbb{A}^1_T$ with respect to the twisted $\mathbb{G}_m$ action.

The morphism $E|_{\mathbb{A}^1_T} \to \mathbb{A}^1_T$ is separated, so a twisted equivariant section is uniquely determined by its restriction to $\mathbb{G}_m \times T$, and by equivariance this is uniquely determined by its restriction to $\{1\} \times T$. Thus we can identify $\mathbb{G}_m$-equivariant sections with the set of maps $T \to E$ such that $\lim_{t\to 0} t \star e$ exists and $T \to E \to \mathbb{A}^1_S$ factors as the given morphism $T \to \{1\} \times S \to \mathbb{A}^1_S$.

If we define the subsheaf of E over $\mathbb{A}^1_S$

$$E'(T) := \left\{ e \in E(T) | \mathbb{G}_m \times T \xrightarrow{t\star e(x)} E \text{ extends to } \mathbb{A}^1 \times T \right\} \subset E(T),$$

[6]To see this, note that a map $T \to \Theta_S$ corresponds to a $\mathbb{G}_m$-bundle $P \to T$ along with a $\mathbb{G}_m$ equivariant map $f : P \to \mathbb{A}^1 \times S$. Then the restrictions $(E_\lambda \times S)_T$ and $E|_T$ correspond (via descent for G-bundles) to the $\mathbb{G}_m$-equivariant bundles $f^{-1}(E_\lambda \times S)$ and $f^{-1}E$ over P. Forgetting the $\mathbb{G}_m$-equivariant structure, the G-bundle $E_\lambda \times S$ is trivial, so an isomorphism $f^{-1}(E_\lambda \times S) \to f^{-1}E$ as G-bundles corresponds to a section of $f^{-1}E$, or equivalently to a lifting



to a map $\tilde{f} : P \to E \to \mathbb{A}^1 \times S$. The isomorphism of G-bundles defined by the lifting $\tilde{f}$ descends to an isomorphism of $\mathbb{G}_m$-equivariant G-bundles $f^{-1}(E_\lambda \times S) \to f^{-1}E$ if and only if the lift $\tilde{f}$ is equivariant with respect to the twisted $\mathbb{G}_m$ action on E.

then we have shown that $E'|_{\{1\}\times S}(T) \simeq \mathrm{Iso}((E_\lambda)_{\Theta_T}, E|_{\Theta_T})$. Next we show in several steps that the subsheaf $E' \subset E$ over $\mathbb{A}^1_S$ is a torsor for the subgroup $P_\lambda \subset G$, so E' is a reduction of structure group to P_λ.

Step 1: *E' is representable:* The functor E' is exactly the functor of Corollary 1.17 below, so Proposition 1.15 implies that E' is representable by a disjoint union of $\mathbb{G}_m$ equivariant locally closed subschemes of E.

Step 2: *$P_\lambda \subset G$ acts simply transitively on $E' \subset E$:* Because E is a G-bundle over $\mathbb{A}^1_S$, right multiplication $(e,g) \mapsto (e, e\cdot g)$ defines an isomorphism $E \times G \to E \times_{\mathbb{A}^1_S} E$. The latter has a $\mathbb{G}_m$ action, which we can transfer to $E \times G$ using this isomorphism.

For $g \in G(T)$, $e \in E(T)$, and $t \in \mathbb{G}_m(T)$ we have $t \star (e \cdot g) = (t \star e) \cdot (\lambda(t) g \lambda(t)^{-1})$. This implies that the $\mathbb{G}_m$ action on $E \times G$ corresponding to the diagonal action on $E \times_{\mathbb{A}^1_S} E$ is given by

$$t \cdot (e,g) = (t \star e, \lambda(t) g \lambda(t)^{-1})$$

The subfunctor of $E \times G$ corresponding to $E' \times_{\mathbb{A}^1_S} E' \subset E \times_{\mathbb{A}^1_S} E$ consists of those points for which $\lim_{t\to 0} t \cdot (e,g)$ exists. This is exactly the subfunctor represented by $E' \times P_\lambda \subset E \times G$. We have thus shown that E' is equivariant for the action of P_λ, and $E' \times P_\lambda \to E' \times_{\mathbb{A}^1_S} E'$ is an isomorphism of sheaves.

Step 3: *$p : E' \to \mathbb{A}^1_S$ is smooth:* Proposition 1.18 implies that E' and $E^{\mathbb{G}_m} \subset E'$ are both smooth over S. The restriction of the tangent bundle $T_{E/S}|_{E^{\mathbb{G}_m}}$ is an equivariant locally free sheaf on a scheme with trivial $\mathbb{G}_m$ action, hence it splits into a direct sum of vector bundles of fixed weight with respect to $\mathbb{G}_m$. The tangent sheaf $T_{E'/S}|_{E^{\mathbb{G}_m}} \subset T_{E/S}|_{E^{\mathbb{G}_m}}$ is precisely the subsheaf with weight ≥ 0. By hypothesis $T_{E/S} \to p^* T_{\mathbb{A}^1_S/S}$ is surjective, and $p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}}$ is concentrated in nonnegative weights, therefore the map

$$T_{E'/S}|_{E^{\mathbb{G}_m}} = (T_{E/S}|_{E^{\mathbb{G}_m}})_{\geq 0} \to p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}} = (p^* T_{\mathbb{A}^1_S/S}|_{E^{\mathbb{G}_m}}))\geq 0$$

is surjective as well.

Thus we have shown that $T_{E'/S} \to p^* T_{\mathbb{A}^1_S/S}$ is surjective when restricted to $E^{\mathbb{G}_m} \subset E'$, and by Nakayama's Lemma it is also surjective in a Zariski neighborhood of $E^{\mathbb{G}_m}$. On the other hand, the only equivariant open subscheme of E' containing $E^{\mathbb{G}_m}$ is E' itself. It follows that $T_{E'/S} \to p^* T_{\mathbb{A}^1_S/S}$ is surjective, and therefore that the morphism p is smooth.

Step 4: *$p : E' \to \mathbb{A}^1_S$ admits sections étale locally:* We consider the $\mathbb{G}_m$ equivariant G-bundle $E|_{\{0\}\times S}$. After étale base change we can assume that $E|_{\{0\}\times S}$ admits a non-equivariant section, hence the $\mathbb{G}_m$-equivariant structure is given by a homomorphism $(\mathbb{G}_m)_{S'} \to G_{S'}$. Lemma 1.10 implies that after further étale base change this homomorphism is conjugate to a constant homomorphism. Thus $E|_{\{0\}\times S'}$ is isomorphic to the trivial equivariant $\mathbb{G}_m$-bundle $(E_\lambda)_{\mathbb{A}^1_S} = \mathbb{A}^1_S \times G \to \mathbb{A}^1_S$ with $\mathbb{G}_m$ acting by left multiplication by $\lambda(t)$.

It follows that $E|_{\{0\}\times S'}$ admits an invariant section with respect to the twisted $\mathbb{G}_m$ action. In other words $(E_{\mathbb{A}^1_{S'}})^{\mathbb{G}_m} \to \{0\} \times S'$ admits a section, and $E^{\mathbb{G}_m} \subset E'$, so we have shown that $E' \to \mathbb{A}^1_{S'}$ admits a section over

$\{0\} \times S'$. On the other hand, because $p : E' \to \mathbb{A}^1_{S'}$ is smooth and $\mathbb{G}_m$-equivariant, the locus over which p admits an étale local section is open and $\mathbb{G}_m$-equivariant. It follows that p admits an étale local section over every point of $\mathbb{A}^1_{S'}$.

□

Remark 1.9. In fact we have shown something slightly stronger than the existence of étale local sections of $E' \to \mathbb{A}^1_S$ in Step 4. We have shown that there is an étale map $S' \to S$ such that $E'|_{S'} \to \mathbb{A}^1_{S'}$ admits a $\mathbb{G}_m$-equivariant section.

We now prove that families of one parameter subgroups of G are étale locally constant up to conjugation, which was the key fact in Step 4.

Lemma 1.10. *Let S be a connected scheme of finite type over a perfect field k, let G be a smooth affine k-group, and let $\phi : (\mathbb{G}_m)_S \to G_S$ be a homomorphism of group schemes over S. Let $\lambda : \mathbb{G}_m \to G$ be a 1PS conjugate to ϕ_s for some $s \in S(k)$. Then the subsheaf*

$$F(T) = \left\{ g \in G(T) | \phi_T = g \cdot (\mathrm{id}_T, \lambda) \cdot g^{-1} : (\mathbb{G}_m)_T \to G_T \right\} \subset G_S(T)$$

is an L_λ-torsor. In particular ϕ is étale-locally conjugate to a constant homomorphism.

Proof. Verifying that $F \times L_\lambda \to F \times_S F$ given by $(g, l) \mapsto (g, gl)$ is an isomorphism of sheaves is straightforward. The more important question is whether $F(T) \neq \emptyset$ étale locally.

As in the proof of Proposition 1.8 we introduce a twisted $\mathbf{G}_m$ action on $G \times S$ by $t \star (g, s) = \phi_s(t) \cdot g \cdot \lambda(t)^{-1}$. Then $G \times S \to S$ is $\mathbf{G}_m$ invariant, and the functor $F(T)$ is represented by the map of schemes $(G \times S)^{\mathbb{G}_m} \to S$. By Proposition 1.18, $(G \times S)^{\mathbb{G}_m} \to S$ is smooth, and in particular it admits a section after étale base change in a neighborhood of a point $s \in S(k)$ for which $(G \times S)^{\mathbb{G}_m}_s = (G \times \{s\})^{\mathbb{G}_m} \neq \emptyset$. By construction this set is nonempty precisely when ϕ_s is conjugate to λ, so by hypothesis it is nonempty for some $s \in S(k)$.

By the same reasoning, for every finite separable extension k'/k, every k'-point has an étale neighborhood on which ϕ is conjugate to a constant homomorphism determined by some one parameter subgroup defined over k'. Because k is perfect and S is locally finite type, we have a cover of S by Zariski opens over each of which ϕ is etale-locally conjugate to a constant homomorphism determined by a one parameter subgroup defined over some finite separable extension of k. Because S is connected, all of these 1PS's are conjugate to λ, possibly after further finite separable field extensions. Thus $(G \times S)^{\mathbb{G}_m} \to S$ admits a global section after étale base change. □

One immediate consequence of Proposition 1.8 is a classification of principal G-bundles over Θ in the case when k is algebraically closed.

Corollary 1.11. *Let $k = \bar{k}$ be an algebraically closed field. Every G-bundle over Θ is isomorphic to E_λ for some one parameter subgroup $\lambda : \mathbb{G}_m \to G$. In addition, $E_{\lambda_0} \simeq E_{\lambda_1}$ if and only if λ_0 and λ_1 are conjugate, and $\mathrm{Aut}(E_\lambda) \simeq P_\lambda$ as an algebraic group.*

Proof. This is essentially exactly statement (2) of Proposition 1.8 applied to the case $S = \operatorname{Spec} k$, combined with the observation that $E'|_{\{1\}}$ is trivializable because k is algebraically closed. □

In fact Proposition 1.8 induces a stronger version of this correspondence – it identifies the category of G-bundles over Θ_S and the category of P_λ torsors over $1 \times S$ by restriction. Furthermore, this identification holds for arbitrary k-schemes S in addition to those of finite type, and it works over arbitrary fields as long as G has a split maximal torus.

Proposition 1.12. *Let k be a field, and let G be a smooth affine k-group which contains a split maximal torus. Let I be a set of 1PS's, $\lambda : \mathbb{G}_m \to G$, which form a complete set of representatives of conjugacy classes upon base change to $\bar{k}$. Then as a stack over the étale site of k-schemes, we have*

$$[\Theta, */G] \simeq \bigsqcup_{\lambda \in I} */P_\lambda.$$

The maps $*/P_\lambda \to [\Theta, */G]$ classify the G-bundles E_λ over Θ.

Proof. The objects E_λ define a 1-morphism $\bigsqcup_{\lambda\in I} */P_\lambda \to [\Theta, */G]$ of stacks over the site of all k-schemes. Both stacks satisfy descent for the fpqc topology, so it suffices to show that this morphism induces an equivalence after base change to $\bar{k}$. Because I represents a complete set of representatives of conjugacy classes of one-parameter subgroups in $G_{\bar{k}}$, it suffices to prove the claim when $k = \bar{k}$, which we assume for the remainder of the proof.

Proposition 1.8 implies that this is an equivalence of stacks over the sub-site of k-schemes of finite type. The functor $[\Theta, */G]$ is limit preserving by the formal observation

$$\operatorname{Hom}(\varprojlim_i T_i, [\Theta, */G]) \simeq \operatorname{Hom}(\varprojlim_i T_i \times \Theta, */G) \simeq \varprojlim_i \operatorname{Hom}(T_i \times \Theta, */G)$$

where the last equality holds because $*/G$ is an algebraic stack locally of finite presentation. The stack $\bigsqcup_I */P_\lambda$ is locally of finite presentation and thus limit preserving as well. Every affine scheme over k can be written as a limit of finite type k schemes, so the isomorphism for finite type k-schemes implies the isomorphism for all k-schemes. □

Scholium 1.13. *With the hypotheses of Proposition 1.12, we have an isomorphism of stacks over the étale site of k-schemes $\bigsqcup_I */L_\lambda \xrightarrow{\simeq} [B\mathbb{G}_m, */G]$. The maps $*/L_\lambda \to [B\mathbb{G}_m, */G]$ classify the trivial G-bundles $G \to \operatorname{Spec} k$ with $\mathbb{G}_m$ equivariant structure defined by left multiplication by $\lambda(t)$.*

Proof. For S of finite type over $k = \bar{k}$, the proof of Proposition 1.8 carries over unchanged for G-bundles over $(*/\mathbb{G}_m) \times S$, showing that étale locally in S they are isomorphic to $S \times G$ with $\mathbb{G}_m$ acting by left multiplication by $\lambda(t)$ on G. In fact, we had to essentially prove this when we considered the $\mathbb{G}_m$-equivariant bundle $E|_{\{0\}\times S}$ in Step 4 of that proof. The amplification of the statement from finite type $\bar{k}$-schemes to all $\bar{k}$-schemes, and from $\bar{k}$-schemes to k-schemes, is identical to the proof of Proposition 1.12. □

Finally, we extend this story to describe G-bundles on Θ^n. For any homomorphism $\psi : \mathbb{G}_m^n \to G$, we let the subgroup $P_\psi \subset G$ be the intersection of P_λ as $\lambda : \mathbb{G}_m \to G$ varies over all 1PS's induced by composition of ψ with 1PS's in the positive cone of the cocharacter lattice of $\mathbb{G}_m^n$. More explicitly, any such ψ is of the form $\psi(t_1, \ldots, t_n) = \lambda_1(t_1) \cdots \lambda_n(t_n)$ for some collection of mutually commuting 1PS's.

We denote this as $\psi = \lambda_1 \times \cdots \times \lambda_n$. Then $P_\psi = P_{\lambda_1} \cap \cdots \cap P_{\lambda_n}$. We define the G-bundle E_ψ over Θ^n as $\mathbb{A}^n \times G$ with G acting by right multiplication and

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n, g) := (t_1 z_1, \ldots, t_n z_n, \psi(t_1, \ldots, t_n) \cdot g)$$

Corollary 1.14. *Let k be a field, and let G be a smooth affine k-group which contains a split maximal torus. Let I be a set of homomorphisms, $\psi : \mathbb{G}_m^n \to G$, which form a complete set of representatives of conjugacy classes upon base change to $\bar{k}$. Then as a stack over the étale site of k-schemes, we have*

$$[\Theta^n, */G] \simeq \bigsqcup_{\psi \in I} */P_\psi.$$

The maps $*/P_\psi \to [\Theta^n, */G]$ classify the G-bundles E_ψ over Θ^n.

Proof. As before each E_ψ defines a morphism $*/P_\psi \to */G$, so it suffices to show that this morphism becomes an equivalence after base change to $\bar{k}$, i.e. we may assume $k = \bar{k}$ for the remainder of the proof. We may consider $[\Theta^n, \mathfrak{X}]$ as an iterated mapping stack

$$[\Theta^n, \mathfrak{X}] = \underline{\mathrm{Map}}(\Theta, \underline{\mathrm{Map}}(\Theta, \ldots, \underline{\mathrm{Map}}(\Theta, \mathfrak{X}) \ldots),$$

so applying Proposition 1.12 multiple times implies that the connected components of $[\Theta^n, \mathfrak{X}]$ are classifying stacks for various subgroups of G. For any smooth affine group, any homomorphism from a torus factors through a fixed choice of maximal torus up to conjugation. If we fix a T and only consider $\psi : \mathbb{G}_m^n \to G$ which factor through T, then T is simultaneously a maximal torus for all P_ψ. It follows that the connected components of $[\Theta^n, \mathfrak{X}]$ are of the form $*/P_\psi$ where ψ ranges over a set of representatives of $\psi : \mathbb{G}_m^n \to G$ up to a certain conjugacy relation.

Let $\lambda, \lambda', \lambda_1, \ldots, \lambda_n$ be a set of one parameter subgroups factoring through T. If λ is conjugate to λ' in $P_{\lambda_1 \times \cdots \times \lambda_n}$, then in fact λ' is conjugate to λ in $L_{\lambda_1 \times \cdots \times \lambda_n}$. If follows that $\lambda \times \lambda_1 \times \cdots \times \lambda_n$ is conjugate to $\lambda' \times \lambda_1 \times \cdots \times \lambda_n$ in $L_{\lambda_1 \times \cdots \times \lambda_n}$, and hence in G. By an inductive argument, this shows that connected components of $[\Theta^n, \mathfrak{X}]$ correspond simply to conjugacy classes of homomorphisms $\psi : \mathbb{G}_m^n \to G$. □

1.3. **Explicit description of $[\Theta, \mathfrak{X}]$ for local quotient stacks.** In this section we will bootstrap Proposition 1.12 into a description of $[\Theta, \mathfrak{X}]$ for all quotient stacks $\mathfrak{X} = X/G$ over k. Our main result applies to the situation where X is quasiprojective with a linearizable action of a group with a split maximal torus. A result of Totaro [T2] implies that any finite type k-stack with the resolution property is equivalent to a stack of the form X/GL_N. Hence we will effectively describe $[\Theta, \mathfrak{X}]$ for any finite type k-stack with the resolution property.

First we recall some basic facts about concentration under the action of $\mathbb{G}_m$.

Proposition 1.15 ([H3],[DG])**.** *Let X be a qc.qs. algebraic k-space of finite type, and let $\mathbb{G}_m$ act on X. Then the functor*

$$\Phi_X(T) = \left\{ \mathbb{G}_m\text{-equivariant maps } \mathbb{A}^1 \times T \to X \right\}$$

is representable by a finite type k-space Y.

Proof. When X is a scheme admitting a $\mathbb{G}_m$-equivariant affine open cover, then this is a special case of the main theorem of section 4 of [H3] for which the "center" is $C = X$ and the "speed" is $m = 1$. The general statement of the existence of Y is the main result of [DG]. □

Remark 1.16. We say that an algebraic group G acting on a scheme X is *locally affine* if for every one parameter subgroup $\lambda : \mathbb{G}_m \to G$, X admits a $\mathbb{G}_m$-equivariant affine cover. The explicit construction of [H3] shows that for a locally affine $\mathbb{G}_m$-action, restriction to $\{0\} \times T \subset \mathbb{A}^1 \times T$ defines a morphism $\pi : Y \to X^{\mathbb{G}_m}$ which is affine. Furthermore, $\pi : Y \to X^{\mathbb{G}_m}$ has connected geometric fibers.

Corollary 1.17. *If X is separated, then restriction of a map $\mathbb{A}^1 \times T \to X$ to $\{1\} \times T \subset \mathbb{A}^1 \times T$ defines a monomorphism of functors, identifying $\Phi_X(T)$ with*

$$\{f : T \to X | \mathbb{G}_m \times T \xrightarrow{t \cdot f(x)} X \text{ extends to } \mathbb{A}^1 \times T\} \subset \operatorname{Hom}(T, X)$$

Hence the inclusion of functors defines a local immersion $j : Y \to X$.

Proof. Restriction to $\{1\} \times T$ identifies the set of equivariant maps $\mathbb{G}_m \times T \to X$ with $\operatorname{Hom}(T, X)$. If the corresponding map extends to $\mathbb{A}^1 \times T$ it will be unique because X is separated. Likewise the uniqueness of the extension of $\mathbb{G}_m \times \mathbb{G}_m \times T \to X$ to $\mathbb{G}_m \times \mathbb{A}^1 \times T \to X$ guarantees the $\mathbb{G}_m$ equivariance of the extension $\mathbb{A}^1 \times T \to X$. □

We will also need the following strengthened version of the Biaynicki-Birula theorem

Proposition 1.18. *If X is a k-scheme with a $\mathbb{G}_m$-action and $X \to S$ is a smooth morphism which is $\mathbb{G}_m$ invariant, then both $X^{\mathbb{G}_m}$ and Y are smooth over S. If S is smooth over k, then $Y \to X^{\mathbb{G}_m}$ is a Zariski-locally trivial bundle of affine spaces with linear $\mathbb{G}_m$ action on the fibers.*

Finally, for later use we record the computation of the tangent space of Y at a point of $X^{\mathbb{G}}_m$, which can be deduced directly from the functor of points description of the tangent space.

Lemma 1.19. *Let X be a separated k-space with $\mathbb{G}_m$-action, and let $p : \operatorname{Spec} k' \to X^{\mathbb{G}}_m$. Then T_pX has a natural action of $(\mathbb{G}_m)_{k'}$. One can identify $T_p(X^{\mathbb{G}}_m)$ with the subspace of weight 0, and $T_pY \subset T_pX$ is the sum of the subspaces of positive weight.*

For most of our discussion, we will focus on the case when G is a linear algebraic group acting on a separated k-scheme, X, in a locally affine manner, and $\lambda : \mathbb{G}_m \to G$ is a 1PS. Proposition 1.15 allows us to define the *blade* corresponding to a connected component $Z \subset X^\lambda$

$$Y_{Z,\lambda} := \pi^{-1}(Z) = \left\{ x \in X | \lim_{t \to 0} \lambda(t) \cdot x \in Z \right\} \tag{4}$$

Note that $Y_{Z,\lambda}$ is connected and $\pi : Y_{Z,\lambda} \to Z$ is affine. When X is smooth then both $Y_{Z,\lambda}$ and Z are smooth, and $Y_{Z,\lambda}$ is a fiber bundle of affine spaces over Z by Proposition 1.18.

We define the subgroup

$$P_{Z,\lambda} := \{p \in P_\lambda | l(Z) \subset Z, \text{ where } l = \pi(p)\}$$

$P_{Z,\lambda} \subset P_\lambda$ has finite index – it consists of the preimage of those connected components of L_λ which stabilize Z. The locally closed subscheme $Y_{Z,\lambda}$ is equivariant with respect to the action of $P_{Z,\lambda}$, because

$$\lim_{t \to 0} \lambda(t) p x = \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} \lambda(t) x = l \cdot \lim_{t \to 0} \lambda(t) x.$$

Lemma 1.20. *There is a map of stacks $\Theta \times (Y_{Z,\lambda}/P_{Z,\lambda}) \to X/G$ which maps the k-point defined by $(z,x) \in \mathbb{A}^1 \times Y$ to the k-point defined by $\lambda(z) \cdot x \in X$. By definition this defines a morphism $Y_{Z,\lambda}/P_{Z,\lambda} \to [\Theta, \mathfrak{X}]$. Likewise there is a map $*/\mathbb{G}_m \times Z/L_{Z,\lambda} \to \mathfrak{X}$ defining a map $Z/L_{Z,\lambda} \to [B\mathbb{G}_m, \mathfrak{X}]$.*

Proof. We will drop the subscripts Z and λ. A morphism $\Theta \times (Y/P) \to X/G$ is a $\mathbf{G}_m \times P$-equivariant G-bundle over $\mathbb{A}^1 \times Y$ along with a G-equivariant and $\mathbf{G}_m \times P$ invariant map to X.

Consider the trivial G-bundle $\mathbb{A}^1 \times Y \times G$, where G acts by right multiplication on the rightmost factor. This principal bundle acquires a $\mathbf{G}_m \times P$-equivariant structure via the action

$$(t,p) \cdot (z,x,g) = (tz, p \cdot x, \lambda(tz) p \lambda(z)^{-1} g)$$

This expression is only well defined when $z \neq 0$, but it extends to a regular morphism because $\lim_{z \to 0} \lambda(z) p \lambda(z)^{-1} = l$ exists. It is straightforward to check that this defines an action of $\mathbf{G}_m \times P$, that the action commutes with right multiplication by G, and that the map $\mathbb{A}^1 \times Y \times G \to X$ defined by

$$(z,x,g) \mapsto g^{-1} \lambda(z) \cdot x$$

is $\mathbf{G}_m \times P$-invariant.

The morphism $*/\mathbb{G}_m \times Z/L_{Z,\lambda} \to X/G$ is simpler. It is determined by the group homomorphism $\mathbb{G}_m \times L_{Z,\lambda} \to G$ given by $(t,l) \mapsto \lambda(t) l \in G$ which intertwines the inclusion of schemes $Z \subset X$. □

Note that for $g \in G$, the subscheme $g \cdot Z$ is a connected component of $X^{\lambda'}$ where $\lambda'(t) = g\lambda(t)g^{-1}$. Furthermore, $g \cdot Y_{Z,\lambda} = Y_{gZ,\lambda'}$ and we have an equivalence $Y_{Z,\lambda}/P_{Z,\lambda} \to Y_{g \cdot Z,\lambda'}/P_{g \cdot Z,\lambda'}$ which commutes up to 2-isomorphism with the morphisms to $[\Theta, \mathfrak{X}]$ constructed in Lemma 1.20.

Theorem 1.21. *Let $\mathfrak{X} = X/G$ be a quotient of a k-scheme X by a locally affine action of a smooth affine k-group G which contains a split maximal torus. Fix a set of one parameter subgroups I as in Proposition 1.12. The natural morphism $Y_{Z,\lambda}/P_{Z,\lambda} \to [\Theta, \mathfrak{X}]$ from Lemma 1.20 induce isomorphisms*

$$[\Theta, \mathfrak{X}] \simeq \bigsqcup_{\substack{\lambda \in I \\ Z \subset X^\lambda}} Y_{Z,\lambda}/P_{Z,\lambda}, \quad \textit{and} \quad [B\mathbb{G}_m, \mathfrak{X}] \simeq \bigsqcup_{\substack{\lambda \in I \\ Z \subset X^\lambda}} Z/L_{Z,\lambda},$$

where Z ranges over the connected components of X^λ.

Proof. Consider the morphism $[\Theta, \mathfrak{X}] \to [\Theta, */G]$, and let $*/P_\lambda \to [\Theta, */G]$ be the morphism classified by the G-bundle E_λ over Θ. By Proposition 1.12, it suffices to show that

$$[\Theta, \mathfrak{X}] \times_{[\Theta, */G]} (*/P_\lambda) \simeq \bigsqcup_{Z \subset X^\lambda} Y_{Z,\lambda}/P_{Z,\lambda}$$

for all $\lambda \in I$.

If we consider the composition $\operatorname{Spec} k \to */P_\lambda \to [\Theta, */G]$, then we have

$$\left([\Theta, \mathfrak{X}] \times_{[\Theta, */G]} \operatorname{Spec} k\right)(T) \simeq \{\mathbb{G}_m\text{-equivariant maps } \mathbb{A}^1 \times T \to X\} = \Phi_X(T),$$

which is represented by a locally closed subscheme $j : Y \hookrightarrow X$ by Corollary 1.17. It follows that $[\Theta, \mathfrak{X}] \times_{[\Theta, */G]} (*/P_\lambda) \simeq Y/P_\lambda$. The claim follows from the observation that $Y = \bigsqcup Y_{Z,\lambda}$, allowing us to rewrite the quotient Y/P_λ.

The same argument implies the statement for $[B\mathbb{G}_m, \mathfrak{X}]$ with little modification. By Scholium 1.13 the mapping stack $[B\mathbb{G}_m, */G]$ is isomorphic to $\bigsqcup */L_\lambda$. The morphism $[B\mathbb{G}_m, \mathfrak{X}] \to [B\mathbb{G}_m, */G]$ is representable, and the preimage of the connected component $*/L_\lambda$ is the global quotient X^λ/L_λ, which can be further decomposed into connected components. □

From the explicit description of the stack $[\Theta, \mathfrak{X}]$ when $\mathfrak{X}$ is a global quotient stack, we obtain explicit descriptions of the morphisms ev_0 and ev_1 from the diagram (2) and deduce some basic properties.

Corollary 1.22. *Let $\mathfrak{X}$ be as in Theorem 1.21. The morphism $ev_0 : [\Theta, \mathfrak{X}] \to [B\mathbb{G}_m, \mathfrak{X}]$ has connected fibers. On each connected component, ev_0 corresponds to the projection $\pi : Y_{Z,\lambda} \to Z$, which intertwines the group homomorphism $P_{Z,\lambda} \to L_{Z,\lambda}$.*

We can also describe the morphism ev_1, whose fiber over k-points will be central to the rest of this paper.

Corollary 1.23. *On the connected component of $[\Theta, \mathfrak{X}]$ corresponding to $[Z, \lambda]$, the restriction morphism $ev_1 : [\Theta, \mathfrak{X}] \to \mathfrak{X}$ is equivalent to the inclusion $j : Y_{Z,\lambda} \hookrightarrow X$ which intertwines the inclusion of groups $P_{Z,\lambda} \subset G$. In particular ev_1 is representable, and $[\Theta, \mathfrak{X}] \times_{\mathfrak{X}} X \simeq \bigsqcup G \times_{P_{Z,\lambda}} Y_{Z,\lambda}$.*

This lets one describe the fiber of ev_1 over $p \in X(k)$. Its k-points are specified by the three pieces of data:

- a one parameter subgroup $\lambda \in I$;
- a point $q \in X$ such that $\lim_{t\to 0} \lambda(t) \cdot q$ exists; and
- a $g \in G$ such that $g \cdot q = p$.

Where two sets of such data specify the same point of the fiber if and only if $(g', q') = (gh^{-1}, h \cdot q)$ for some $h \in P_\lambda$.

Remark 1.24. Alternatively, given such a datum we define the one parameter subgroup $\lambda'(t) := g\lambda(t)g^{-1}$, and $\lim_{t\to 0} \lambda'(t) \cdot p$ exists. The point in $ev_1^{-1}(p)$ is uniquely determined by this data, thus we can specify a point in the fiber by one parameter subgroup λ, not necessarily in I, for which $\lim_{t\to 0} \lambda(t) \cdot p$ exists. Two one parameter subgroups specify the same point in the fiber if and only if $\lambda' = h\lambda h^{-1}$ for some $h \in P_\lambda$.

2. Existence and uniqueness of optimal destabilizers

In this section, we revisit Kempf's original construction [K1] of the stratification of the unstable locus in GIT from an intrinsic perspective. Let X be a projective over affine variety with a reductive group action. Let $\mathcal{L}$ be a G-linearized ample invertible sheaf on X, and let $|\bullet|$ be a Weil-group-invariant positive definite bilinear form on the cocharacter lattice of G. Given a point $p \in X$ and one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $q := \lim_{t\to 0} \lambda(t) \cdot p$ exists, we have the Hilbert-Mumford numerical invariant

$$\mu(p, \lambda) = \frac{-1}{|\lambda|} \operatorname{weight}_\lambda \mathcal{L}|_q \in \mathbb{R}. \tag{5}$$

The point p is unstable if and only if $\mu(p, \lambda) > 0$ for some λ, and Kempf's result states that in this case there is a λ which maximizes μ and is unique up to conjugation by an element of P_λ.

Theorem 1.21 allows us to identify the data of the pair (p, λ) with a morphism $f : \Theta \to \mathfrak{X} = X/G$ along with an isomorphism $f(1) \simeq p$. If $\mathfrak{X}$ is a general algebraic k-stack with a Θ stratification, then every unstable point $p \in \mathfrak{X}(k)$ extends canonically to a morphism $f : \Theta \to \mathfrak{X}$ with an isomorphism $f(1) \simeq p$. The canonical f in the Kempf-Ness stratification is the unique maximizer of Hilbert-Mumford numerical invariant (5).

Let us assume for the moment that $k = \mathbb{C}$. One can compute the cohomology (of the topological stack underlying the analytification) of Θ as $H^*(\Theta; \mathbb{Q}) = \mathbb{Q}[[q]]$, where $q = c_1(\mathcal{O}_\Theta(1))$. For a cohomology class $\eta \in H^{2n}(\mathfrak{X}; \mathbb{Q})$ we have $f^*\eta = r \cdot q^n$ for some $r \in \mathbb{Q}$. The key observation for an intrinsic description of the Hilbert-Mumford numerical invariant is the following

Lemma 2.1. *The numerical invariant* (5) *can be expressed in terms of a class* $l \in H^2(\mathfrak{X}; \mathbb{Q})$ *and* $b \in H^4(\mathfrak{X}; \mathbb{Q})$ *via the formula.*[7]

$$\mu(f) = f^*l / \sqrt{f^*b} \in \mathbb{R} \tag{6}$$

Proof. We define $l = c_1(\mathcal{L}) \in H^2(\mathfrak{X}; \mathbf{q})$, so that $f^*l = c_1(f^*\mathcal{L}) = -\operatorname{weight}(f^*\mathcal{L})_{\{0\}} \cdot q$, which follows from the fact[8] that $f^*\mathcal{L} \simeq \mathcal{O}_\Theta(w)$ where $w = -\operatorname{weight}(f^*\mathcal{L})_{\{0\}}$. For the denominator, $|\bullet|$ can be interpreted as a class in $H^4(*/G; \mathbb{C})$ under the identification $H^*(*/G; \mathbb{C}) \simeq (\operatorname{Sym}(\mathfrak{g}^\vee))^G$, and we let b be the image of this class under the map $H^4(*/G) \to H^4(X/G)$. For a morphism $f : \Theta \to X/G$, f^*b is the pullback of the class in $H^4(*/G)$ under the composition $\Theta \to X/G \to */G$. We therefore have $f^*b = |\lambda|^2 q^2 \in H^4(\Theta)$. □

Now suppose that $\mathfrak{X}$ is a general algebraic stack locally of finite type over $\mathbb{C}$, and $l \in H^2(\mathfrak{X})$ and $b \in H^4(\mathfrak{X})$. If $p \in \mathfrak{X}(\mathbb{C})$ is an unstable point in the sense that there is some $f : \Theta \to \mathfrak{X}$ with $f(1) \simeq p$ and $f^*l \in \mathbb{R}_{>0} \cdot q$, then we may define a numerical invariant on $\mathfrak{X}$ via (6), and we may ask

Question 2.2. *Does there exist a morphism* $f : \Theta \to \mathfrak{X}$ *along with an isomorphism* $f(1) \simeq p$ *which maximizes the numerical invariant* $\mu(f)$ *and is unique up to the ramified coverings* $\Theta \to \Theta$ *defined by* $x \mapsto x^n$*?*

The answer is no in full generality, but in this section we describe conditions which guarantee affirmative answers to both the existence and uniqueness parts of Question 2.2. The existence of a maximizer is guaranteed by a "boundedness" principle (B), and the uniqueness is guaranteed by a "convexity" principle (C). Both the implication that (B) implies existence and (C) implies uniqueness are trivial, but they are useful as organizational tools. To illustrate this, we verify that Principle (B) holds for any quotient stack (Proposition 2.25), regardless of the numerical invariant or choice of $p \in X/G$.

We also introduce the notion of a *reductive k-stack* in Definition 2.27 – the primary example is V/G where V is an affine scheme and G a reductive group, but we will see a modular example in Section 4 as well. We show that Principle (C)

[7]Here our b is chosen in such a way that $f^*b \in \mathbb{Q}_{>0} \cdot q^2$ for any $f : \Theta \to \mathfrak{X}$ such that the homomorphism $\mathbb{G}_m \to \operatorname{Aut} f(0)$ is nontrivial. We present a slightly more robust definition of a numerical invariant using classes in H^2 and H^4 in Construction 2.19.

[8]Every invertible sheaf on Θ is of the form $\mathcal{O}_\Theta(n)$, which correspond to the free $k[t]$ module with generator in degree $-n$. Note that the isomorphism $\operatorname{Pic}(\Theta) \simeq \mathbb{Z}$ is canonical, because $\Gamma(\Theta, \mathcal{O}_\Theta(n)) = 0$ for $n > 0$ whereas $\Gamma(\Theta, \mathcal{O}_\Theta(n)) \simeq k$ for $n \leq 0$. This holds in contrast to $\operatorname{Pic}(*/\mathbb{G}_m)$. The stack $*/\mathbb{G}_m$ has an automorphisms exchanging $\mathcal{O}(1)$ and $\mathcal{O}(-1)$.

holds automatically for a reductive stack (Corollary 2.31) for any $l \in H^2(\mathfrak{X})$ and certain $b \in H^4(\mathfrak{X})$, and thus we recover (a slight strengthening of) the main result of [K1] for V/G. When X is projective-over-affine and G is reductive, X/G is not a reductive stack. Nevertheless in Proposition 2.33 we show that Principle (C) holds in this case for classes $l \in H^2(X/G)$ which are NEF, obtaining a slight weakening of the usual hypotheses under which Question 2.2 is known to have an affirmative answer.

Both principles (B) and (C) are stated in terms of a certain topological space, $\mathcal{D}(\mathfrak{X}, p)$, which we associate to a k-stack and a point $p \in \mathfrak{X}(k)$. The "rational" points of $\mathcal{D}(\mathfrak{X}, p)$ correspond to maps $f : \Theta \to \mathfrak{X}$ along with an isomorphism $f(1) \simeq p$, up to n-fold ramified coverings of Θ. In Construction 2.19 we use classes $l \in H^2(\mathfrak{X})$ and $b \in H^4(\mathfrak{X})$ to define a numerical invariant, $\mu : \mathcal{D}(\mathfrak{X}, p) \to \mathbb{R} \cup \{\pm\infty\}$, which is continuous on the open locus corresponding to points where $f^*b \neq 0$ and agrees with Equation 6 on the rational points of $\mathcal{D}(\mathfrak{X}, p)$. The point p is defined to be unstable if $\mu > 0$ at some point, and in this case, answering Question 2.2 in the affirmative amounts to showing that μ obtains a unique maximum at a rational point of $\mathcal{D}(\mathfrak{X}, p)$.

More precisely, the space $\mathcal{D}(\mathfrak{X}, p)$ is constructed as follows: We start with a stack $\mathfrak{X}$ over an arbitrary field k. For an extension K/k and a map $p : \operatorname{Spec} K \to \mathfrak{X}$, we consider the (isomorphism classes of) K-points of the fiber product $[\Theta^n, \mathfrak{X}]_p := [\Theta^n, \mathfrak{X}] \times_{ev_1, \mathfrak{X}, p} \operatorname{Spec} K$, which classify maps $\Theta^n_K \to \mathfrak{X}$ with a choice of isomorphism $f(1, \ldots, 1) \simeq p$. To each K-point of $[\Theta^n, \mathfrak{X}]_p$ we associate a standard $n-1$ simplex Δ_{n-1}, and we define a topological space, $\mathcal{D}(\mathfrak{X}, p)$, obtained by gluing these standard simplices along embeddings $\Delta_{k-1} \to \Delta_{n-1}$ corresponding to certain maps of stacks $\Theta^k \to \Theta^n$. This space will typically be the geometric realization of a simplicial complex, but we will introduce a different combinatorial structure, which we call a *formal fan*, which is better suited to our application.

2.1. **Formal fans and their geometric realizations.** In this section we define combinatorial objects which we call formal fans. They are analogous to semisimplicial sets.

Definition 2.3. We define a category of *integral simplicial cones* $\mathfrak{C}$ to have

- objects: positive integers $[n]$ with $n > 0$,
- morphisms: a morphism $\phi : [k] \to [n]$ is an injective group homomorphism $\mathbb{Z}^k \to \mathbb{Z}^n$ which maps the standard basis of $\mathbb{Z}^k$ to the cone spanned by the standard basis of $\mathbb{Z}^n$.

We define the category of *formal fans*

$$\mathrm{Fan} := \mathrm{Fun}(\mathfrak{C}^{op}, \mathrm{Set})$$

For $F \in \mathrm{Fan}$ we use the abbreviated notation $F_n = F([n])$ and refer to this as the set of *cones*. We refer to the elements of F_1 as *rays*.

For any $F \in \mathrm{Fan}$, we can define two notions of geometric realization. First form the comma category $(\mathfrak{C}|F)$ whose objects are elements $\xi \in F_n$ and morphisms $\xi_1 \to \xi_2$ are given by morphisms $\phi : [n_1] \to [n_2]$ with $\phi^* \xi_2 = \xi_1$. There is a canonical functor $(\mathfrak{C}|F) \to \mathrm{Top}$ assigning $\xi \in F_n$ to the cone $(\mathbb{R}_{\geq 0})^n$ spanned by the standard basis of $\mathbb{R}^n$. Using this we can define the geometric realization of F

$$|F| := \operatorname*{colim}_{(\mathfrak{C}|F)} (\mathbb{R}_{\geq 0})^n = \operatorname*{colim}_{[n] \in \mathfrak{C}} F_n \times (\mathbb{R}_{\geq 0})^n$$

This is entirely analogous to the geometric realization functor for semisimplicial sets.

Given a map $[k] \to [n]$ in $\mathfrak{C}$, the corresponding linear map $\phi : \mathbb{R}^k \to \mathbb{R}^n$ is injective. Thus ϕ descends to a map $\Delta_{k-1} \to \Delta_{n-1}$, where $\Delta_{n-1} = ((\mathbb{R}_{\geq 0})^n - \{0\}) / (\mathbb{R}_{>0})^\times$ is the standard $(n-1)$-simplex realized as the space of rays in $(\mathbb{R}_{\geq 0})^n$. Thus for any $F \in \mathrm{Fan}$ we have a functor $(\mathfrak{C}|F) \to \mathrm{Top}$ assigning $\xi \in F_n$ to Δ_{n-1}. We define the *projective realization* of F to be

$$\mathbb{P}(F) := \operatorname*{colim}_{(\mathfrak{C}|F)} \Delta_{n-1} = \operatorname*{colim}_{[n]\in\mathfrak{C}} F_n \times \Delta_{n-1}$$

Alternatively, we observe that injective maps $(\mathbb{R}_{\geq 0})^k \to (\mathbb{R}_{\geq 0})^n$ arising in the construction of $|F|$ are equivariant with respect to the action of $\mathbb{R}^\times_{>0}$ on $(\mathbb{R}_{\geq 0})^n$ by scalar multiplication, and thus $|F|$ has a canonical continuous $\mathbb{R}^\times_{>0}$-action. Commuting colimits shows that $\mathbb{P}(F) \simeq (|F| - \{*\})/\mathbb{R}^\times_{>0}$ as topological spaces, where $* \in |F|$ is the cone point corresponding to the origin in each copy of $(\mathbb{R}_{\geq 0})^n$.

For a representable fan, $h_{[n]}(\bullet) = \mathrm{Hom}_{\mathfrak{C}}(\bullet, [n])$, the category $(\mathfrak{C}|h_{[n]})$ has a single terminal object, the identity map on $[n]$. Hence $|h_{[n]}| \simeq (\mathbb{R}_{\geq 0})^n$ and $\mathbb{P}(h_{[n]}) \simeq \Delta_{n-1}$. It follows that any $\xi \in F_n$ defines a map $\Delta_{n-1} \to \mathbb{P}(F_\bullet)$. We call such a map a *rational simplex* of $\mathbb{P}(F_\bullet)$ and when $n = 1$ we call this a rational point of $\mathbb{P}(F_\bullet)$.

The terminology of formal fans and cones is motivated by the following construction, which establishes a relationship between formal fans and the classical notion of a fan in a vector space.

Construction 2.4. A subset $K \subset \mathbb{R}^N$ which is invariant under multiplication by $\mathbb{R}^\times_{\geq 0}$ is called a *cone in* $\mathbb{R}^N$. Given a set of cones $K_\alpha \subset \mathbb{R}^N$, we define

$$R_n(\{K_\alpha\}) := \left\{ \begin{array}{c} \text{injective homomorphisms } \phi : \mathbb{Z}^n \to \mathbb{Z}^N \text{ s.t.} \\ \exists \alpha \text{ s.t. } \phi(e_i) \subset K_\alpha, \forall i \end{array} \right\} \tag{7}$$

These sets naturally form an object $R_\bullet(\{K_\alpha\}) \in \mathrm{Fan}$.

Remark 2.5. We use the phrase *classical fan* to denote a collection of rational polyhedral cones in $\mathbb{R}^N$ such that a face of any cone is also in the collection, and the intersection of two cones is a face of each. We expect that if $K_\alpha \subset \mathbb{R}^N$ are the cones of a classical fan Σ, it is possible to reconstruct Σ from the data of $R_\bullet(\{K_\alpha\})$.

Lemma 2.6. *Let $K_\alpha \subset \mathbb{R}^N$ be a finite collection of cones and assume that there is a classical fan $\Sigma = \{\sigma_i\}$ in $\mathbb{R}^N$ such that each K_α is the union of some collection of σ_i. Then the canonical map $|R_\bullet(\{K_\alpha\})| \to \bigcup K_\alpha$ is a homeomorphism. Furthermore, $\mathbb{P}(R_\bullet(\{K_\alpha\})) \simeq S^{N-1} \cap \bigcup_\alpha K_\alpha$ via the evident quotient map $\mathbb{R}^N - \{0\} \to S^{N-1}$.*

Proof. By refinement we may assume that Σ is simplicial, and that the ray generators of each σ_i form a basis for the lattice generated by $\sigma_i \cap \mathbb{Z}^N$. Consider the formal fans $F_\bullet = R_\bullet(\{K_\alpha\})$ and $F'_\bullet = R_\bullet(\{\sigma_i\})$. By hypothesis $F'_\bullet$ is a subfunctor of $F_\bullet$. Hence we have a functor of comma categories $(\mathfrak{C}|F') \to (\mathfrak{C}|F)$ and thus a map of topological spaces $|F'| \to |F|$ which commutes with the map to $\mathbb{R}^N$.

We can reduce to the case when $\{K_\alpha\} = \{\sigma_i\}$. Indeed the map $|F'| \to |F|$ is surjective because any point in K_α lies in some σ_i. If the composition $|F'| \to |F| \to \bigcup_\alpha K_\alpha = \bigcup \sigma_i$ were a homeomorphism it would follow that $|F'| \to |F|$ was injective as well, and one could use the inverse of $|F'| \to \bigcup_\alpha K_\alpha$ to construct and inverse for $|F| \to \bigcup_\alpha K_\alpha$.

For a single simplicial cone $\sigma \subset \mathbb{R}^N$ of dimension n whose ray generators form a basis for the lattice generated by $\sigma \cap \mathbb{Z}^N$, we have $R_\bullet(\sigma) \subset R_\bullet(\mathbb{R}^N)$ is isomorphic to $h_{[n]}(\bullet)$. The terminal object of $(\mathfrak{C}|h_{[n]})$ corresponds to the linear map $\mathbb{R}^n \to \mathbb{R}^N$ mapping the standard basis vectors to the ray generators of σ, and thus $|R_\bullet(\sigma)| \to \sigma$ is a homeomorphism.

Let $\sigma_1^{\max}, \ldots, \sigma_r^{\max} \in \Sigma$ be the cones which are maximal with respect to inclusion and let n_i be the dimension of each. Let $\sigma'_{ij} := \sigma_i^{\max} \cap \sigma_j^{\max} \in \Sigma$, and let n_{ij} denote its dimension. Then by construction

$$F := R_\bullet(\{\sigma_i\}) \simeq \operatorname{coeq}\left(\bigsqcup_{i,j} h_{n_{ij}} \rightrightarrows \bigsqcup_i h_{n_i} \right)$$

as functors $\mathfrak{C}^{op} \to \mathrm{Set}$. Our geometric realization functor commutes with colimits, so it follows that

$$|F| = \operatorname{coeq}\left(\bigsqcup_{i,j} \sigma'_{ij} \rightrightarrows \bigsqcup_i \sigma_{ij}^{\max} \right)$$

Which is homeomorphic to $\bigcup \sigma_i$ under the natural map $|F| \to \mathbb{R}^N$. The final claim follows from the fact that $\mathbb{P}(F) \simeq (|F| - \{0\})/\mathbb{R}^\times_{>0}$. $\square$

Example 2.7. Objects of Fan describe a wider variety of structures than classical fans. For instance if K_1 and K_2 are two simplicial cones which intersect but do not meet along a common face, then $R_\bullet(K_1, K_2)$ will not be equivalent to $R_\bullet(\{\sigma_i\})$ for any classical fan $\Sigma = \{\sigma_i\}$.

Example 2.8. While objects of Fan are more general than classical fans, the definition is broad enough to include some pathological examples. For instance, if $K \subset \mathbb{R}^3$ is the cone over a circle which is not contained in a linear subspace, then $|R_\bullet(K)|$ consists of the rational rays of K equipped with the discrete topology and is not homeomorphic to K.

Another example: if $K \subset \mathbb{R}^2$ is a convex cone generated by two irrational rays, then $|R_\bullet(K)|$ is the interior of that cone along with the origin. There are also examples of fans whose geometric realizations are not Hausdorff, such as multiple copies of the standard cone in $\mathbb{R}^2$ glued to each other along the set of rational rays.

2.2. **Formal fan associated to a point in a stack.** We introduce a 2-category of pointed k-stacks, St_k as follows: objects are algebraic k-stacks $\mathfrak{X}$ with separated diagonal along with a fixed $p : \operatorname{Spec} k \to \mathfrak{X}$ over $\operatorname{Spec} k$. A 1-morphism in this category is a 1-morphism of k-stacks $f : \mathfrak{X} \to \mathfrak{X}'$ along with an isomorphism $p' \simeq f \circ p$, subject to the constraint that $f_* : \mathrm{Aut}(q) \to \mathrm{Aut}(f(q))$ has finite kernel for all $q \in \mathfrak{X}(k)$. A 2-isomorphism is a 2-isomorphism $f \to \tilde{f}$ which is compatible with the identification of marked points.

We let the point 1^n denote the canonical k-point in Θ_k^n coming from $(1, \ldots, 1) \in \mathbb{A}_k^n$, and regard Θ^n as an object of St_k. Sometimes we use the term "non-degenerate" to refer to a morphism $f : \Theta^n \to \mathfrak{X}$ which defines a map $(\Theta^n, 1^n) \to (\mathfrak{X}, f(1^n))$ in St_k. We introduce the functor $\mathbf{D} : \mathrm{St}_k \to \mathrm{Fan}$, where we regard the latter as a 2-category whose morphisms all have trivial 2-automorphism groups.

Definition 2.9. Let $(\mathfrak{X}, p) \in \mathrm{St}_k$. We define the *degeneration fan*

$$\mathbf{D}(\mathfrak{X}, p)_n := \mathrm{Map}_{\mathrm{St}_k}(\Theta^n, (\mathfrak{X}, p)) \tag{8}$$

We define the *degeneration space* $\mathcal{D}(\mathfrak{X}, p) := \mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet)$. For any field extension k'/k and point $p \in \mathfrak{X}(k')$, we define $\mathbf{D}(\mathfrak{X}, p)_\bullet := \mathbf{D}(\mathfrak{X}_{k'}, p)$ and $\mathcal{D}(\mathfrak{X}, p) := \mathcal{D}(\mathfrak{X}_{k'}, p)$, both formed in the category $\mathrm{St}_{k'}$.

This definition implicitly assumes that the expression (8) describes a functor $\mathfrak{C}^{op} \to \mathrm{Set}$, which depends on two observations. First, the morphisms between two pointed stacks forms a groupoid, so the expression (8) is not literally a set. However, Remark 1.3 shows that $\mathbf{D}(\mathfrak{X}, p)_n$ is equivalent to a set, and may be regarded as such. Second, we must explain how (8) is functorial in n, which is an immediate consequence of the following

Lemma 2.10. *The assignment $[n] \mapsto \Theta^n$ extends to a functor $\mathfrak{C} \to \mathrm{St}_k$.*

Proof. A morphism $\phi : [k] \to [n]$ in $\mathfrak{C}$ is represented by a matrix of nonnegative integers ϕ_{ij} for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. One has a map of stacks $\phi_* : \Theta^k \to \Theta^n$ defined by the map $\mathbb{A}^k \to \mathbb{A}^n$

$$(z_1, \ldots, z_k) \mapsto (z_1^{\phi_{11}} \cdots z_k^{\phi_{1k}}, \ldots, z_1^{\phi_{n1}} \cdots z_k^{\phi_{nk}})$$

which is intertwined by the group homomorphism $\mathbb{G}_m^k \to \mathbb{G}_m^n$ defined by the same formula. It is clear that these 1-morphisms are compatible with composition in $\mathfrak{C}$, and that this construction gives canonical identifications $\phi_*(1^k) \simeq 1^n \in \Theta^n(k)$, hence these are morphisms in St_k. □

In particular (8) defines a functor $\mathfrak{C}^{op} \to \mathrm{Set}$ via pre-composition: a morphism $\phi : [k] \to [n]$ of $\mathfrak{C}$ defines a map $\mathbf{D}(\mathfrak{X}, p)_n \to \mathbf{D}(\mathfrak{X}, p)_k$ by $f \mapsto f \circ \phi_*$.

Remark 2.11. When $\mathfrak{X}$ is a Tannakian stack, then Proposition 1.1 implies that (8) is the set of k-points of an algebraic space. When $\mathfrak{X}$ is not an algebraic stack, we can still discuss the groupoid of k-points of $[\Theta^n, \mathfrak{X}]_p := [\Theta^n, \mathfrak{X}] \times_{ev_1, \mathfrak{X}, p} \operatorname{Spec} k$, but this groupoid need not be equivalent to a set. For instance, if $\mathfrak{X}$ is the stack which associates every k-scheme T to the groupoid of coherent sheaves on T and isomorphisms between them, the homotopy fiber of $ev_1 : [\Theta, \mathfrak{X}] \to \mathfrak{X}$ has non-trivial automorphism groups. These automorphisms arise as automorphisms of coherent sheaves on Θ which are supported on the origin in $\mathbb{A}^1$.

Example 2.12. If $G = T$ is a torus, then $\mathbf{D}(*/T, *)_n$ is the set of all injective homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^r$ where $r = \operatorname{rank} T$. This fan is equivalent to $R_\bullet(\mathbb{R}^r)$ where $\mathbb{R}^r \subset \mathbb{R}^r$ is thought of as a single cone. Because this cone admits a simplicial subdivision, Lemma 2.6 implies that $|\mathbf{D}(*/T, *)| \simeq \mathbb{R}^r$ and $\mathbb{P}(\mathbf{D}(*/T, *)) \simeq S^{r-1}$.

Now consider the action of a torus T a k-scheme X, let $p \in X(k)$, and define $T' = T/\operatorname{Aut}(p)$. Define $Y \subset X$ to be the closure of $T \cdot p$ and $\tilde{Y} \to Y$ its normalization. $\tilde{Y}$ is a toric variety for the torus T' and thus determines a classical fan consisting of cones $\sigma_i \subset N'_\mathbb{R}$, where N' is the cocharacter lattice of T'.

Proposition 2.13. *Let $\pi : N_\mathbb{R} \to N'_\mathbb{R}$ be the linear map induced by the surjection of cocharacter lattices $N \to N'$ corresponding to the quotient homomorphism $T \to T'$. Then the cones $\pi^{-1}\sigma_i \subset N_\mathbb{R}$ define a classical fan, and*

$$\mathbf{D}(X/T, p) \simeq R_\bullet\{\pi^{-1}(\sigma_i)\}$$

Proof. The map $Y/T \to X/T$ is a closed immersion, so by Proposition 1.4 the map $\mathbf{D}(Y/T, p) \to \mathbf{D}(X/T, p)$ is an isomorphism, so it suffices to consider the case when X is the closure of a single open orbit, i.e. $X = Y$.

A morphism of stacks $f : \mathbb{A}^n/\mathbb{G}_m^n \to Y/T$ along with an isomorphism $f(1) \simeq p$ is determined uniquely up to unique isomorphism by the group homomorphism

$\psi : \mathbb{G}_m^n = \mathrm{Aut}((0, \ldots, 0)) \to T$. Given such a group homomorphism ψ, the morphism f is determined by the equivariant map

$$f(t_1, \ldots, t_k) = \psi(t) \cdot f(1) = \psi(t) \cdot p \in Y$$

This map is defined on the open subset $\mathbb{G}_m^n \subset \mathbb{A}^n$, and if it extends equivariantly to all of $\mathbb{A}^n$ then the extension is unique because Y is separated. We will use the term "equivariant morphism" $f : \mathbb{A}^n \to Y$ to denote the data of the morphism along with a group homomorphism $\psi : \mathbb{G}_m^n \to T$ which intertwines it.

In the language of fans, this observation says that $\mathbf{D}(Y/T, p)$ is a sub-fan of the fan $\mathbf{D}(*/T, *) \simeq R_\bullet(N_\mathbb{R})$ discussed in Example 2.12. The open orbit $T \cdot p \subset Y$ is smooth, so the map from the normalization $\tilde{Y} \to Y$ is an isomorphism over this open subset. The projection $\tilde{Y} \to Y$ is finite, and $\tilde{Y}$ has a unique T action covering the T action on Y. If an equivariant morphism $\mathbb{A}^n \to Y$ lifts to $\tilde{Y}$, it does so uniquely because $\tilde{Y} \to Y$ is separated.

Now fix an equivariant morphism $f : \mathbb{A}^n \to Y$ with $f(1, \ldots, 1) = p$. Using the T action on Y we can extend this to a morphism

$$T \times \mathbb{A}^n \to T \times Y \to Y$$

which is equivariant with respect to the action of $T \times \mathbb{G}_m^n$ and is dominant. Thus the morphism factors through $\tilde{Y}$ by the universal property of the normalization. We can then restrict this lift to get an equivariant lift $\mathbb{A}^n \times \{1\} \subset \mathbb{A}^n \times T \to \tilde{F}$ of our original $f : \mathbb{A}^n \to Y$. Thus we have shown that the canonical map

$$\mathbf{D}(\tilde{Y}/T, p)_n \to \mathbf{D}(Y/T, p)_n$$

is a bijection. It thus suffices to prove the proposition when Y is normal.

If Y is normal, then it is a toric variety under the action of T', and it is determined by a fan $\Sigma = \{\sigma_i\}$ in $N'_\mathbb{R} = N' \otimes_\mathbb{Z} \mathbb{R}$. Equivariant maps between toric varieties preserving a marked point in the open orbit are determined by maps of lattices such that the image of any cone in the first lattice is contained in some cone of the second [F]. Applying this to the toric variety $\mathbb{A}^n$ under the torus $\mathbb{G}_m^n$ and to Y under the torus T', equivariant maps from $\mathbb{A}^n$ to Y correspond exactly to homomorphisms $\phi : \mathbb{Z}^n \to N'$ such that the image of the standard cone in $\mathbb{Z}^N$ lies in some cone of Σ.

Because the T action on Y factors through T', a group homomorphism $\mathbb{G}_m^n \to T$ determines a map $\Theta^n \to Y/T$ if and only if the composite $\mathbb{G}_m^n \to T'$ determines a map $\Theta^n \to Y/T'$. Thus $\mathbf{D}(Y/T, p)_n$ consists of injective group homomorphisms $\phi : \mathbb{Z}^n \to N$ such that the image of the standard basis under the composite $\mathbb{Z}^n \to N \to N'$ lies in some cone of Σ. This is exactly $R_n\{\pi^{-1}\sigma_i\}$. □

Example 2.14. Let X be an affine toric variety defined by a rational polyhedral cone $\sigma \subset \mathbb{R}^n$ and let $p \in X$ be generic. Then $\mathbf{D}(X/T, p)_\bullet \simeq R_\bullet(\sigma)$ as defined in (7). For instance, $\mathbf{D}(\mathbb{A}^n/\mathbb{G}_m^n, 1^n)_\bullet \simeq h_{[n]}$ is represented by the object $[n] \in \mathfrak{C}$.

Next we show that the notion of a formal fan generalizes the construction of the spherical building of a semisimple group. Recall that the spherical building $\Delta(G)$ is a simplicial complex whose vertices are the maximal parabolic subgroups of G, where a set of vertices spans a simplex if and only if the corresponding maximal parabolics contain a common parabolic subgroup of G.

Proposition 2.15. *Let $\mathfrak{X} = */G$ where G is a split semisimple group, and let p be the unique k point. Then $\mathcal{D}(\mathfrak{X}, p)$ is homeomorphic to the spherical building $\Delta(G)$.*

Proof. Corollary 1.14 implies that $[\Theta^n, \mathfrak{X}]_p \simeq \bigsqcup_{\psi \in I} G/P_\psi$ where I a set of representatives for conjugacy classes of homomorphisms $\psi : \mathbb{G}_m^n \to G$, and $\mathbf{D}(\mathfrak{X}, p)_n$ is the set of k-points of those connected components of $[\Theta^n, \mathfrak{X}]$ corresponding to homomorphisms ψ with finite kernel. As a consequence, elements of $\mathbf{D}(\mathfrak{X}, p)_1$ correspond to parabolic subroups, but not all elements of $\mathbf{D}(\mathfrak{X}, p)_n$ correspond to parabolics.

Let $F_\bullet = \mathbf{D}(\mathfrak{X}, *)_\bullet$, and let $F_\bullet^{par}$ be the sub-fan consisting of elements of $\mathbf{D}(\mathfrak{X}, *)_n$ lying on connected components of $[\Theta^n, \mathfrak{X}]_p$ corresponding to homomorphisms $\psi : \mathbb{G}_m^n \to G$ such that P_ψ is parabolic. We claim that the map induced by the inclusion $F_\bullet^{par} \subset F_\bullet$ is a homeomorphism

$$\operatorname*{colim}_{(\mathfrak{C}|F_\bullet^{par})} \Delta_{\bullet-1} \to \operatorname*{colim}_{(\mathfrak{C}|F_\bullet)} \Delta_{\bullet-1} = \mathbb{P}(F_\bullet).$$

To prove this, it suffices to show that for any element $\xi \in F_n$, the canonical map

$$\operatorname*{colim}_{((\mathfrak{C}|F_\bullet^{par})|\xi)} \Delta_{\bullet-1} \to \Delta_{n-1} \tag{9}$$

is a homeomorphism, where the overcategory $((\mathfrak{C}|F_\bullet^{par})|\xi)$ consists of elements $\xi' \in F_k$ along with a morphism $\phi : [k] \to [n]$ in $\mathfrak{C}$ such that $\xi' = \phi^* \xi$.

For any $\xi \in F_n$, we can choose a maximal torus $T \subset G$ such that the pointed map $(\Theta^n, 1) \to (BG, *)$ factors through $(BT, *) \to (BG, *)$. The over-category of ξ in $(\mathfrak{C}|F_\bullet)$ is canonically equivalent to the over-category of ξ in $(\mathfrak{C}|\mathbf{D}(BT, *)_\bullet)$. More explicitly it is equivalent (See Example 2.12) to the over-category $(\mathfrak{C}|[n])$, with terminal object corresponding to ξ itself.

Now let N be the cocharacter lattice of T, so that ξ corresponds to an injective homomorphism $\mathbb{Z}^n \to N$. An element $\xi' \in ((\mathfrak{C}|F_\bullet)|\xi)$ lies in F_k^{par} if and only if the corresponding pointed map $(\Theta^k, 1) \to (\Theta^n, 1) \to (BT, *) \to (BG, *)$ is parabolic. If we regard the decomposition of $N_\mathbb{R}$ into Weyl chambers as a classical fan $\mathcal{W} = \{\sigma\}$, then parabolic subgroups containing T correspond to cones of $\mathcal{W}$.[9] It follows that ξ' is parabolic if and only if it defines an element of $R_\bullet \mathcal{W} \subset R_\bullet(N_\mathbb{R})$. Taking the preimage of this fan under the homomorphism $\mathbb{Z}^n \to N$ induced by ξ and intersecting with the cone $(\mathbb{R}_{\geq 0})^n$, we get a classical fan $\mathcal{W}'$ in $\mathbb{R}^n$. We conclude that $((\mathfrak{C}|F_\bullet^{par})|\xi) \simeq R_\bullet \mathcal{W}' \subset R_\bullet((\mathbb{R}_{\geq 0})^n)$. Because $\mathcal{W}$ is complete, the support of $\mathcal{W}'$ is $(\mathbb{R}_{\geq 0})^n$, hence Proposition 2.13 implies that (9) is an equivalence.

To conclude the argument, we must show that colimit in (9) is homeomorphic to $\Delta(G)$. Fix a maximal torus $T \subset G$, and let $\{v_1, \ldots, v_n\}$ be the set of minimal ray generators for a chosen dominant chamber in the Weyl fan $\mathcal{W}$ (where we have implicitly chosen an ordering of the rays). Every subset of $\{v_1, \ldots, v_n\}$ of cardinality k defines a conjugacy class of homomorphism $\psi : \mathbb{G}_m^k \to G$ such that P_ψ is parabolic. Let $\mathcal{S} \subset (\mathfrak{C}|F_\bullet^{par})$ be the full subcategory consisting of elements of $\mathbf{D}(\mathfrak{X}, *)_n$ which lie on connected components of $[\Theta^n, \mathfrak{X}]$ corresponding to these ψ.

We claim that $\mathcal{S} \subset (\mathfrak{C}|F_\bullet^{par})$ is cofinal. Let $\xi \in F_n^{par}$, and let $\psi : \mathbb{G}_m^n \to G$ be the homomorphism corresponding to the connected component of $[\Theta^n, \mathfrak{X}]_p$ which contains ξ. Note that up to conjugation, any homomorphism $\psi : \mathbb{G}_m^n \to G$ factors through T, and because $\xi \in F_n^{par}$, ψ must be in $R_\bullet \mathcal{W}$ and therefore up to further conjugation by an element of the Weyl group we may assume that ψ maps $(\mathbb{R}_{\geq 0})^n$ to the dominant Weyl chamber of $\mathcal{W}$.

[9] For each λ in the interior of σ, $P_\lambda = P_\sigma$ is constant, and as λ specializes to the rays generating the cone, P_λ specializes to the minimal parabolics contained in P_σ.

Let $\sigma \in \mathcal{W}$ be the smallest cone containing the image of $(\mathbb{R}_{\geq 0})^n$. $P_\sigma = P_\psi$, and because cones in the dominant Weyl chamber classify conjugacy classes of parabolics, σ is thus uniquely determined by the conjugacy class of ψ. It follows that there is a unique n-cone $\xi' \in \mathcal{S}$ and a morphism $\phi : [n] \to [n]$ such that $\xi = \phi^* \xi'$. A similar argument shows that any morphism in $(\mathfrak{C}|F_\bullet^{par})$ from ξ to an element of $\mathcal{S}$ factors uniquely through this ξ'. It follows that the inclusion $\mathcal{S} \subset (\mathfrak{C}|F_\bullet^{par})$ admits a left adjoint, and hence the inclusion is cofinal.

By construction, objects in $\mathcal{S}$ correspond bijectively to proper parabolic subgroups of G, and a morphism in $\mathcal{S}$ corresponds to a containment of parabolics in the opposite direction. It follows from the construction of the spherical building that

$$\Delta(G) \simeq \operatorname*{colim}_{\mathcal{S}} \Delta_{\bullet - 1}$$

and hence we have verified the claim.

□

2.3. **Numerical invariants from cohomology classes.** Now that we have defined a formal fan whose rays correspond to points of $[\Theta, \mathfrak{X}]_p$, we will show how cohomology classes define continuous functions on the topological space $\mathcal{D} = \mathbb{P}(\mathbf{D}(\mathfrak{X}, p)_\bullet)$.

When $k \subset \mathbb{C}$ and $\mathfrak{X}$ is locally of finite type over k, we may discuss the classical topological stack underlying the analytification of $\mathfrak{X}$. This is defined by taking a presentation of $\mathfrak{X}$ by a groupoid in schemes and then taking the analytification, which is a groupoid in topological spaces. The cohomology is then defined as the cohomology of the classifying space of this topological stack [N2]. For global quotient stacks $\mathfrak{X} = X/G$ this agrees with the equivariant cohomology $H^*_{G^{an}}(X^{an})$ which agrees with $H^*_K(X^{an})$ when $K \subset G$ is a maximal compact subgroup weakly homotopy equivalent to G.

To work over an arbitrary base field, one must choose a different cohomology theory, such as Chow cohomology. We thus use H^* to denote any contravariant functor from some subcategory of k-stacks, which must at least contain quotient stacks, to graded modules over some coefficient ring $A \subset \mathbb{R}$ such that

(1) $H^*(\Theta^n) \simeq A[x_1, \ldots, x_n]$, which we regard as polynomial A-valued functions on $(\mathbb{R}_{\geq 0})^n$, and
(2) for any $\phi : [k] \to [n]$ in $\mathfrak{C}$, the restriction homomorphism $H^*(\Theta^n) \to H^*(\Theta^k)$ induced by the morphism $\Theta^k \to \Theta^n$ agrees with the restriction of polynomial functions along the corresponding inclusion $(\mathbb{R}_{\geq 0})^k \subset (\mathbb{R}_{\geq 0})^n$.

See [P1] for a verification of these properties in the case of equivariant Chow cohomology.

For $k \subset \mathbb{C}$, the stack $\Theta^n = \mathbb{A}^n / \mathbb{G}_m^n$ equivariantly deformation retracts onto $*/\mathbb{G}_m^n$, so (1) and (2) are standard computations in equivariant cohomology. Note that while $H^*(*/\mathbb{G}_m^n) \simeq A[x_1, \ldots, x_n]$ as well, the generators in H^2 are only canonical up to the action of $GL_n(\mathbb{Z})$ via automorphisms of $*/\mathbb{G}_m^n$. Automorphisms of Θ^n correspond to $M \in \mathrm{GL}_n(\mathbb{Z})$ such that M and M^{-1} both fix $(\mathbb{R}_{\geq 0})^n$, which implies that M is a permutation matrix. It follows that the generators of $H^2(\Theta^n)$ are canonical up to permutation. We encode this distinction by regarding $A[x_i]$ as functions on $(\mathbb{R}_{\geq 0})^n$ rather than $\mathbb{R}^n$.

Lemma 2.16. *Let $\mathfrak{X}$ be a stack, $p \in \mathfrak{X}(k)$, and let $\eta \in H^{2l}(\mathfrak{X})$. For any $f : \Theta \to \mathfrak{X}$ with $f(1) \simeq p$, regarded as an element of $[\Theta, \mathfrak{X}]_p$, we define*

$$\hat{\eta}(f) := \frac{1}{q^n} f^* \eta \in \mathbb{R}$$

Then $\hat{\eta}$ extends uniquely to a continuous function $\hat{\eta} : |\mathbf{D}(\mathfrak{X}, p)_\bullet| \to \mathbb{R}$ which is homogeneous of degree l with respect to scaling, i.e. $\hat{\eta}(e^t x) = e^{lt} \hat{\eta}(x)$.

Proof. The geometric realization is a colimit, so a continuous function $|F| \to \mathbb{R}$ is defined by a family of continuous functions $(\mathbb{R}_{\geq 0})^n \to \mathbb{R}$ for each $\xi \in F_n$ which is compatible with the continuous maps $(\mathbb{R}_{\geq 0})^k \to (\mathbb{R}_{\geq 0})^n$ for each morphism in $(\mathfrak{C}|F)$. Because our cohomology theory H^* satisfies (1) and (2) above, we get such a family of functions by regarding $A[x_1, \ldots, x_n] \subset \mathbb{R}[x_1, \ldots, x_n]$ as continuous functions on $(\mathbb{R}_{\geq 0})^n$.

□

Remark 2.17. Let $b \in H^4(\mathfrak{X})$ be a class such that for any element of $\mathbf{D}(\mathfrak{X}, p)_2$, $\gamma : \Theta^2 \to \mathfrak{X}$, the pullback $\gamma^* b \in H^4(\Theta^2; \mathbb{R}) \simeq \mathrm{Sym}^2(\mathbb{R}^2)$ corresponds to a positive definite bilinear form. This happens, for instance, if $\phi^* b \in H^4(B\mathbb{G}_m; \mathbb{R})$ is positive for every morphism $\phi : B\mathbb{G}_m \to \mathfrak{X}$ with finite kernel on automorphism groups. In this case given $\gamma \in \mathbf{D}(\mathfrak{X}, p)_2$ we can define the length of the line segment in $\mathcal{D}(\mathfrak{X}, p)$ defined by γ as

$$\mathrm{length}(\gamma) = \arccos\left(\frac{b(\gamma_{1,1}) - b(\gamma_{1,0}) - b(\gamma_{0,1})}{2\sqrt{b(\gamma_{0,1}) b(\gamma_{1,0})}}\right)$$

where we define $\gamma_{m,n} : \Theta \to \mathfrak{X}$ for $m, n \geq 0$ to be the morphism induced by the morphism $\Theta \to \Theta^2$ corresponding to the ray (m, n).

We can then define a *spherical metric* on $\mathcal{D}(\mathfrak{X}, p)$ by the formula

$$d(f, g) := \inf \left\{ \sum_i \mathrm{length}(\gamma^{(i)}) \right\}$$

Where the infimum is taken over all piecewise linear paths, meaning sequences $\gamma^{(0)}, \ldots, \gamma^{(n)}$ of elements of $\mathbf{D}(\mathfrak{X}, p)_2$ such that $\gamma^{(0)}_{1,0} = f$, $\gamma^{(n)}_{0,1} = g$ and $\gamma^{(i)}_{0,1} = \gamma^{(i+1)}_{1,0}$.

Remark 2.18. The locally polynomial functions $\hat{\eta}$ induced by $\eta \in H^{2l}(\mathfrak{X})$ generalize a construction in toric geometry. Let X be a normal toric variety with torus T defined by a collection of strictly convex rational polyhedral cones $\sigma_i \subset N_\mathbb{R}$, where N is the cocharacter lattice of T. Let $p \in X$ be a generic point. Proposition 2.13 implies that $|\mathbf{D}(X/T, p)_\bullet| = \bigcup \sigma_i \subset \mathbb{R}^d$, the support of the fan of X, and letting H^* be the operational Chow cohomology, Lemma 2.16 constructs a homomorphism from $A^*(X/T)$ to the space of functions on the support of the fan of X which are homogeneous polynomials of degree l (with integer coefficients) on each cone of the fan. In fact this morphism is an isomorphism, and it agrees with the isomorphism constructed in [P1], as both homomorphisms are natural with respect to toric morphisms and are uniquely determined by, and agree in, the case when X/T is smooth.

Our primary use of Lemma 2.16 is to define stability functions on $\mathcal{D} = \mathcal{D}(\mathfrak{X}, p)$. We define an element $b \in H^4(\mathfrak{X})$ to be *positive semi-definite* if $f^* b \in \mathbb{R}_{\geq 0} q^2$ for all $f : \Theta \to \mathfrak{X}$.

Construction 2.19. Fix a class $b \in H^4(\mathfrak{X})$ which is positive semi-definite, and $l \in H^2(\mathfrak{X})$. For $p \in \mathfrak{X}$, let $\mathcal{D} = (|\mathbf{D}(\mathfrak{X}, p)_\bullet| - \{*\})/\mathbb{R}^\times_{>0}$. We define an open subset $\mathcal{U} \subset \mathcal{D}$ and a continuous function $\mu : \mathcal{U} \to \mathbb{R}$ by

$$\mathcal{U} := \left\{ x \in \mathcal{D} \middle| \hat{b}(\tilde{x}) > 0 \right\}, \quad \text{and} \quad \mu(x) = \frac{\hat{l}(\tilde{x})}{\sqrt{\hat{b}(\tilde{x})}},$$

where $\tilde{x} \in |\mathbf{D}(\mathfrak{X}, p)_\bullet|$ is some lift of $x \in \mathcal{D}$. Both $\mathcal{U}$ and μ are well-defined by the homogeneity properties of $\hat{l}$ and $\hat{b}$ in Lemma 2.16.

We say that a class $b \in H^4(\mathfrak{X})$ is *positive definite at* p if $\mathcal{U} = \mathcal{D}(\mathfrak{X}, p)$, and we say that b is *positive definite* if this condition holds for all $p \in \mathfrak{X}(k)$. We say that a point $p \in \mathfrak{X}$ is *unstable* if there is a map $f : \Theta \to \mathfrak{X}$ with $f(1) \simeq p$ which corresponds to a point in $\mathcal{U}$ such that $\mu(f) > 0$. In GIT, one can choose a fixed cohomology class $b \in H^4(BG)$ which is positive definite, and this recovers the notion of instability with respect to an invertible sheaf, L, with $c_1(L) = l$. The notion of instability only depends on the class l when b is positive definite.

2.4. **Boundedness implies existence of maximal destabilizers.** Now we address the question of when a stability function μ, defined as in Construction 2.19, achieves a maximum on $\mathcal{D}$. Consider the following property for a fan $F_\bullet$ and a function $\mu : \mathbb{P}(F_\bullet) \to \mathbb{R} \cup \{\pm\infty\}$:

(B) There exists a finite collection of rational simplices $\sigma_i : \Delta_{n_i} \to \mathbb{P}(F_\bullet)$ such that for any rational point $x \in \mathbb{P}(F_\bullet)$, there is a j and an $x' \in \sigma_j(\Delta_{n_j})$ such that $\mu(x') \geq \mu(x)$.

Let us assume that $\mu \circ \sigma_i$ is a continuous real-valued function on Δ_{n_i}, and therefore obtains a maximum. In this case it is immediate from (B) that μ obtains a maximum on $\mathbb{P}(F_\bullet)$. For instance, when $F_\bullet = \mathbf{D}(\mathfrak{X}, p)_\bullet$ and μ is defined by Construction 2.19, then one should look for rational simplices such that $\sigma_i(\Delta_{n_i}) \subset \mathcal{U}$. If $\mathcal{U} \neq \mathcal{D}$, then μ need not be bounded above, so care must be taken to ensure that the optimization problem has a solution.

We shall show that when $\mathfrak{X} = X/G$ is a quotient of a quasiprojective variety by a linearizable action of an algebraic group, then (B) holds for any $p \in \mathfrak{X}(k)$ and any numerical invariant, μ. The key tool is the construction of a second type of fan associated to any map $p : S \to \mathfrak{X}$.

Definition 2.20. Let $\mathfrak{X}$ be a quasi-geometric stack locally of finite presentation over k, so that $[\Theta^n, \mathfrak{X}]$ is algebraic and locally of finite presentation over k as well. We say that a connected component of $[\Theta^n, \mathfrak{X}]$ is *non-degenerate* if it contains a point $f : \Theta^n_{k'} \to \mathfrak{X}$ such that the map $(\mathbb{G}_m)^n_{k'} \to \mathrm{Aut}_{k'} f(0)$ has a finite kernel (as an algebraic group over k'). For any k-scheme S and map $p : S \to \mathfrak{X}$, we define

$$\mathrm{Comp}(\mathfrak{X}, p)_n := \{\alpha \in \pi_0([\Theta^n, \mathfrak{X}] \times_{\mathfrak{X}} S) \text{ whose image in } \pi_0 [\Theta^n, \mathfrak{X}] \text{ is non-degenerate}\}.$$

Where π_0 denotes the set of connected components of an algebraic space. $\mathrm{Comp}(\mathfrak{X}, p)_\bullet$ naturally defines a functor $\mathfrak{C} \to \mathrm{Set}$. We let $\mathcal{C}omp(\mathfrak{X}, p) := \mathbb{P}(\mathrm{Comp}(\mathfrak{X}, p)_\bullet)$.

Remark 2.21. The fact that ev_1 is locally finite type implies that for any map from a finite type k-scheme $\phi : S \to \mathfrak{X}$, we have $\mathrm{Comp}(\mathfrak{X}, \phi)_\bullet = \bigcup_{p \in S(\bar{k})} \mathrm{Comp}(\mathfrak{X}, p)_\bullet$.

From this point forward, we will consider cohomology theories, H^*, satisfying stronger properties than (1) and (2) above: We assume that for any S which is a disjoint union of integral k-schemes of finite type, we have isomorphisms

(†) $H^*(S \times \Theta^n) \simeq H^*(S)[x_1, \ldots, x_n]$ and $H^0(S) \simeq \pi_0(S)^\vee$

which are natural in S. Note that the first isomorphism follows from (1) in any cohomology theory in which the Künneth decomposition holds, $H^*(\Theta \times S) \simeq H^*(\Theta) \otimes H^*(S)$.

Let $p : \operatorname{Spec} k \to \mathfrak{X}$ be a k-point of $\mathfrak{X}$. Every k point of $[\Theta^n, \mathfrak{X}] \times_{\mathfrak{X}} \operatorname{Spec} k$ lies on a unique connected component, hence we have a map of fans $\mathbf{D}(\mathfrak{X}, p)_\bullet \to \operatorname{Comp}(\mathfrak{X}, p)_\bullet$.[10]

Lemma 2.22. *Let H^* be a cohomology theory satisfying (†). Then for any k-point of $\mathfrak{X}$, any $\eta \in H^{2l}(\mathfrak{X})$ defines a continuous function $|\operatorname{Comp}(\mathfrak{X}, p)_\bullet| \to \mathbb{R}$ which gives $\hat{\eta}$ after restricting along the map $|\mathbf{D}(\mathfrak{X}, p)_\bullet| \to |\operatorname{Comp}(\mathfrak{X}, p)_\bullet|$.*

Proof. The property (†) guarantees that for any connected finite type k-scheme S, the restriction homomorphism $H^*(\mathfrak{X}) \to H^*(S \times \Theta^n) \to H^*(\Theta^n)$ induced by a morphism $S \times \Theta^n \to \mathfrak{X}$ and a k-point $s \in S$ is actually independent of s. Consequently a class in $H^{2l}(\mathfrak{X})$ defines a function $\pi_0 [\Theta^n, \mathfrak{X}] \to H^{2l}(\Theta^n)$, the latter of which is canonically a subspace of the space of symmetric polynomials with real coefficients of degree l in n variables (in particular when $n = 1$ the function takes values in $\mathbb{R}q^l$). From this point the proof is identical to that of Lemma 2.16. □

We denote this function $|\operatorname{Comp}(\mathfrak{X}, p)_\bullet| \to \mathbb{R}$ by $\hat{\eta}$ as well. Furthermore, Lemma 2.22 shows that Construction 2.19 actually defines a continuous function on $\mathcal{C}omp(\mathfrak{X}, p)$ which gives μ after restricting along the map $\mathcal{D}(\mathfrak{X}, p) \to \mathcal{C}omp(\mathfrak{X}, p)$. We note the following

Lemma 2.23. *There is a finite collection of cones $\sigma_i \in \mathbf{D}(\mathfrak{X}, p)_{n_i}$ such that the corresponding morphism $\sqcup \sigma_i : \coprod h_{[n_i]} \to \operatorname{Comp}(\mathfrak{X}, p)_\bullet$ is surjective on the sets of rays, as well as the projective realizations $\coprod \Delta_{n_i} \to \mathcal{C}omp(\mathfrak{X}, p)$.*

Proof. Without loss of generality we may assume that G is connected. By iterated applications of Theorem 1.21, the connected components of $[\Theta^n, \mathfrak{X}]$ are of the form $Y_{Z,\psi}/P_\psi$ as $\psi : \mathbb{G}_m^n \to G$ varies over conjugacy classes, Z is a connected component of $X^{\psi(\mathbb{G}_m^n)}$, and $Y_{Z,\psi} = Y_{Z,\lambda_1} \cap \cdots \cap Y_{Z,\lambda_n}$, where $\psi = \lambda_1 \times \cdots \times \lambda_n$.

Let $T \subset G$ be a maximal torus. Applying the same analysis, the connected components of $[\Theta^n, X/T]$ are of the form $Y_{Z,\psi}/T$ as ψ varies over homomorphisms $\psi : \mathbb{G}_m^n \to T$. Because $Y_{Z,\psi}/T \to Y_{Z,\psi}/P_\psi$ is a locally trivial bundle with a connected fiber, P_ψ/T, and because every homomorphism to G is conjugate to one which factors through T, it follows that $\pi_0 [\Theta^n, X/T] \to \pi_0 [\Theta^n, X/G]$ is surjective. Likewise, $\operatorname{Comp}(X/T, p)_n \to \operatorname{Comp}(X/G, p)_n$ is surjective for all n.

Thus it suffices to prove the Lemma for $G = T$. In this case $[\Theta^n, X/T]_p$ is an infinite disjoint union of copies of $\operatorname{Spec} k$, one for each pair Z, ψ for which $p \in Y_{Z,\psi}$. It follows that $\mathbf{D}(X/T, p)_n \to \operatorname{Comp}(X/T, p)_n$ is bijective. Lemma 2.13 describes $\mathbf{D}(X/T, p)_\bullet$ explicitly as $R_*(\{\pi^{-1}\sigma_i\})$ where $\pi : N_\mathbb{R} \to N'_\mathbb{R}$ is a linear map and σ_i is a fan of strictly convex rational polyhedral cones in $N'_\mathbb{R}$. Our claim follows from the fact that each cone $\pi^{-1}(\sigma_i)$ can be covered by finitely many simplicial cones. □

[10] Note also that the group of k-points of $\operatorname{Aut}(p)$ also acts on $\mathbf{D}(\mathfrak{X}, p)_\bullet$ and if G is connected, then the morphism $\mathbf{D}(\mathfrak{X}, p)_\bullet \to \operatorname{Comp}(\mathfrak{X}, p)_\bullet$ factors through $\mathbf{D}(\mathfrak{X}, p)_\bullet / \operatorname{Aut}(p)$. One can define a fan which is still coarser than $\operatorname{Comp}(\mathfrak{X}, \phi)_\bullet$ whose set of n-cones is $\operatorname{im}(\operatorname{Comp}(\mathfrak{X}, \phi)_n \to \pi_0 [\Theta^n, \mathfrak{X}])$, but $\operatorname{Comp}(\mathfrak{X}, \phi)_\bullet$ will suffice for our purposes.

Remark 2.24. Note that the map $\bigsqcup_i h_{[n_i]} \to \mathrm{Comp}(\mathfrak{X}, p)_\bullet$ is not surjective as a natural transformation of functors. Even the fan $R_*(\sigma)$, where $\sigma \subset \mathbb{R}^N$ is a strictly convex rational polyhedral classical cone, does not admit a surjection from a finite collection of cones unless σ is simplicial.

As an immediate consequence of Lemmas 2.22 and 2.23, we have

Proposition 2.25. *Let $\mathfrak{X} = X/G$ be a quotient of a quasiprojective k-scheme by a linearizable action of a smooth affine k-group which contains a split maximal torus, and let $\mathcal{D} = \mathcal{D}(\mathfrak{X}, p)$ for some point $p \in \mathfrak{X}(k)$. Let $b \in H^4(\mathfrak{X})$ be positive definite at p, and let $l \in H^2(\mathfrak{X})$ be arbitrary. Then principal (B) holds for μ on $\mathcal{D}$, and hence μ obtains a maximum.*

Remark 2.26 (positive definite classes)**.** A class $b \in H^4(\mathfrak{X})$ is positive definite if and only if the following holds: for every morphism $f : B\mathbb{G}_m \to \mathfrak{X}$ whose map on stabilizer groups has a finite kernel, the class $f^*b \in H^4(B\mathbb{G}_m) \simeq \mathbb{R}q^2$ is positive. Note that the pullback of a positive definite class along the morphism $X/G \to BG$ is still positive definite. Also, when $\mathfrak{X} = X/G$ is a global quotient of a quasiprojective variety by a linearizable group action, an argument analogous to the proof of Lemma 2.23 shows that there is a finite set of morphisms

$$f_i : B(\mathbb{G}_m)^{n_i} \to \mathfrak{X}$$

with finite kernels such that $b \in H^4(\mathfrak{X})$ is positive definite if and only if $f_i^* b \in H^4(B(\mathbb{G}_m)^{n_i}) \simeq \mathrm{Sym}^2(\mathbb{R}^{n_i})$ is a positive definite bilinear form for all i. Hence the set of positive definite forms is the interior of a convex cone of full dimension in $H^4(X/G)$, and small perturbations of a positive definite class remain positive definite.

2.5. Reductive stacks, convexity, and uniqueness of maximal destabilizers.

Recall that a real-valued function on an interval $[0, N]$ is *convex upward* if $\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x + (1-\lambda) y)$ for all $\lambda \in [0, 1]$. It is immediate that such a function either obtains a unique maximum on $[0, N]$ or is constant. Therefore the uniqueness of a maximal destabilizer follows from the following principle, which can be stated for any fan $F_\bullet$:

- (C) For any pair of rational points $x_1, x_2 \in \mathbb{P}(F_\bullet)$ there is a path $\gamma : [0, N] \to \mathbb{P}(F_\bullet)$, obtained by gluing rational 1-simplices, such that $\gamma(0) = x_1$ and $\gamma(N) = x_2$, and $\mu \circ \gamma$ is convex upward and non-constant on $[0, N]$.

We will see that this convexity property holds for $\mathbf{D}(\mathfrak{X}, p)_\bullet$ in two different situations. First we introduce a type of stack which we call *weakly reductive*. For a weakly reductive stack $\mathfrak{X}$, Principle (C) holds automatically for point in $\mathfrak{X}$ and any numerical invariant arising from Construction 2.19. Second, if $\mathfrak{X} = X/G$ where X is projective over affine, $l \in H^2(\mathfrak{X})$ restricts to a NEF class on X, and $b \in H^4(\mathfrak{X})$ is induced from a positive definite class in $H^4(BG)$, then the principal (C) applies as well.

Definition 2.27. Let $\mathfrak{X}$ be a quasi-geometric k-stack locally of finite type. We say $\mathfrak{X}$ is *weakly reductive* if the representable morphism $ev_1 : [\Theta, \mathfrak{X}] \to \mathfrak{X}$ satisfies the valuative criterion for properness. We say that $\mathfrak{X}$ is *reductive* if furthermore the connected components of $T \times_{\mathfrak{X}} [\Theta, \mathfrak{X}]$ are quasi-compact for any finite type k-scheme T and map $T \to \mathfrak{X}$.

Example 2.28. Any stack of the form V/G, where G is a reductive group acting on an affine scheme V. It suffices to verify that a stack is algebraic after base change to an algebraic closure $\bar{k}$. In this case, Theorem 1.21 shows that the connected components of $[\Theta, V/G]$ are over the form Y/P, where Y is a closed subvariety of V, and P is a parabolic subgroup of G. In this case ev_1 is the composition $Y/P \to V/P \to V/G$. The first morphism is a closed immersion, and the second is a locally trivial fiber bundle with fiber G/P, hence proper.

Let $p \in \mathfrak{X}(k)$, and consider two k-points f_1, f_2 of $[\Theta, \mathfrak{X}]_p$. Let $U = \mathbb{A}^1 - \{0\}$. We consider f_1 and f_2 as morphisms $U \times \mathbb{A}^1/\mathbb{G}_m^2 \to \mathfrak{X}$ and $\mathbb{A}^1 \times U/\mathbb{G}_m^2 \to \mathfrak{X}$ respectively with a fixed isomorphism of their restrictions to $U \times U/\mathbb{G}_m^2 \simeq *$, so we can glue them to define

$$f_1 \cup f_2 : (\mathbb{A}^2 - \{0\})/\mathbb{G}_m^2 \to \mathfrak{X}$$

And the data of the morphism $f_1 \cup f_2$ is equivalent to the data of the pair f_1, f_2. To state this more formally, we have an equivalence

$$\underline{\mathrm{Map}}\left((\mathbb{A}^2 - \{0\})/\mathbb{G}_m^2, \mathfrak{X}\right) \xrightarrow{\simeq} [\Theta, \mathfrak{X}] \times_{\mathfrak{X}} [\Theta, \mathfrak{X}],$$

where the fiber product is taken with respect to ev_1. For weakly reductive stacks, any such morphism from $(\mathbb{A}^2 - \{0\})/\mathbb{G}_m^2$ extends uniquely to $\mathbb{A}^2/\mathbb{G}_m^2$.

Proposition 2.29. *Let $\mathfrak{X}$ be a weakly reductive k-stack, and let $p \in \mathfrak{X}(k')$ for some extension k'/k. Then any two rational points in $\mathcal{D}(\mathfrak{X}, p)$ are connected by a unique rational 1-simplex.*

Proposition 2.29 can be interpreted as stating that $\mathcal{D}(\mathfrak{X}, p)$ is convex in a strong sense for weakly reductive $\mathfrak{X}$. The technical heart of the proof is the following lemma.

Lemma 2.30. *Let $f : \Theta^2 \to \mathfrak{X}$ be a morphism such that $\ker(\mathbb{G}_m^2 \to \mathrm{Aut}\, f(0))$ has positive dimensional kernel. Then there is a commutative diagram of the form*

$$\begin{array}{ccccc} \Theta^2 & \longrightarrow & \Theta^2 & \xrightarrow{f} & \mathfrak{X} \\ & \searrow & & \nearrow & \\ & & \Theta & & \end{array}$$

where $\Theta^2 \to \Theta^2$ is of the form $(z_1, z_2) \mapsto (z_1^a, z_2^b)$ for some $a, b > 0$.

Proof. As $G = \ker(\mathbb{G}_m^2 \to \mathrm{Aut}\, f(0))$ is a positive dimensional subgroup, it contains a one dimensional torus G_0 of rank 1. This subgroup is determined by a vector $[a, b]$ in the cocharacter lattice of $\mathbb{G}_m^2$. We prove the claim in three cases:

Case 1: $ab < 0$

By permuting the two coordinates we may assume that $a > 0$ and $b < 0$. Then composing with the non-degenerate map $\Theta^2 \to \Theta^2$ defined by (z_1^a, z_2^{-b}), the new map $\Theta^2 \to \mathfrak{X}$ has a subgroup G_0 corresponding to the vector $[1, -1]$ in the cocharacter lattice of G_0. Let us assume then that $a = 1$ and $b = -1$. So that G_0 is precisely the kernel of the homomorphism $\phi : \mathbb{G}_m^2 \to \mathbb{G}_m$ given by $(z_1, z_2) \mapsto z_1 z_2$.

Consider the morphism $\pi : \Theta^2 \to \Theta$ defined by $(z_1, z_2) \mapsto z_1 z_2$, which is equivariant with respect to ϕ. We claim that the pullback functor π^* induces an isomorphism

$$\pi^* : \mathrm{Perf}(\Theta) \xrightarrow{\simeq} \mathcal{C} := \left\{ F^\bullet \in \mathrm{Perf}(\Theta^2) \,\middle|\, F^\bullet|_{\{0\}} \text{ has weight } 0 \text{ w.r.t. } G_0 \right\} \qquad (10)$$

The proof uses the main structure theorem of [HL2]. The one parameter subgroup $\lambda(t) = z_1^t z_2^{-t}$ generating G_0 defines a KN-stratum $Y = \{(z_1, z_2) | z_2 = 0\} \subset \mathbb{A}^2$, with fixed locus $Z = \{(0,0)\} \in \mathbb{A}^2$. According to the main theorem of [HL2], the restriction functor $\mathcal{C} \to \operatorname{Perf}(\Theta^2 \setminus (Y/\mathbb{G}_m^2)) \simeq \operatorname{Perf}(\mathbb{A}^1_{z_1}/\mathbb{G}_m)$ is a fully faithful functor of symmetric monoidal ∞-categories. Furthermore the composition

$$(\mathbb{A}^1_{z_1} \times \{1\})/\mathbb{G}_m \to \mathbb{A}^2/\mathbb{G}_m^2 \xrightarrow{\pi} \mathbb{A}^1/\mathbb{G}_m$$

is an isomorphism of stacks. It follows that the functor $\pi^* : \operatorname{Perf}(\Theta) \to \mathcal{C}$ is an equivalence.

It follows that any symmetric monoidal functor $\operatorname{Perf}(\mathfrak{X}) \to \operatorname{Perf}(\Theta^2)$ landing in the subcategory $\mathcal{C}$ factors uniquely as a symmetric monoidal functor through $\pi^* : \operatorname{Perf}(\Theta) \to \operatorname{Perf}(\Theta^2)$. The same argument applies to the categories APerf and $\operatorname{APerf}^{cn}$. Hence the Tannakian formalism [L2] implies that the morphism $f : \Theta^2 \to \mathfrak{X}$ must factor uniquely through $\pi : \Theta^2 \to \Theta$.

Case 2: $ab > 0$

In this case we can choose a generator for G_0 of the form $[a, b]$ with $a, b > 0$. Again we use the results of [HL2]. We can consider Θ^2 to consist of a single Θ-stratum with one parameter subgroup $\lambda(t) = z_1^{at} z_2^{bt}$, and $Z = \{(0)\} \in \mathbb{A}^2$. In this case the baric structure on the KN-stratum Θ^2 shows that the pullback along the projection $\pi : \Theta^2 \to B\mathbb{G}_m^2$ induces an equivalence

$$\pi^* : \{F^\bullet \in \operatorname{Perf}(B\mathbb{G}_m^2) \text{ with } \lambda\text{-weight0}\} \to \{F^\bullet \in \operatorname{Perf}(\Theta^2) | F^\bullet|_Z \text{ has } \lambda\text{-weight } 0\}$$

The subcategory of $\operatorname{Perf}(B\mathbb{G}_m^2)$ can be identified with the category $\operatorname{Perf}(B\mathbb{G}_m)$ via the pullback functor along the morphism $\phi : B\mathbb{G}_m^2 \to B\mathbb{G}_m$, where ϕ is the group homomorphism $(z_1, z_2) \mapsto z_1^{-b} z_2^a$. As in the previous case, this argument extends to the categories APerf and $\operatorname{APerf}^{cn}$ as well, so the Tannakian formalism implies that the map $\Theta^2 \to \mathfrak{X}$ factors through the composition $\Theta^2 \to B\mathbb{G}_m^2 \xrightarrow{\phi} B\mathbb{G}_m$.

Case 3: $ab = 0$

In this case, we may assume, by permuting coordinates, that $a = 0$, so that G_0 is generated by the one parameter subgroup $\lambda(t) = z_2^t$. Let $Z = \mathbb{A}^1_{z_1} \times \{0\} \subset \mathbb{A}^2$, and note that this subvariety is equivariant with respect to $\mathbb{G}_m^2$ and fixed by λ. Any $F^\bullet \in \operatorname{Perf}(Z/\mathbb{G}_m^2)$ decomposes canonically into a direct sum of objects on which λ acts with fixed weight. Because any $\mathbb{G}_m^2$ equivariant open subset containing (0) contains all of Z, semicontinuity implies that if $F^\bullet|_{\{0\}}$ is concentrated in weight 0 with respect to λ, then it is concentrated in weight 0 everywhere on Z. Therefore, if $F^\bullet \in \operatorname{Perf}(\Theta^2)$ and $F^\bullet|_{\{0\}}$ has weight 0 with respect to λ, then $F^\bullet|_Z$ has weight 0 with respect to λ as well.

If we consider the projection $\pi : \Theta^2 \to \Theta$ given by $(z_1, z_2) \mapsto z_1$, then the counit of adjunction is an isomorphism $\operatorname{id}_{\operatorname{Perf}(\Theta)} \simeq R\pi_* L\pi^*$. It follows that π^* is fully faithful. Its image is exactly the full subcategory $\{F^\bullet \in \operatorname{Perf}(\Theta^2) | F|_Z \text{ has } \lambda\text{-weight } 0\}$, which we have seen is precisely the subcategory $\{F^\bullet \in \operatorname{Perf}(\Theta^2) | F^\bullet|_{\{0\}} \text{ has } \lambda\text{-weight } 0\}$. As in the previous cases, the Tannakian formalism implies that $f : \Theta^2 \to \mathfrak{X}$ factors uniquely through π.

□

Proof of Proposition 2.29. Fix two maps $f_1, f_2 : \Theta_{k'} \to \mathfrak{X}$. Then finding a morphism $g : \Theta^2_{k'} \to \mathfrak{X}$ along with an isomorphism of the restriction of g to $(\mathbb{A}^2_{k'} - \{0\})/\mathbb{G}_m^2$

with $f_1 \cup f_2$ is equivalent to finding a lift in the diagram

$$\begin{array}{ccc} (\mathbb{A}^1_{k'} - \{0\})/\mathbb{G}_m \simeq \operatorname{Spec} k' & \longrightarrow & [\Theta, \mathfrak{X}] \\ \downarrow & \nearrow & \downarrow {\scriptstyle ev_1} \\ \mathbb{A}^1_{k'}/\mathbb{G}_m & \xrightarrow{f_1} & \mathfrak{X} \end{array}$$

where the left vertical morphism is the inclusion of the point $\{1\}$, and the top horizontal morphism classifies the morphism $f_2 : \Theta_{k'} \to \mathfrak{X}$.

After restricting f_1 along $\mathbb{A}^1_{k'} \to \mathbb{A}^1_{k'}/\mathbb{G}_m$, the valuative criterion for properness for the morphism ev_1 guarantees the existence and uniqueness of a lift compatible with the given map $\mathbb{A}^1 - \{0\} \to [\Theta, \mathfrak{X}]$. Because ev_1 is separated, the restriction of this lift to $\mathbb{G}_m \times \mathbb{A}^1_{k'}$ along either morphism in the groupoid Equation 1 is uniquely determined by its restriction to $\mathbb{G}_m \times (\mathbb{A}^1_{k'} - \{0\})$. This provides descent datum satisfying the cocycle condition for the lift $\mathbb{A}^1_{k'} \to [\Theta, \mathfrak{X}]$. Hence there exists a unique lift $\Theta_{k'} \to [\Theta, \mathfrak{X}]$.

In order to complete the proof, we must show that if f_1 and f_2 are non-degenerate and distinct, then the unique extension of $f_1 \cup f_2$ to $\Theta^2_{k'}$ is also non-degenerate. Because f_1 and f_2 are non-degenerate, the only point at which the maps of stabilizer groups can have a positive dimensional kernel is the origin, and Lemma 2.30 implies that this can not happen if f_0 and f_1 are distinct. □

Corollary 2.31. *Let $\mathfrak{X}$ be a weakly reductive stack. Let $l \in H^2(\mathfrak{X})$ and let $b \in H^4(\mathfrak{X})$ be positive definite. Then for any extension k'/k and each $p \in \mathfrak{X}(k')$, the numerical invariant μ of Construction 2.19 satisfies Principal (C).*

Proof. Let $\Delta_{n-1} \to \mathcal{D}(\mathfrak{X}, p)$ be a rational simplex. Then $\mu = \hat{l}/\sqrt{\hat{b}}$ restricted to Δ_{n-1} is the quotient of a rational linear form by the square root of a positive definite rational quadratic form on $(\mathbb{R}_{\geq 0})^n - \{0\}$. It is an elementary fact that such a function is convex upward and non-constant, and in fact its maximum is obtained at a unique rational point on Δ_{n-1}.

Proposition 2.25 implies that Principle (B) holds for μ on $\mathcal{D}(\mathfrak{X}, p)$, hence a global maximum of μ is obtained at some rational point. Proposition 2.29 implies that any two points with $\mu > 0$ are connected by a rational 1-simplex, and we have observed that μ is convex upward on this segment. Hence the maximizer is unique by Principle (C).

□

Example 2.32. Continuing Example 2.28, we may apply Corollary 2.31 to the quotient of an affine scheme by a reductive group V/G. Combined with Proposition 2.25, this recovers the main result of [K1]: an affirmative answer to Question 2.2 for all unstable points in V/G. In fact we have a slight generalization of this result – $l \in H^2(V/G)$ can be arbitrary, and $b \in H^4(V/G)$ can be any positive definite class.

Principle (C) fails for a general quotient stack $\mathfrak{X} = X/G$ and cohomology classes l and b, where X is a quasiprojective scheme with a linearizable action of a split reductive G. To construct a counterexample, one can consider quotients of toric varieties X/T where $p \in X$ is a point in the open orbit, and $l = c_1(L)$ is the first Chern class of an equivariant invertible sheaf (see also Example 3.11 above). Such classes l correspond to arbitrary integral piecewise linear functions on the fan of the toric variety, and it is straightforward to construct examples where the subset $\{\hat{l} > 0\}$ is not connected, and also examples where $\hat{l}(x)/|x|$ has a non-unique maximizer.

Nevertheless when X is projective-over-affine, and $l \in H^2(X/G)$ restricts to a suitably positive class in $H^2(X)$, one can still verify Principle (C).

Proposition 2.33. *Let G be a split reductive group acting on an affine k-scheme, V, of finite type, let $X \to V$ be a G-equivariant projective morphism, and consider the stack $\mathfrak{X} = X/G$. Let $l \in H^2(\mathfrak{X})$ be such that its restriction to X is NEF, and let $b \in H^4(\mathfrak{X})$ be pulled back from a positive definite class on V/G. Then for every unstable point $p \in \mathfrak{X}(k)$, $\mu(f)$ obtains a maximum at a unique rational point $f \in \mathcal{D}(\mathfrak{X}, p)$.*

Proof. In light of Proposition 2.25, one only needs to verify (C). Let $f_1, f_2 \in \mathbf{D}(\mathfrak{X}, p)_1$ be distinct points with $\mu(f_1), \mu(f_2) > 0$. From the proof of Proposition 2.29, we see that the composition $f_1 \cup f_2 : (\mathbb{A}^2 - \{0\})/\mathbb{G}_m^2 \to X/G \to V/G$ extends uniquely to $\mathbb{A}^2/\mathbb{G}_m^2$. Furthermore by Theorem 1.21, we may assume that this morphism factors through V/T for some maximal torus $T \subset G$. We also have a lift of the restriction of this map to $(\mathbb{A}^2 - \{0\})/\mathbb{G}_m^2$ to X/T.

We claim that the induced homomorphism $\lambda_1 \times \lambda_2 : \mathbb{G}_m^2 \to T$ has finite kernel. If not then $\lambda_1^m = \lambda_2^{-n}$ for some $m, n > 0$, and the normalization of the orbit closure of p under λ_1 is $\mathbb{P}^1 \to X$ with $\mathbb{G}_m$ acting on $\mathbb{P}^1$ by rotation. The map to X is finite, hence $l|_{\mathbb{P}^1}$ is NEF, and in fact is the Chern class of a $\mathbb{G}_m$-equivariant invertible sheaf. If we fix our coordinates and $\mathbb{G}_m$ action on $\mathbb{P}^1$ such that $\lim_{t\to 0} t \cdot z = \{0\}$ for generic $z \in \mathbb{P}^1$, then it is an elementary computation that a $\mathbb{G}_m$-equivariant NEF invertible sheaf on $\mathbb{P}^1$ can not have the property that $\mathrm{wt}_{\mathbb{G}_m}(L_{\{0\}}) < 0$ and $\mathrm{wt}_{\mathbb{G}_m}(L_{\{\infty\}}) > 0$. This contradicts the hypothesis that $\mu(f_1), \mu(f_2) > 0$.

For some smooth toric blowup, $S \to \mathbb{A}^2$, there is a $\mathbb{G}_m^2$-equivariant map $\phi : S \to X$ extending the map $\mathbb{A}^2 - \{0\} \to X$ and lying above the map $\mathbb{A}^2 \to V$.

$$\begin{array}{ccccc} & & S & \xrightarrow{\phi} & X \\ & \nearrow & \downarrow & & \downarrow \\ \mathbb{A}^2 - \{0\} & \hookrightarrow & \mathbb{A}^2 & \longrightarrow & V \end{array}$$

The fan of S is a subdivision of the cone $(\mathbb{R}_{\geq 0})^2$. Because $\lambda_1 \times \lambda_2$ has finite kernel, we have a map of fans $\mathbf{D}(S/\mathbb{G}_m^2, 1)_\bullet \to \mathbf{D}(\mathfrak{X}, p)_\bullet$. Each two-dimensional cone in the fan of S defines a rational 1-simplex in $\mathcal{D}(\mathfrak{X}, p)$, and $\gamma : [0, N] \simeq \mathcal{D}(S/T, 1) \to \mathcal{D}(\mathfrak{X}, p)$ is a path connecting the points corresponding to f_1 and f_2.

We claim that μ is convex upward and non-constant along this path. Note that the restriction of μ to $\mathcal{D}(S/T, 1)$ is the numerical invariant defined, via Construction 2.19, by a NEF class $l \in H^2(S/T)$ and $b \in H^4(S/T)$ induced by a positive definite class in $H^4(BT)$. By Proposition 2.13, $\mathcal{D}(S/T, p)$ is the support of the fan of Y, and by Remark 2.18 for any NEF class l, $\hat{l}$ is a convex upward piecewise linear function on the the fan. Furthermore, because b is the restriction of a class in $H^4(\mathbb{A}^2/\mathbb{G}_m^2) \simeq H^4(B\mathbb{G}_m^2)$, it follows that $\hat{b}(x) = |x|^2$ for some inner product norm on $\mathbb{R}^2$. We leave it as an exercise that $\mu(x) = \hat{l}(x)/|x|$ defines a convex upward and non-constant function on $\mathcal{D}(S/\mathbb{G}_m^2, p) \simeq ((\mathbb{R}^2_{\geq 0}) - \{0\})/\mathbb{R}^\times$. □

3. The notion of a Θ-stratification

Let us recall the construction of the Kempf-Ness (KN) stratification of the unstable locus in GIT. Let X be a projective over affine variety with a reductive group action, and let μ be the numerical invariant (5). We think of this as a function on pairs (Z, λ), where $\lambda : \mathbb{G}_m \to G$ and Z is a connected component of X^λ, so we let $\mu(Z, \lambda) = \frac{-1}{|\lambda|} \operatorname{weight}_\lambda \mathcal{L}|_Z \in \mathbb{R}$, and $\mu(X, \lambda) = 0$ for the trivial λ.

One constructs the KN stratification iteratively by selecting a pair $(Z_\alpha, \lambda_\alpha)$ which maximizes μ among those (Z, λ) for which Z is not contained in the union of the previously defined strata. One defines the open subset $Z_\alpha^\circ \subset Z_\alpha$ not intersecting any higher strata, and the attracting set $Y_\alpha^\circ := \pi^{-1}(Z_\alpha^\circ) \subset Y_{Z_\alpha,\lambda_\alpha}$. One also defines $P_\alpha = P_{Z_\alpha,\lambda_\alpha}$ and the new strata is defined to be $S_\alpha = G \cdot Y_\alpha^\circ$.

The strata are pre-ordered by the value of the numerical invariant $\mu_\alpha := \mu(Z_\alpha, \lambda_\alpha)$. Note that by construction, for any unstable point $p \in X$, the λ_α corresponding to the stratum containing p is conjugate to an optimal destabilizer of p. However, the stratification has some properties which are not immediate from its construction. For instance

$$\overline{S_\alpha} \subset S_\alpha \cup \bigcup_{\mu_\beta > \mu_\alpha} S_\beta,$$

and the canonical morphism $G \times_{P_\alpha} Y_\alpha^\circ \to S_\alpha$ is an isomorphism for all α. As a consequence $S_\alpha/G \simeq Y_\alpha^\circ/P_\alpha$, and by Theorem 1.21 and Proposition 1.4 this is a connected component of $[\Theta, X_\alpha/G]$, where we let $X_\alpha = X \setminus \bigcup_{\mu_\beta > \mu_\alpha} S_\beta$.

In this section we shall generalize the construction of the Kempf-Ness stratification to arbitrary quasi-geometric stacks using the modular interpretation of subsection 1.3. We first generalize the notion of the numerical invariant $\mu(Z, \lambda)$, present a formal definition of a Θ-stratification as an open substack of $[\Theta, \mathfrak{X}]$ satisfying additional hypotheses, and discuss how a numerical invariant defines a subset of $|[\Theta, \mathfrak{X}]|$ which is a candidate for a Θ-stratification. It is more difficult to specify conditions under which a general numerical invariant on a general stack defines a Θ-stratification. One result in that direction is given in Theorem 3.16.

3.1. **Formal definition of a numerical invariant and a Θ-stratification.** Let us recast the construction of the Kempf-Ness stratification intrinsically. We regard $\mu(Z, \lambda)$ as an $\mathbb{R}$-valued function on the set of connected components, $\pi_0\,[\Theta, \mathfrak{X}]$, or equivalently a locally constant function on $|\,[\Theta, \mathfrak{X}]\,|$.[11] The $(Z_\alpha, \lambda_\alpha)$ which maximize μ at a given stage in the construction of the KN stratification are not unique, because $\mu(Z_\alpha, \lambda_\alpha) = \mu(Z_\alpha, \lambda_\alpha^n)$, even though this does not change S_α/G. More precisely, the locally constant function μ is invariant with respect to the $\mathbf{N}^\times$ action on $[\Theta, \mathfrak{X}]$, so we can think of μ as a function $\mu : \pi_0\,[\Theta, \mathfrak{X}]\,/\mathbf{N}^\times \to \mathbb{R}$.

Recall from Definition 2.20 the constrtuction of the formal fan $\operatorname{Comp}(\mathfrak{X}, \phi)_\bullet$ from any morphism $\phi : S \to \mathfrak{X}$. There we regarded S as a scheme, but if fact S can be any stack or pre-stack, and $\mathfrak{X}$ need not be an algerbaic stack either. The set of connected components of a stack can be defined as the set of points modulo the smallest equivalence relation identifying any two points on a connected scheme mapping to $\mathfrak{X}$. Thus the set $\pi_0(S \times_{\mathfrak{X}} [\Theta, \mathfrak{X}])_\bullet$ is well-defined. The fan $\operatorname{Comp}(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})_\bullet$ admits a canonical map of fans $\operatorname{Comp}(\mathfrak{X}, \phi)_\bullet \to \operatorname{Comp}(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})_\bullet$ for any map from a pre-stack

[11] Recall that the topological space $|\mathfrak{Y}|$ associated to an algebraic stack consists of all field valued points of $\mathfrak{Y}$ up to the equivalence relation generated by field extension and 2-isomorphism of points.

$\phi : S \to \mathfrak{X}$. Note that the rational points of $\mathcal{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})$ correspond exactly to the set $\pi_0 [\Theta, \mathfrak{X}] / \mathbf{N}^\times$.

Definition 3.1. A *numerical invariant* for the k-stack $\mathfrak{X}$ is a continuous real-valued function[12] defined on an open subset $U \subset \mathcal{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})$. Given a numerical invariant we define the *stability function* $M^\mu : |\mathfrak{X}| \to \mathbb{R}$ as

$$M^\mu(p) = \sup \{ \mu(f) \,|\, f \in U \text{ with } f(1) = p \in |\mathfrak{X}| \} ,$$

where we are abusing notation by letting $f \in U$ to denote that the connected component of $[\Theta, \mathfrak{X}]$ containing f, when regarded as a rational point of $\mathcal{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})$, lies in U, and by letting $\mu(f)$ denote the value of μ in this case.

Remark 3.2. When $\mathfrak{X}$ is quasi-geometric and locally of finite presentation, $ev_1 : [\Theta, \mathfrak{X}] \to \mathfrak{X}$ is representable by algebraic spaces, locally of finite presentation. It follows that for a geometric point $p : \operatorname{Spec} l \to \mathfrak{X}$, every connected component of the fiber $[\Theta, \mathfrak{X}]_p$ contains an l-point, and hence

$$M^\mu(p) = \sup \{ \mu(f) \,|\, f : \Theta_l \to \mathfrak{X} \text{ with } f(1) \simeq p \in \mathfrak{X}(l) \}$$

Furthermore, if there is some point $f \in |\,[\Theta, \mathfrak{X}]\,|$ for which $M^\mu(f(1)) = \mu(f)$, then there is an l-point of $[\Theta, \mathfrak{X}]$ with this property.

Remark 3.3. Given a numerical invariant, μ, for $\mathfrak{X}/k$ and a field extension k'/k, we can define a numerical invariant, μ', for $\mathfrak{X}_{k'} := \mathfrak{X} \times_{\operatorname{Spec} k} \operatorname{Spec} k'$, regarded as a stack over $\operatorname{Spec} k'$. Indeed we have

$$\underline{\mathrm{Map}}_{k'}(\Theta^n_{k'}, \mathfrak{X}_{k'}) \simeq \underline{\mathrm{Map}}_k(\Theta^n_k, \mathfrak{X}) \times_k k',$$

and hence a map $\pi_0 \mathrm{Map}_{k'}(\Theta^n_{k'}, \mathfrak{X}_{k'}) \to \pi_0 [\Theta^n, \mathfrak{X}]$. $M^{\mu'}$ is the restriction of M^μ along the map $|\mathfrak{X}_{k'}| \to |\mathfrak{X}|$, hence we will unambiguously denote both as M^μ.

Our primary examples of numerical invariants come from Construction 2.19. Lemma 2.22 shows that for any $p \in \mathfrak{X}(k)$, the function μ defined on $\mathcal{D}(\mathfrak{X}, p)$ is induced by a function on $\mathcal{C}omp(\mathfrak{X}, p)$ under the canonical map $\mathcal{D}(\mathfrak{X}, p) \to \mathcal{C}omp(\mathfrak{X}, p)$. It is clear from this construction that in fact the function μ is in fact restricted via the map $\mathcal{C}omp(\mathfrak{X}, p) \to \mathcal{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})$, and thus μ is a numerical invariant in the sense of Definition 3.1.

Recall from Lemma 1.5 and the discussion which followed it that for any set of connected components $\{\mathfrak{Y}_\alpha \subset \mathfrak{X}\}$ on which $\mathbf{N}^\times$ acts freely, we can canonically construct an algebraic stack $\operatorname{colim}_{\mathbf{N}} \mathfrak{Y}$ with an evaluation map $ev_1 : \operatorname{colim}_{\mathbf{N}} \mathfrak{Y} \to \mathfrak{X}$ extending ev_1 on each of the open substacks $\mathfrak{Y}_\alpha \subset \operatorname{colim}_{\mathbf{N}} \mathfrak{Y}$.

Definition 3.4. Let $\mathfrak{X}$ be a locally finite type quasi-geometric k-stack. A Θ-*stratification* of $\mathfrak{X}$ consists of:

(i) a family of open substacks $\mathfrak{X}_{\leq c}$ for $c \in \mathbb{R}_{\geq 0}$ such that $\mathfrak{X}_{\leq c} \subset \mathfrak{X}_{\leq c'}$ for $c' > c$ and $\mathfrak{X} = \bigcup_c \mathfrak{X}_{\leq c}$, and

[12] By continuity such a function is uniquely determined by its restriction to rational points, which is simply the data of a function $\mu : \pi_0 [\Theta, \mathfrak{X}] / \mathbf{N}^\times \to \mathbb{R}$, or a locally constant function on $[\Theta, \mathfrak{X}]$ which is invariant under the $\mathbf{N}^\times$-action. For many of the definitions and constructions which follow, one only requires sucha function, without it extending continuously to any subset of $\mathcal{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})$. However, the continuity of μ appears crucial to the analysis of numerical invariants in examples, and thus we include it as part of the definition.

(ii) a $\mathbf{N}^\times$-equivariant collection of connected components $\mathfrak{S}_c \subset [\Theta, \mathfrak{X}_{\leq c}]$ for all $c > 0$ such that $\mathbf{N}^\times$ acts freely on $\mathfrak{S}_c$ and $ev_1 : \operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c \to \mathfrak{X}_{\leq c}$ is a closed immersion whose image is the complement of $\mathfrak{X}_{<c}$ in $\mathfrak{X}_{\leq c}$,

where $\mathfrak{X}_{<c} := \bigcup_{c'<c} \mathfrak{X}_{\leq c'}$. A structure meeting all of these properties except that $ev_1 : \operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c \to \mathfrak{X}_{\leq c}$ is merely finite and radicial will be denoted a *weak* Θ*-stratification*. Given a (weak) Θ-stratification of $\mathfrak{X}$, we refer to the open substack $\mathfrak{X}_{\leq 0}$ as the *semistable locus* and its complement in $|\mathfrak{X}|$ as the *unstable locus*.

Lemma 3.5. *A weak* Θ*-stratification is a* Θ*-stratification if and only if the map on tangent spaces,* $T_f \mathfrak{S}_c \to T_{f(1)}\mathfrak{X}$*, is injective for all* $f \in \mathfrak{S}_c(\bar{k})$ *which lie in the image of* $\sigma : [B\mathbb{G}_m, \mathfrak{X}] \to [\Theta, \mathfrak{X}]$*,*

Proof. This follows from the openness of condition that ev_1 is an immersion, and the fact that any open set of $|\,[\Theta,\mathfrak{X}]\,|$ which contains $\sigma \circ ev_0(f)$ also contains f. This is proven most succiently (albeit a little confusingly) by the fact that any k-point $f \in [\Theta, \mathfrak{X}]\,(k)$ canonically defines a morphism $\Theta \to [\Theta, \mathfrak{X}]$ whose generic point corresponds to f and whose special point corresponds to $\sigma \circ ev_0(f)$. The morphism is classified by the diagram

$$\begin{array}{ccccc} \Theta \times \Theta & \xrightarrow{(x,y)\mapsto xy} & \Theta & \xrightarrow{f} & \mathfrak{X} \\ \downarrow & & & & \\ \Theta & & & & \end{array}$$

The preimage of any open subset of $[\Theta, \mathfrak{X}]$ must be an open subset of Θ, and any open subset of Θ containing the origin is all of Θ. $\square$

Example 3.6. The relevant modular example for the failure of the map on tangent spaces to be injective is the failure of Behrend's conjecture for the moduli of G-bundles on a curve in finite characteristic [H1]. In that example, the moduli of G bundles on a smooth projective curve C is stratified by the type of the canonical parabolic reduction of a given unstable G-bundle. In finite characteristic there are examples where the map $H^1(C, \mathfrak{p}) \to H^1(C, \mathfrak{g})$ is not injective, where $\mathfrak{g}$ is the adjoint bundle of a principle G-bundle and $\mathfrak{p}$ the adjoint bundle of its canonical parabolic reduction.

If the connected components of $\operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c$ are quasi-compact – this happens, for instance, when $\mathfrak{X}$ is finite type, because $\operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c \to \mathfrak{X}_{\leq c}$ is a closed immersion – then the colimit must stabilize. So $\operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c$ can be identified with a suitable set of connected components of $\mathfrak{S}_c$, one for each $\mathbf{N}^\times$-orbit on $\pi_0(\mathfrak{S}_c)$. The colimit need not stabilize in general, although the colimit stabilizes after base change to any finite type open substack $\mathfrak{X}' \subset \mathfrak{X}$.

In the case of a global quotient stack, however, the situation is better. Theorem 1.21 shows that the maps $(\bullet)^n$ between connected components of $[\Theta, X/G]$ are isomorphisms, hence $\operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c$ can be identified with *any* subset of $\pi_0(\mathfrak{S}_c)$ which form a complete set of representatives of $\mathbf{N}^\times$-orbits. The natural generalization of this is the following

Lemma 3.7. *Let* $\mathfrak{X} \subset \mathfrak{X}'$ *be an open immersion, with* $\mathfrak{X}'$ *reductive. Then the map* $(\bullet)^n$ *between any two connected components of* $[\Theta, \mathfrak{X}]$ *is an isomorphism.*

Proof. Let $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ are two connected components of $[\Theta, \mathfrak{X}']$ related by $(\bullet)^n$. After base change along an fppf morphism $T \to \mathfrak{X}'$, $\mathfrak{Y}_1 \times_{\mathfrak{X}'} T \to \mathfrak{Y}_2 \times_{\mathfrak{X}'} T$ is an open immersion of spaces which are proper over T, hence an isomorphism.

Now let $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ be two connected components of $[\Theta, \mathfrak{X}]$ related by $(\bullet)^n$. Then there images in $\pi_0([\Theta, \mathfrak{X}'])$ are related by $(\bullet)^n$ as well, and we denote these connected components $\mathfrak{Y}'_1, \mathfrak{Y}'_2 \subset [\Theta, \mathfrak{X}']$. By Proposition 1.4, the open substack $[\Theta, \mathfrak{X}] \subset [\Theta, \mathfrak{X}']$ can be described as the preimage of $\mathfrak{X}$ under the morphism $[\Theta, \mathfrak{X}'] \to [B\mathbb{G}_m, \mathfrak{X}'] \to \mathfrak{X}'$. It follows from this description that $(\bullet)^n : \mathfrak{Y}'_1 \cap [\Theta, \mathfrak{X}] \to \mathfrak{Y}'_2 \cap [\Theta, \mathfrak{X}]$ is an isomorphism as well. Now $\mathfrak{Y}'_1 \cap [\Theta, \mathfrak{X}]$ and $\mathfrak{Y}'_2 \cap [\Theta, \mathfrak{X}]$ are disjoint unions of connected components of $[\Theta, \mathfrak{X}]$, so because the morphism $(\bullet)^n$ is an isomorphism it must also be an isomorphism on each of these connected components. In particular $(\bullet)^n : \mathfrak{Y}_1 \to \mathfrak{Y}_2$ is an isomorphism. □

In the following section, we will be mostly concerned with stacks as in the previous lemma, so for our purposes we can consider $\operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c$ to be a selection of connected components of $\mathfrak{S}_c$ representing each of the $\mathbf{N}^\times$-orbits.

Next, by Proposition 1.4 the canonical map $[\Theta, \mathfrak{X}_{\leq c}] \to [\Theta, \mathfrak{X}]$ is an open immersion, so we can consider the open substack

$$\mathfrak{S} := \bigcup_c \mathfrak{S}_c \subset [\Theta, \mathfrak{X}] \,.$$

Evidently this open substack, along with the information of the locally constant function[13] $\mu : \mathfrak{S} \to \mathbb{R}$ whose value on $\mathfrak{S}_c$ is c, uniquely determine the Θ-stratification. We also note the inductive nature of this definition, that $\mathfrak{X}_{\leq c}$ admits a Θ-stratification by $\mathfrak{S}_{c'}$ with $c' \leq c$. Also, Proposition 1.4 implies that if $\mathfrak{X}$ admits a Θ-stratification and $\mathfrak{Y} \subset \mathfrak{X}$ is a closed substack, then $\mathfrak{Y}_{\leq c} := \mathfrak{X}_{\leq c} \times_{\mathfrak{X}} \mathfrak{Y}$ defines a Θ-stratification as well.

Example 3.8. The KN stratification of the unstable locus in GIT is the motivating example of a Θ-stratification. Via Theorem 1.21 the morphism $Y^\circ_\alpha / P_\alpha \to X/G$ coming from the immersion $Y^\circ_\alpha \subset X$ which intertwines the inclusion $P_\alpha \subset G$ corresponds to the restriction ev_1. We will see that Kempf's theorem implies that when the choices are organized according to the numerical invariant μ as above, and $\operatorname{char} k = 0$, the morphism ev_1 is a locally closed immersion.

More generally when $\mathfrak{X}$ is a quotient stack, then Theorem 1.21 shows that a Θ-stratification is exactly what was referred to as a KN stratification in [HL2].

Example 3.9. Let X be a scheme with a locally affine action of a torus G, and let $\lambda : \mathbb{G}_m \to T$ be a one parameter subgroup such that $G = P_\lambda$. Then the strata S_α of X with respect to the action of $\mathbb{G}_m$ (See Proposition 1.15) are G-equivariant, and hence their disjoint union modulo G gives a Θ-stratification.

Let $\mathfrak{S}^\star \subset [\Theta, \mathfrak{X}]$ be a union of connected components containing $\mathfrak{S}$, and assume that $ev_1^{-1}(\mathfrak{X}_{\leq c}) \cap \mathfrak{S}^\star = \mathfrak{S}$ and the locally constant function $\mu : \mathfrak{S} \to \mathbb{R}$ extends[14] to $\mathfrak{S}^\star$, the Θ-stratification can be reconstructed uniquely from the set of connected components $\mathfrak{S}^\star$ and μ as follows: We let $|\mathfrak{X}|_{>c}$ be the image of $\mathfrak{S}^\star_{c'}$ for $c' > c$ under

[13] At this point, we are overloading the symbol μ, but as we will see in Construction 3.10, this function will often be the restriction of a numerical invariant to $\mathfrak{S}$, so the notation is justified.

[14] Typically μ will be the restriction of a function on $[\Theta, \mathfrak{X}]$, so it will extend to $\mathfrak{S}^\star$.

ev_1,[15] then we define $\mathfrak{X}_{\leq c}$ to be the complement of $|\mathfrak{X}|_{>c}$, and define $\mathfrak{S}_c$ to be $ev_1^{-1}(\mathfrak{X}_{\leq c}) \cap \mathfrak{S}^\star$.

On the other hand, if we are given an $\mathbf{N}^\times$-equivariant collection of connected components $\mathfrak{S}^\star \subset [\Theta, \mathfrak{X}]$ such that $\mathbf{N}^\times$ acts freely on $\pi_0(\mathfrak{S}^\star)$, and given a locally constant function $\mu : \mathfrak{S}^\star \to \mathbb{R}_{>0}$, we can check whether this defines a Θ-stratification. Specifically, we must check that

(1) the subset $|\mathfrak{X}|_{\mu > c} := \mathrm{im}\left(ev_1 : \mathfrak{S}^\star_{\mu > c} \to \mathfrak{X}\right) \subset |\mathfrak{X}|$ is a closed, and
(2) if we define the open substack $\mathfrak{X}_{\leq c} := \mathfrak{X} \setminus |\mathfrak{X}|_{>c}$, then $\mathfrak{S}_c := \mathfrak{S}^\star_c \cap ev_1^{-1}(\mathfrak{X}_{\leq c})$ is contained in the open substack $[\Theta, \mathfrak{X}_{\leq c}] \subset [\Theta, \mathfrak{X}]$, and the morphism $ev_1 : \mathrm{colim}_{\mathbf{N}} \mathfrak{S}_c \to \mathfrak{X}_{\leq c}$ is a closed immersion.

This shows that a large class of Θ-stratification admit purely topological descriptions, in the sense that specifying the Θ-stratification amounts to selecting a sequence of connected components of $[\Theta, \mathfrak{X}]$ such that (1) and (2) hold.

Given a numerical invariant on $\mathfrak{X}$, there is a canonical method of selecting $\mathfrak{S}^\star$ and μ which directly imitates GIT, although not every numerical invariant will actually result in a Θ-stratification (i.e. $\mathfrak{S}^\star$ will not necessarily satisfy (1) and (2)).

Construction 3.10 (Potential Θ-stratification)**.** Given a numerical invariant $\mu : U \subset \mathcal{C}omp(\mathfrak{X}, \mathrm{id}_\mathfrak{X}) \to \mathbb{R}$, let $\mathfrak{S}^\star \subset [\Theta, \mathfrak{X}]$ consist of the connected components which correspond to a rational point of U on which $\mu > 0$.

Let us assume that μ is a numerical invariant on $\mathfrak{X}$, and Construction 3.10 does define a Θ-stratification. Then the unstable locus $|\mathfrak{X}|_{>0}$ defined in Definition 3.4 is exactly the set of points of the form $f(1)$ for some f with $\mu(f) > 0$, so the notion of instability agrees with that of Section 2. Furthermore, for every unstable point, the supremum defining $M^\mu(p)$ is achieved at some point in $[\Theta, \mathfrak{X}]$. It follows that $\mathfrak{S} \subset [\Theta, \mathfrak{X}]$ is the open substack consisting of points such that $M^\mu(f(1)) = \mu(f)$.

Despite the large class of examples coming from GIT, let us highlight some of the perils of attempting to construct a Θ-stratification using Construction 3.10. When $\mathfrak{X}$ is a global quotient by an abelian group, then ev_1 is a local immersion, however this can fail for global quotients by non-abelian G. For instance, for $\mathfrak{X} = BG$, the fiber of ev_1 over $* \to BG$ is $\bigsqcup G/P_\lambda$. Furthermore, ev_1 need not be proper on connected components:

Example 3.11. Let $V = \operatorname{Spec} k[x,y,z]$ be a linear representation of $\mathbb{G}_m$ where x, y, z have weights $-1, 0, 1$ respectively, and let $X = V - \{0\}$ and $\mathfrak{X} = X/\mathbb{G}_m$. The fixed locus is the punctured line $Z = \{x = z = 0\} \cap X$, and the connected component of $[\Theta, \mathfrak{X}]$ corresponding to the pair $[Z, \lambda(t) = t]$ is the quotient $S/\mathbb{G}_m$ where

$$S = \{(x,y,z) | z = 0 \text{ and } y \neq 0\}$$

$S \subset X$ is not closed. Its closure contains the points where $x \neq 0$ and $y = 0$. These points would have been attracted by λ to the missing point $\{0\} \in V$. It follows that $S/\mathbb{G}_m \to X/\mathbb{G}_m$ is not proper.

3.2. **Construction of Θ-stratifications.** For any extension k'/k and any point $p \in \mathfrak{X}(k')$, we let $\mathcal{D}(\mathfrak{X}, p) := \mathcal{D}(\mathfrak{X} \times_{\operatorname{Spec} k} \operatorname{Spec} k', p)$, where the latter is constructed in the category of k'-stacks. Any non-degenerate map $f : \Theta^n_{k'} \to \mathfrak{X}$ corresponds to a point in a non-degenerate connected component of $[\Theta, \mathfrak{X}]$, so there is a canonical

[15] This agrees with the image of $\mathfrak{S}_{c'}$ for $c' > c$ by the hypothesis that $ev_1^{-1}\mathfrak{X}_{\leq c} \cap \mathfrak{S}^\star = \mathfrak{S}$.

map of spaces $\mathcal{D}(\mathfrak{X}, p) \to \mathcal{C}omp(\mathfrak{X}, p)$ induced by a map of fans. It follows that any numerical invariant simultaneously induces a continuous function on an open subset of $\mathcal{D}(\mathfrak{X}, p)$ for every point of $\mathfrak{X}$. The following is an immediate consequence of Definition 3.4.

Lemma 3.12. *Let μ be a numerical invariant on $\mathfrak{X}$ such that Construction 3.10 defines a Θ-stratification. Then for any extension k'/k and unstable point $p \in \mathfrak{X}(k')$, there is a unique $f : \Theta_{k'} \to \mathfrak{X}$ along with an isomorphism $f(1) \simeq p$, up to the action of $\mathbf{N}^\times$, such that $M^\mu(p) = \mu(f)$.*

Remark 3.13. If μ defines a weak Θ-stratification, then for any unstable point $p \in \mathfrak{X}(k')$ there is a unique purely inseparable extension l/k' and a unique $f : \Theta_l \to \mathfrak{X}$ with an isomorphism $f(1) \simeq p|_l$ such that $M^\mu(p) = \mu(f)$. There need not be a k'-point with this property, however.

In this subsection we discuss to what extent this property, an affirmative answer to Question 2.2 for all $p \in \mathfrak{X}(k')$, for certain field extensions k'/k, is sufficient to ensure that Construction 3.10 defines a Θ-stratification. We shall restrict our focus to reductive stacks, so that the uniqueness part of Question 2.2 is essentially automatic (See Corollary 2.31). Instead of Principle (B), which was enough to guarantee the existence part of Question 2.2 in Section 2, we will need a strengthened version:

(B+) For all maps from a finite type k-scheme $\phi : T \to \mathfrak{X}$, $\exists$ a finite collection of rational simplices $\sigma_i : \Delta_{n_i} \to \mathcal{C}omp(\mathfrak{X}, \phi)$ such that $\forall p \in |T|$ with $M^\mu(p) > 0$, the sub-collection $\{\sigma_i | \sigma_i(\Delta_{n_i}) \subset \mathcal{C}omp(\mathfrak{X}, p)\}$ satisfies Principle (B) for μ on $\mathcal{C}omp(\mathfrak{X}, p)$.

We will find in examples that (B+) follows from an apparently stronger property. There will often be a (potentially infinite) sequence of rational simplices $\sigma_i : \Delta_{n_i} \to \mathcal{C}omp(\mathfrak{X}, \mathrm{id}_\mathfrak{X})$ such that

(i) for any $\phi : T \to \mathfrak{X}$ with T finite type, only finitely many σ_i lie in the image of $\mathrm{Comp}(\mathfrak{X}, \phi)_\bullet \to \mathrm{Comp}(\mathfrak{X}, \mathrm{id}_\mathfrak{X})_\bullet$, and
(ii) for every $p \in |\mathfrak{X}|$, the set of σ_i which lie in the image of $\mathrm{Comp}(\mathfrak{X}, p)_\bullet \to \mathrm{Comp}(\mathfrak{X}, \mathrm{id}_\mathfrak{X})_\bullet$ satisfy Principal (B) on $\mathcal{C}omp(\mathfrak{X}, p)$.

When one has such a collection, one can satisfy Principal (B+) by considering, for any $T \to \mathfrak{X}$, the finite set of σ_i lying in $\mathrm{Comp}(\mathfrak{X}, \phi)_\bullet$.

Example 3.14. Besides the case of global quotient stacks, the primary example of Principle (B+) arises in the moduli of vector bundles on a smooth projective curve, $\mathcal{M}_\Sigma$. We will see in Proposition 4.10 that there are connected components of $[\Theta^n, \mathfrak{X}]$ corresponding to filtrations of length n whose corresponding rank-degree sequence in the upper half plane is convex, and for every p, $M^\mu(p)$ is realized by a unique point on one of these connected components. A Lemma of Grothendieck bounding the slopes of subbundles occuring in a bounded family of vector bundles [HL5] implies that $[\Theta^n, \mathfrak{X}] \times_\mathfrak{X} T$ is empty for all but finitely many of these connected components.

In order to construct Θ-stratifications, we will need to identify another property of the stack $[\Theta, \mathfrak{X}]$. We observe that $\Theta \times B\mathbb{G}_m \simeq \mathbb{A}^1_{z_1}/(\mathbb{G}_m)_{z_1} \times (\mathbb{G}_m)_{z_2}$ has a large group of automorphisms. For any $a \in \mathbb{Z}$ we define a morphism $\Theta \times B\mathbb{G}_m \to \Theta \times B\mathbb{G}_m$ which is defined by the group homomorphism

$$(z_1, z_2) \mapsto (z_1, z_1^a z_2),$$

which is equivariant with respect to the identity map on $\mathbb{A}^1$. This action canonically commutes with the inclusion of the point $\{1\} \times (B\mathbb{G}_m)_{z_2}$, so in fact the group $\mathbb{Z}$ acts on $\mathrm{Map}(\Theta \times B\mathbb{G}_m, \mathfrak{X})$ over $[B\mathbb{G}_m, \mathfrak{X}]$ with respect to the morphism $ev_1 : \mathrm{Map}(\Theta \times B\mathbb{G}_m, \mathfrak{X}) \to [B\mathbb{G}_m, \mathfrak{X}]$.

This construction is a little strange, but it is something we have seen before. Regarding $\mathrm{Map}(\Theta \times B\mathbb{G}_m, \mathfrak{X}) \simeq [\Theta, [B\mathbb{G}_m, \mathfrak{X}]]$, we see that the morphism is representable by algebraic spaces and is locally of finite presentation. What we have shown is that when the target is of the form $[B\mathbb{G}_m, \mathfrak{X}]$, then for any $T \to [B\mathbb{G}_m, \mathfrak{X}]$, the algebraic space $[\Theta, [B\mathbb{G}_m, \mathfrak{X}]] \times_{[B\mathbb{G}_m, \mathfrak{X}]} T$ has a canonical $\mathbb{Z}$-action (unrelated to the standard $\mathbf{N}^\times$ action).

We will consider k-stacks $\mathfrak{X}$ such that for every commutative square of the form

$$\begin{array}{ccc} (\mathbb{A}^1 - \{0\})/\mathbb{G}_m \simeq \operatorname{Spec} \bar{k} & \longrightarrow & [\Theta, \mathfrak{X}] \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle ev_0} \\ \Theta_{\bar{k}} & \xrightarrow{f} & [B\mathbb{G}_m, \mathfrak{X}] \end{array} \tag{11}$$

there is some twist of the map f by the action of a nonnegative $N \in \mathbb{Z}$ described above such that the twisted diagram admits a dotted arrow filling commutative square as shown.

Concretely, if we consider the $\mathbb{G}_m^2$-equivariant closed subvariety $Y = \{(z_1, z_2) \in \mathbb{A}^2 | z_2 = 0\}$ and the open subvariety $U = \{(z_1, z_2) \in \mathbb{A}^2 | z_1 \neq 0\}$. The lifting property (11) states that given maps $U/\mathbb{G}_m^2 \to \mathfrak{X}$ and $Y/\mathbb{G}_m^2 \to \mathfrak{X}$ along with an isomorphism of the restrictions to $U \cap Y/\mathbb{G}_m^2$, then possibly after composing the map $Y/\mathbb{G}_m^2 \to \mathfrak{X}$ with one of the automorphisms of $Y/\mathbb{G}_m^2$ constructed above, there is an extension to a morphism $\mathbb{A}^2/\mathbb{G}_m^2 \to \mathfrak{X}$.

Lemma 3.15. *The lifting property* (11) *holds for any local quotient stack.*

Proof. Any square as in (11) factors through any open substack containing the image of $f(0)$ under the forgetful map $[B\mathbb{G}_m, \mathfrak{X}] \to \mathfrak{X}$. Therefore we may assume that $\mathfrak{X}$ is a quotient stack. Furthermore, it suffices to consider a single connected component of $[B\mathbb{G}_m, \mathfrak{X}]$. By Theorem 1.21, on connected components ev_0 is modelled by the projection $Y_{Z,\lambda}/P_{Z,\lambda} \to Z/L_{Z,\lambda}$ for some one-parameter-subgroup λ and connected component $Z \subset X^\lambda$. We shall suppress the subscript (Z, λ) from our notation.

Now the morphism $f : \Theta \to Z/L$ is determined by a choice of one-parameter-subgroup λ' in L, and a point $x' \in Z$ under which $x'' := \lim_{t\to 0} \lambda'(t) \cdot x'$ exists. The top horizontal arrow in (11) is determined by a point $x \in Y$, and the commutativity of (3.10) up to 2-isomorphism guarantees that $\lim_{t\to 0} \lambda(t) \cdot x = l \cdot x'$ for some $l \in L$. Using the splitting $L \subset P$, we may represent the point in Y/P by an x such that $\lim_{t\to 0} \lambda(t) \cdot x = x'$.

Forgetting all but the action of the torus, T, generated by λ and λ', we may find an open equivariant affine neighborhood $U \subset X$ containing x'', and therefore also containing x' and x. Furthermore, we can consider U to be a locally closed subvariety of a linear representation of T on $\mathbb{A}^n$ in which T acts diagonally. In these coordinates, we may write

$$x = x_{\lambda>0} + x',$$

where $x_{\lambda>0}$ is the projection onto those coordinates which have strictly positive weight w.r.t. λ, and x' is fixed by λ. Now x' has a limit under λ', but there is no apriori bound on the weights of $x_{\lambda>0}$ w.r.t. λ'. Nevertheless there is some $N > 0$

such that for all $n \geq N$, $\lim \lambda(t)^n \lambda'(t) \cdot x_{\lambda > 0} = 0$, and thus $\lim_{t \to 0} \lambda(t)^n \lambda'(t) \cdot x = x''$. In particular for any $a, b \geq 0$, we have the existence of all limits of the form

$$\lim_{t \to 0} \lambda(t)^a (\lambda^N(t) \lambda'(t))^b \cdot x = \left\{ \begin{array}{ll} x', & a > 0, b = 0 \\ x'', & b > 0 \end{array} \right.$$

It follows that we have a map $\mathbb{A}^2/\mathbb{G}_m^2 \to U/T \to X/G$ corresponding to the point $x \in U$ and group homomorphism $(z_1, z_2) \mapsto \lambda^N(z_1) \cdot \lambda'(z_2)$. This morphism $\Theta^2 \to \mathfrak{X}$ corresponds to a map $\Theta \to [\Theta, \mathfrak{X}]$ providing a lift in (11) after twisting the morphism $\Theta \times B\mathbb{G}_m \to \mathfrak{X}$ by the automorphism of $\Theta \times B\mathbb{G}_m$ corresponding to $N \in \mathbb{Z}$. □

We shall refer to a numerical invariant $\mu : U \subset \mathfrak{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}}) \to \mathbb{R}$ as *locally convex* if the subset $U \subset \mathfrak{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})$ is locally convex, and the restriction of μ to any rational 1-simplex in U is strictly convex upward. We can now state our main criterion for constructing Θ-stratifications:

Theorem 3.16. *Let $\mathfrak{X}$ be a reductive k-stack (Definition 2.27) which satisfies the lifting property* (11)*. Let $\mu : U \subset \mathfrak{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}}) \to \mathbb{R}$ be a locally convex numerical invariant, and consider the collection of connected components $\mathfrak{S}^\star \subset [\Theta, \mathfrak{X}]$ of Construction 3.10. If μ satisfies Principal (B+) and $\mathbf{N}^\times$ acts freely on $\pi_0 \mathfrak{S}^\star$, then $\mathfrak{S}^\star$ defines a weak Θ-stratification.*

Remark 3.17. Of the conditions on μ in Theorem 3.16, in practice only Principle (B+) requires work to establish. Any numerical invariant arising from construction Construction 2.19 with a positive definite $b \in H^4(\mathfrak{X})$ will satisfy the other hypotheses automatically. We have already seen that such a μ is locally convex in our sense. Furthermore, if $S \subset \pi_0(\mathfrak{X})$ is a set of connected components for which there exists an $\eta \in H^{2p}(\mathfrak{X})$ such that $\hat{\eta} \neq 0$ on S, then $\mathbf{N}^\times$ acts freely on S. This follows immediately from the fact that $\hat{\eta}$ is homogeneous of degree p with respect to the action of $\mathbf{N}^\times$.

We shall prove this theorem at the end of the section, but first let us note some consequences.

Proposition 3.18. *Let $\mathfrak{X}$ be a local quotient stack over a field of characteristic* 0*, and let $\mathfrak{S}^\star \subset [\Theta, \mathfrak{X}]$ be defined by a numerical invariant via Construction 3.10. Then $\mathfrak{S}^\star$ defines a Θ-stratification if and only if it defines a weak Θ-stratification.*

Proof. We must show that for any $f : \Theta_{\bar{k}} \to \mathfrak{X}$ such that $\mu(f) = M^\mu(f(1))$, the map on tangent spaces $T_f [\Theta, \mathfrak{X}] \to T_{f(1)} \mathfrak{X}$ is injective and hence $ev_1 : \mathfrak{S}_{\mu = c} \to \mathfrak{X}_{\mu \leq c}$ is a closed immersion. f factors through a quotient stack X/G, and thus can be thought of as a pair (p, λ) with $p \in X(\bar{k})$, and $p_0 := \lim_{t \to 0} \lambda(t) \cdot p$ exists. Let $G \times_{P_\lambda} Y_{Z,\lambda} \to X$ be the connected component of $[\Theta, X/G] \times_{X/G} X$ containing f, which corresponds to the point $(1, p)$. Then we have

$$T_{(1,p_0)}(G \times_{P_\lambda} Y_{Z,\lambda}) \simeq \left(\mathfrak{g} \oplus (T_{p_0} X)^{\lambda \geq 0} \right) / (\mathfrak{g})^{\lambda \geq 0},$$

so if the map on tangent spaces at the point $(1, p_0)$ fails to be injective, there there is a nonzero subspace of $(\mathfrak{g})^{\lambda < 0} \oplus 0$ contained in the kernel. Let U be the unipotent group integrating that subspace. Then U stabilizes the point p_0.

On the other hand, for any pair, (p, λ), representing an optimal destabilizer, one must have an inclusion of k-points $\mathrm{Stab}(p) \subset P_\lambda(\bar{k})$. Indeed, let $G \times_{P_\lambda} Y_{Z,\lambda} \to X$ be the connected component of $[\Theta, \mathfrak{X}] \times_{\mathfrak{X}} X$ containing the optimal destabilizer, which corresponds to the point $(1,p)$ as above. Then for any $g \in \operatorname{Stab}(p)$, the point (g,p) lies in the fiber of p and is not equivalent to $(1,p)$ unless $g \in P_\lambda$.

Applying this to the pair (p_0, λ), which must be an optimal destabilizer because $p_0 \in \mathfrak{X}_{\leq \mu(Z,\lambda)}$, we find that $\operatorname{Stab}(p_0) \subset P_\lambda$. This contradicts the fact that U cannot be contained in P_λ by construction. Thus the map on tangent spaces is injective at p_0, and every G-equivariant open subset of p_0 contains p, so the map on tangent spaces is injective there as well. □

Corollary 3.19. *Let V be an affine k-scheme of finite type, with* $\operatorname{char} k = 0$*, and let G be a reductive k-group acting on V. Then Construction 3.10 defines a Θ-stratification*[16] *for any numerical invariant μ constructed from a class $l \in H^2(V/G)$ and a positive definite $b \in H^4(V/G)$ via Construction 2.19.*

Proof. To verify Principle (B+), we claim that there is a finite collection of non-degenerate connected components of $[\Theta^n, X/G]$ for varying n such that for any $p : \operatorname{Spec} \bar{k} \to X/G$, the subset of these rational simplices which lie in $\mathcal{C}omp(X/G,p)$ cover it. The argument in Lemma 2.23 shows that $\operatorname{Comp}(X/T,p)_\bullet \to \operatorname{Comp}(X/G,p)_\bullet$ is surjective, where in the former we regard p as a morphism $\operatorname{Spec} \bar{k} \to X/T$. Thus we may assume $G = T$ is a torus. As λ varies over all one parameter subgroups of T, there are finitely many closed subvarieties Z arising as connected components of X^λ. And for every such Z, there are finitely many locally closed subvarieties Y arising as $Y = Y_{Z,\lambda}$ for some λ fixing Z. For every such Y and Z there is a unique rational polyhedral cone in $N_\mathbb{R}$, where N is the cocharacter lattice of T, generated by 1PS's for which $Y \subset Y_{Z,\lambda}$. Choosing a finite simplicial decomposition of each of these cones defines homomorphisms $\mathbb{G}_m^k \to T$, each identifying a connected component of $\left[\Theta^k, X/T\right]$, and the connected components of $\left[\Theta^k, X/T\right] \times_{X/T} \operatorname{Spec} \bar{k}$ lying over this finite list of connected components of $\left[\Theta^k, X/T\right]$ provides a set of rational simplices which covers $\mathcal{C}omp(\mathfrak{X},p)$ for any $p \in X/T(\bar{k})$. □

Remark 3.20. For a quotient stack X/G where $\pi : X \to V$ is projective and V is affine with relatively G-ample invertible sheaf L, one can use Corollary 3.19 to deduce the existence of Θ-stratifications for classes $c_1(L) + \pi^* l \in H^2(X/G)$ and $\pi^* b \in H^4(X/G)$, where $l \in H^2(V/G)$ is arbitrary and $b \in H^4(V/G)$ is positive definite. One does this by extending these classes to the affine quotient $\tilde{X}/G \times \mathbb{G}_m := \underline{\operatorname{Spec}}_V \bigoplus_{n \geq 0} \pi_*(L^n)$ by letting

$$\tilde{l} = c_1(\mathcal{O}_{\tilde{X}}(1)) + \pi^* l, \quad \text{and} \quad \tilde{b} = k\pi^* b + c_1(\mathcal{O}_{\tilde{X}}(1))^2.$$

One can show that for $k \gg 0$ the stratum with highest value of μ is $V \subset \tilde{X}$ with destabilizing one-parameter subgroup which is trivial in G. This is the first stratum that is removed, and the rest gives a Θ-stratification of $\tilde{X} \setminus V/G \times \mathbb{G}_m \simeq X/G$.

We shall now prove Theorem 3.16. Let $\bar{k}/k$ denote an algebraic closure of k.

Lemma 3.21. *Let μ be a numerical invariant for $\mathfrak{X}$. Then Construction 3.10 defines a Θ-stratification of $\mathfrak{X}$ for μ if and only if it defines a Θ-stratification of $\mathfrak{X}_{\bar{k}}$.*

[16] In Theorem 13.5 of [K2] this was claimed over fields of arbitrary characteristic, which is not quite correct. In finite characteristic, one can only guarantee a weak Θ-stratification. A similar statement to the one here was described in Theorem 4.7 of [ADK], but the proof does not seem to be in the literature.

Proof. Let $\pi : [\Theta_{\bar{k}}, \mathfrak{X}_{\bar{k}}] \to [\Theta, \mathfrak{X}]$ be the projection, and let $\mathfrak{S}^\star \subset [\Theta, \mathfrak{X}]$ (respectively $\mathfrak{S}^{\star\prime} \subset [\Theta_{\bar{k}}, \mathfrak{X}_{\bar{k}}]$) denote the collection of connected components selected by Construction 3.10. It follows from the construction that $\mathfrak{S}^{\star\prime} \subset \pi^{-1}(\mathfrak{S}^\star)$. If $\mathfrak{S}^{\star\prime}$ defines a Θ-stratification of $\mathfrak{X}_{\bar{k}}$, so that every point of $\mathfrak{X}_{\bar{k}}$ extends to a unique point of $\mathfrak{S}^{\star\prime} \subset [\Theta_{\bar{k}}, \mathfrak{X}_{\bar{k}}]$ up to the $\mathbf{N}^\times$ action which maximizes μ, then

$$\operatorname{im}(\pi^{-1}(\mathfrak{S}^\star_{\mu>c}) \to \mathfrak{X}_{\bar{k}}) = \operatorname{im}(\mathfrak{S}^{\star\prime}_{\mu>c} \to \mathfrak{X}_{\bar{k}})$$

Because $M^\mu(p) > c$ for any p in $\pi^{-1}(\mathfrak{S}^\star_{\mu>c})$.

For any morphism of algebraic k-stacks $f : \mathfrak{Z} \to \mathfrak{Y}$, the preimage of $\operatorname{im}(f) \subset |\mathfrak{Y}|$ under the projection $|\mathfrak{Y}_{\bar{k}}| \to |\mathfrak{Y}|$ is $\operatorname{im}(f_{\bar{k}} : \mathfrak{Z}_{\bar{k}} \to \mathfrak{Y}_{\bar{k}})$. This shows that the closed subset $|\mathfrak{X}_{\bar{k}}|_{\mu>c}$ is the preimage of the subset $\operatorname{im}(\mathfrak{S}^\star_{\mu>c} \to \mathfrak{X}) \subset |\mathfrak{X}|$, which is therefore also closed. This verifies (1), but it also shows that the open set $|\mathfrak{X}_{\bar{k}}|_{\mu\le c}$ is the preimage of $|\mathfrak{X}|_{\mu\le c}$ and hence $\pi^{-1}(\mathfrak{S}^\star_{\mu=c}) = \mathfrak{S}^{\star\prime}_{\mu=c}$ for all c. Property (2) now follows immediately from the fact that (2) holds for $\pi^{-1}\mathfrak{S}^\star$ and the fact that one can check containment in an open substack and a morphism being a closed immersion after base change to $\bar{k}$. □

Thus we may assume throughout that $k = \bar{k}$. Our first partial construction of a weak Θ-stratification is the following:

Lemma 3.22. *Let μ be a convex numerical invariant on a reductive k-stack, $\mathfrak{X}$, let $\mathfrak{S}^\star \subset [\Theta, \mathfrak{X}]$ be provided by Construction 3.10, and assume that $\mathbf{N}^\times$ acts freely on $\pi_0(\mathfrak{S}^\star)$. Suppose that Principle (B+) holds. Then for any $c \ge 0$, $\operatorname{im}(ev_1 : \mathfrak{S}^\star_{\mu>c} \to \mathfrak{X}) \subset |\mathfrak{X}|$ is closed, so we may define the open substacks $\mathfrak{X}_{\le c} \subset \mathfrak{X}$ and $\mathfrak{S}_c := \mathfrak{S}^\star_c \cap ev_1^{-1}(\mathfrak{X}_{\le c}) \subset [\Theta, \mathfrak{X}]$. Furthermore the morphism $ev_1 : \operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c \to \mathfrak{X}_{\mu\le c}$ is finite and radicial.*

Proof. Fix a k-scheme of finite type and a morphism $\phi : T \to \mathfrak{X}$. Let $\sigma_i : \Delta_{n_i} \to \mathcal{C}omp(\mathfrak{X}, \phi)$ be the collection of rational simplices provided by (B+). Choosing a rational point on each σ_i which maximizes μ, we note that there is a finite collection of connected components of $[\Theta, \mathfrak{X}] \times_{\mathfrak{X}} T$ such that for any $p \in T(k')$, the maximum of μ on $\mathcal{C}omp(\mathfrak{X}, p)$ is achieved on one of these connected components (by Lemma 3.7 it suffices to consider at most one connected component in each $\mathbf{N}^\times$ orbit).

Let $Y \subset [\Theta, \mathfrak{X}] \times_{\mathfrak{X}} T$ be the union of those connected components in our collection for which μ is maximal. Then $Y \to T$ is a proper algebraic space over T. Furthermore, because we have chosen only one connected component in each $\mathbf{N}^\times$ orbit, Corollary 2.31 implies that $Y(\bar{k}) \to T(\bar{k})$ is an injection, hence $Y \to T$ is radicial. Removing the closed image of Y, we can repeat this procedure finitely many times until there are no remaining unstable points in T.

□

Proof of Theorem 3.16. In light of Lemma 3.22, the morphism $\operatorname{colim}_{\mathbf{N}} \mathfrak{S}_c \to \mathfrak{X}_{\le c}$ is proper and radicial $\forall c > 0$, so all that remains is to show that $\mathfrak{S}_c \subset [\Theta, \mathfrak{X}_{\mu\le c}]$. It suffices to check this inclusion on $\bar{k}$-points. By Proposition 1.4, this amounts to checking that for every map $f : \Theta_{\bar{k}} \to \mathfrak{X}$ with $\mu(f) = M^\mu(f(1))$, $f(0) \in \mathfrak{X}_{\mu\le c}$ as well. We suppose that $f(0) \notin |\mathfrak{X}_{\le c}|$, and derive a contradiction. Letting $f' : \Theta \to \mathfrak{X}$ with $f'(1) \simeq f(0)$ correspond to the unique rational point of $\mathcal{D}(\mathfrak{X}, f(0))$ maximizing μ, the hypothesis that $f(0) \notin \mathfrak{X}_{\le c}$ implies $\mu(f') > \mu(f)$.

Let $\mathfrak{Y} \subset \mathfrak{S}^\star_{\mu(f')} \subset [\Theta, \mathfrak{X}]$ be a union of connected components which form a complete set of representatives of $\mathbf{N}^\times$-orbits on $\pi_0(\mathfrak{S}^\star_{\mu(f')})$. We know that the morphism ev_1 is locally of finite presentation and relatively representable by algebraic spaces, hence we have a cartesian square

$$\begin{array}{ccc} Y/(\mathbb{G}_m)_{\bar{k}} & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow ev_1 \\ B(\mathbb{G}_m)_{\bar{k}} & \xrightarrow{f|_{B(\mathbb{G}_m)_{\bar{k}}}} & \mathfrak{X} \end{array}$$

where Y is a locally finite type $\bar{k}$-space with an action of $\mathbb{G}_m$. On the other hand, the existence and uniqueness of a maximal destabilizer for $f(0)$ and the selection of connected components in $\mathfrak{Y}$ guarantees that $Y^{red} \simeq \operatorname{Spec} \bar{k}$. The resulting map $B(\mathbb{G}_m)_{\bar{k}} \to \mathfrak{Y}$ is such that its restriction to $\bar{k}$ classifies f', and the composition with $ev_1 : \mathfrak{Y} \to \mathfrak{X}$ is canonically isomorphic to $f|_{B(\mathbb{G}_m)_{\bar{k}}}$.

Regarding $B(\mathbb{G}_m)_{\bar{k}} \to [\Theta, \mathfrak{X}]$ as a morphism $\Theta \times B(\mathbb{G}_m)_{\bar{k}} \to \mathfrak{X}$, we can regard it as a morphism $\Theta_{\bar{k}} \to [B\mathbb{G}_m, \mathfrak{X}]$. We are thus in the situation of the lifting property (11), where the top horizontal arrow classifies the map f. Thus we can twist this map by an automorphism of $\Theta \times B\mathbb{G}_m$, say $(z_1, z_2) \mapsto (z_1, z_1^N z_2)$, so that the new diagram has a lift. This lift provides us with a morphism $g : \Theta^2 \to \mathfrak{X}$ such that the restriction to $\{(z_1, z_2) | z_2 = 0\}/\mathbb{G}_m \times \{1\}$ is canonically isomorphic to f', and the restriction to $\{(z_1, z_2) | z_1 \neq 0\}/\mathbb{G}_m^2$ is canonically isomorphic to f.

The kernel of the induced homomorphism $\mathbb{G}_m^2 \to \operatorname{Aut} g(0)$ must be finite, or else Lemma 2.30 would imply that, after passing to a non-degenerate cover, g factors through a morphism $\Theta^2 \to \Theta$. This cannot happen: given a non-degenerate morphism $h : \Theta \to \mathfrak{X}$, the only non-degenerate morphisms one can get by precomposing with a map $\Theta \to \Theta$ correspond to h itself and $\sigma \circ ev_0(h)$, up to the action of $\mathbf{N}^\times$. The latter lies on the same connected component of $[\Theta, \mathfrak{X}]$ as h (see the proof of Lemma 3.5) and thus μ has the same value on both morphisms. Thus $\mathbb{G}_m^2 \to \operatorname{Aut} g(0)$ must have trivial kernel, because $\mu(f) \neq \mu(f')$.

It follows that g is a non-degenerate map and defines a rational 1-simplex Δ_g of $\mathcal{D}(\mathfrak{X}, f(1))$ with one endpoint corresponding to the map f. We can compose g with the map $\mathbb{A}^1 \times \Theta^2 \to \Theta^2$ defined by $(t, z_1, z_2) \mapsto (tz_1, tz_2)$, which shows that g lies on the same connected component of $\left[\Theta^2, \mathfrak{X}\right]$ as the composition $\Theta^2 \to B\mathbb{G}_m^2 \to \Theta^2 \xrightarrow{g} \mathfrak{X}$. Hence $\Delta_{g'} = \Delta_g$ in $\mathcal{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})$. On the other hand we can compose the map $B\mathbb{G}_m^2 \to \mathfrak{X}$ with the automorphism $(z_1, z_2) \mapsto (z_1, z_1^{-N} z_2)$ and consider the resulting morphism $g'' : \Theta^2 \to B\mathbb{G}_m^2 \to \mathfrak{X}$, and the simplex $\Delta_{g''}$ in $\mathcal{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}})$.

By construction, $\Delta_{g'}$ is a sub-interval of $\Delta_{g''}$ which shares the endpoint corresponding to f, and the other endpoint of $\Delta_{g''}$ corresponds to f'. Because μ is locally convex, the interval $\Delta_{g''} \subset U$. Furthermore, the convexity of μ and the hypothesis that $\mu(f) < \mu(f')$ implies that $\mu > \mu(f)$ on any point in the interior of the interval $\Delta_{g''}$. As a consequence $\mu > \mu(f)$ at any point in the interior of the interval $\Delta_g \subset \mathcal{D}(\mathfrak{X}, f(1))$ as well, which contradicts the fact that $\mu(f) = M^\mu(f(1))$. □

4. Modular examples of $\mathfrak{X}(\Theta)$ and stability functions

In this section we study $[\Theta, \mathfrak{X}]$ and the degeneration space $\mathcal{D}(\mathfrak{X}, p)$ for some algebraic stacks $\mathfrak{X}$ representing commonly studied moduli problems. Our main example will be the stack, $\mathcal{M}^{\mathcal{C}}(v)$, of flat families of objects in a subcategory $\mathcal{C} \subset D^b(X)$, where X is a projective k-scheme and $\mathcal{C}$ is contained in the heart of some t-structure on the derived category of coherent sheaves, $D^b(X)^\heartsuit$. Given an object $E \in \mathcal{C}$, we describe an explicit covering of $\mathcal{D}(\mathcal{M}^{\mathcal{C}}(v), [E])$ by $(p-1)$-simplices corresponding to descending filtrations of E of length p (Proposition 4.10).

We discuss the notion of a "pre-stability condition" for objects in $D^b(X)^\heartsuit$ which generalizes the notion of Bridgeland stability as well as the more classical notion of Gieseker stability. Given a pre-stability condition, we can define a category of "torsion-free objects," $\mathcal{F} \subset D^b(X)^\heartsuit$. The pre-stability condition induces cohomology classes on $\mathcal{M}^{\mathcal{F}}(v)$, and one of the main results of this section, Theorem 4.14, shows that the existence and uniqueness of a rational point maximizing the associated numerical invariant on $\mathcal{D}(\mathcal{M}^{\mathcal{F}}(v), [E])$ is equivalent to the existence of a Harder-Narasimhan filtration for $E \in \mathcal{F}$. Finally, in Proposition 4.20 we establish conditions under which this numerical invariant defines a Θ-stratification of $\mathcal{M}^{\mathcal{F}}(v)$.

We also formulate K-stability in our framework for schemes over a field of characteristic 0. We identify a stack $\mathcal{V}ar'_{\mathbb{Q}}$, classifying flat families of rationally polarized projective schemes, and we identify classes in $H^2(\mathcal{V}ar'_{\mathbb{Q}}; \mathbb{Q})$ and $H^4(\mathcal{V}ar'_{\mathbb{Q}}; \mathbb{Q})$ such that the corresponding numerical invariant recovers the definition of K-stability. To the author's surprise, this numerical invariant was already studied by Donaldson in [D], where it was called the "normalized Futaki invariant."

4.1. **Moduli of objects in derived categories.** Let X be a projective variety over an algebraically closed field k of characteristic 0. We denote the bounded derived category of coherent sheaves on X by $\mathrm{D}^b(X)$. We let $\mathcal{N}(X)$ denote the numerical K-group of coherent sheaves on X.[17] Our discussion will start with a t-structure on $D^b(X)$, a full abelian subcategory of its heart $\mathcal{A} \subset \mathrm{D}^b(X)^\heartsuit$, and a linear map $Z : \mathcal{N}(X) \to \mathbb{C}$, known as the central charge, which maps $\mathcal{A}$ to $\mathbb{H} \cup \mathbb{R}_{\leq 0}$. We shall refer to the pair $(\mathcal{A}, Z)$ as a *pre-stability condition*.

Example 4.1. The simplest example is where X is a smooth curve, and we let $\mathcal{A}$ be the category of coherent sheaves on X. The central charge is $Z(E) = -\deg(E) + i\,\mathrm{rank}(E)$.

We can generalize this fundamental example in two directions.

Example 4.2. We can let $(\mathcal{A}, Z)$ be an arbitrary numerical Bridgeland stability condition on a smooth projective variety X. In addition to the Harder-Narasimhan property, which we discuss below, a Bridgeland stability condition must have $Z(E) \neq 0$ for nonzero $E \in \mathcal{A}$. We will not need this nondegeneracy hypothesis in our discussion.

Example 4.3. We can fix a projective variety X and $0 < d \leq \dim X$. Let $\mathcal{A}$ be the category of coherent sheaves whose support has dimension d. This is a full abelian subcategory of $\mathrm{Coh}(X)$. The assignment

$$E \mapsto p_E(n) := \chi(E \otimes L^n) = \sum \frac{p_{E,k}}{k!} n^k$$

[17]This is defined to be $K_0(X) \otimes \mathbb{R}$ modulo the subspace spanned by $[E] \in K_0(\mathrm{D}^b(X))$ such that $\chi([F] \otimes [E]) = 0$ for any perfect complex F. Similarly we can define $\mathcal{N}^{perf}(X)$ to be the quotient of the K-group of perfect complexes $K^0(X) \otimes \mathbb{R}$ by those complexes whose Euler pairing with any $[E] \in K_0(X)$ vanishes. One can show that $\mathcal{N}^{perf}(X)$ is a quotient of the image of the Chern character and is thus finite dimensional. A basic linear algebra argument then shows that $\chi : \mathcal{N}^{perf}(X) \otimes \mathcal{N}(X) \to \mathbb{R}$ is a perfect pairing.

defines a group homomorphism $K_0(X) \to \mathbb{Z}[n]$. We define our central charge to be $Z(E) := ip_{E,d} - p_{E,d-1}$. This has the property that $Z(E) \subset \mathbb{H} \cup \mathbb{R}_{\leq 0}$ for any $E \in \mathcal{A}$: if $\dim(\operatorname{supp}(E)) < d-1$ then $Z(E) = 0$, if $\dim(\operatorname{supp} E) = d-1$ then $p_{E,d} = 0$ and $p_{E,d-1} > 0$, and if $\dim(\operatorname{supp} E) = d$ then $p_{E,d} > 0$.

4.1.1. *The moduli stack $\mathcal{M}(v)$ and its degeneration space.* The the notion of torsion free objects in $\mathcal{A}$ will feature prominently in our discussion. We will refer to objects $E \in \mathcal{A}$ with $Z(E) \in \mathbb{R}_{\leq 0}$ as *torsion*, and we call an object *torsion-free* if it has no torsion subobjects (in particular a torsion-free object has $\Im Z(E) > 0$). We define $\mathcal{T} \subset \mathcal{A}$ (resp. $\mathcal{F} \subset \mathcal{A}$) to be the full subcategory consisting of torsion objects (resp. torsion-free objects).

Lemma 4.4. *The category $\mathcal{T} \subset \mathcal{A}$ is closed under extensions and subquotients, and if $\mathcal{A}$ is Noetherian then we have a torsion theory $(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$.*

Proof. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathcal{A}$, then $Z(M) = Z(M') + Z(M'')$. The fact that $Z \in \mathbb{H} \cup \mathbb{R}_{\leq 0}$ implies that $\Im Z(M) = 0$ if and only if $\Im Z(M') = \Im Z(M'') = 0$. The fact that $\mathcal{T}$ is closed under quotients and extensions, along with the fact that $\mathcal{A}$ is Noetherian, shows that any $F \in \mathcal{A}$ has a unique maximal subobject in $\mathcal{T}$, which we denote F_{tor}. The assignment $F \mapsto F_{tor}$ defines a right adjoint for the inclusion $\mathcal{T} \subset \mathcal{A}$, thus $(\mathcal{T}, \mathcal{F})$ is a torsion theory on $\mathcal{A}$. □

Example 4.5. Continuing Example 4.3, let $\mathcal{A}$ is the category of sheaves whose support has dimension $\leq d$. The torsion subcategory $\mathcal{T}$ consists of sheaves whose support has dimension $\leq d-1$, and the $\mathcal{F}$ consists of sheaves which are pure of dimension d.

For each $v \in \mathcal{N}(X)$ and any subcategory $\mathcal{C} \subset \mathrm{D}^b(X)^\heartsuit$, we introduce the moduli functor $\mathcal{M}^{\mathcal{C}}(v)$ as a subfunctors of the (locally finitely presented, algebraic) stack of "universally gluable perfect complexes on X" as in [L1, Definition 2.1.8]. We define $\mathcal{M}^{\mathcal{C}}(v)$ on finite type k schemes[18] as

$$\mathcal{M}^{\mathcal{C}}(v)(S) := \left\{ F \in \mathrm{D}^b(X \times S) | \forall \text{ closed points } s \in S, F|_{X_s} \in \mathcal{C}, \text{ of class } v \right\}$$

We will write $\mathcal{M}(v)$ for $\mathcal{M}^{D^b(X)^\heartsuit}(v)$. The fact that $\mathcal{M}^{\mathcal{C}}(v) \subset \mathcal{M}(v)$ is defined by a fppf-local condition on $F \in \mathrm{D}^b(X \times S)$ implies that it is still an fppf sheaf. This allows us to interpret maps $\Theta \to \mathcal{M}^{\mathcal{C}}(v)$.

Lemma 4.6. *A morphism $f : \Theta \to \mathcal{M}^{\mathcal{C}}(v)$ with $f(1) \simeq E \in \mathcal{C}$ corresponds to a weighted descending filtration $\cdots E_w \supset E_{w+1} \supset \cdots$ such that $\bigoplus_w \operatorname{gr}_w E_\bullet \in \mathcal{C}$, and in particular $\operatorname{gr}_w E_\bullet$ vanishes for all but finitely many w.*

Proof. By fppf-descent $\operatorname{Map}(\Theta, \mathcal{M}(v))$ is the groupoid of $\mathbb{G}_m$-equivariant objects on $X \times \mathbb{A}^1$. Unwinding the descent data, this is equivalent to a graded $\mathcal{O}_X[t]$-module object $F \in \mathrm{D}_{qc}(X)^\heartsuit$, where the t-structure on $\mathrm{D}_{qc}(X)$ is induced from that on $\mathrm{D}^b(X)$ as in [P2], which satisfies additional conditions: 1) that F has bounded cohomology sheaves which are coherent as $\mathcal{O}_X[T]$ modules, and 2) that $F \otimes_{\mathcal{O}_X[t]} \mathcal{O}_X$ setting $t = 1$ and $t = 0$ both lie in $\mathcal{C} \subset \mathrm{D}^b(X)$.

A graded $\mathcal{O}_X[t]$-module is a graded object $F = \bigoplus F_i \in \mathrm{D}_{qc}(X)^\heartsuit$, along with a morphism $F_i \mapsto F_{i-1}$ corresponding to multiplication by t. Coherent, bounded

[18] Technically, we can regard them as functors on the site of all schemes by Kan extension.

cohomology implies that $F_i = 0$ for $i \gg 0$, and $F_i \in \mathrm{D}^b(X)$ for all i. The fact that

$$F \otimes_{\mathcal{O}_X[t]} \mathcal{O}_{X,t=0} \simeq \mathrm{Cone}(\bigoplus_i F_i \to \bigoplus_i F_{i-1}) \in \mathcal{A}$$

implies that each morphism $F_i \to F_{i-1}$ is injective in the t-structure on $\mathrm{D}^b(X)$, and $F_i \to F_{i-1}$ is an equivalence for all but finitely many i.

The condition that a map $\Theta \to \mathcal{M}(v)$ factors through $\mathcal{M}^{\mathcal{C}}(v)$ amounts to the requirement that $\bigoplus \mathrm{gr}_j F_\bullet = F \otimes_{\mathcal{O}_X[t]} \mathcal{O}_{X,t=0} \in \mathcal{C}$ and $F \otimes_{\mathcal{O}_X[t]} \mathcal{O}_{X,t=1} \simeq \mathrm{colim}\, F_w \in \mathcal{C}$. □

Notation 4.7. Because $E_w = E_{w+1}$ for all but finitely many values of w, we shall encode this data more concisely (but equivalently) as a finite decreasing filtration $E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_p \supsetneq E_{p+1} = 0$ along with a choice of strictly increasing weights $w_0 < w_1 < \cdots < w_p$, which record the w in the infinite descending filtration for which $E_w \neq E_{w+1}$. When we say $E_\bullet$ is a descending filtration, we will always use this indexing convention. Also, we let $\mathrm{gr}_j(E_\bullet) := E_j/E_{j+1}$.

We can elaborate on this to get a description of $\mathcal{D}(\mathcal{M}(v), [E])$ for any $E \in \mathcal{A}$. As above $D_{qc}(\Theta^p \times X)$ can be identified with the derived category of p-multigraded quasicoherent sheaves of modules over $\mathcal{O}_X[t_1, \ldots, t_p]$. We will sometimes abbreviate t for the set of variables, and $t^{\underline{m}}$ for $t_1^{m_1} \cdots t_p^{m_p}$. By an argument analogous to the previous lemma, we have:

Lemma 4.8. *A morphism $f : \Theta^p \to \mathcal{M}^{\mathcal{C}}(v)$ corresponds to a p-multigraded descending filtration in $\mathcal{C}$, $E_\bullet$, i.e. $E_{w_1,\ldots,w_j+1,\ldots,w_p} \subset E_{w_1,\ldots,w_j,\ldots,w_p}$ for all j, such that $\mathrm{colim}\, E_w \in \mathcal{C}$, and $\forall j = 1, \ldots, p$ the associated graded in the j^{th} direction, $\mathrm{gr}^{(j)} E_\bullet$, defines a family over Θ^{p-1} when regarded as a $p-1$-multigraded filtration.*

Example 4.9. When $p = 2$ in the previous lemma, the condition amounts to the requirement that all maps $E_{v+1,w} \to E_{v,w}$ and $E_{v,w+1} \to E_{v,w}$ be injective, and that every square

$$\begin{array}{ccc} E_{v+1,w+1} & \longrightarrow & E_{v+1,w} \\ \downarrow & & \downarrow \\ E_{v,w+1} & \longrightarrow & E_{v,w} \end{array}$$

is a fiber product.

Given a family $\tilde{E} \simeq \bigoplus_{w \in \mathbb{Z}^p} E_p \in D_{qc}(\Theta^p \times X)$, the isomorphism $\tilde{E}|_{(1,\ldots,1)} \simeq E$ gives a map $\tilde{E} \to \mathcal{O}_X[t_1^{\pm}, \ldots, t_p^{\pm}] \otimes E$ which is an injection in each weight space, so one can think of a rational simplex in $\mathcal{D}(\mathcal{M}^{\mathcal{C}}(v), [E])$ as a sub-$\mathcal{O}_X[t]$ module of $\mathcal{O}_X[t^{\pm}] \otimes E$ satisfying the conditions of the previous lemma.

Proposition 4.10. *Let $E \in \mathcal{C} \subset \mathcal{A}$. For any decreasing filtration $E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_p \supsetneq 0$ with $\mathrm{gr}\, E_\bullet \in \mathcal{C}$ and any $l \in \mathbb{Z}^p$, we can define a canonical morphism $f : \Theta^p \to \mathcal{M}^{\mathcal{C}}(v)$ which defines a simplex*

$$\sigma(E_\bullet, l) : \Delta_{p-1} \to \mathcal{D}(\mathcal{M}^{\mathcal{C}}(v), [E]).$$

These simplices cover $\mathcal{D}(\mathcal{M}^{\mathcal{C}}(v), [E])$.

Proof. Given a descending filtration, we claim that the following object in $\mathrm{D}_{qc}(\Theta^p \times X)$ defines a family of objects in $\mathcal{C}$ over Θ^p:

$$\begin{aligned}\tilde{E} &:= \sum_{j=0}^{p} \mathcal{O}_X[t_1, \ldots, t_p] \cdot (t_1 \cdots t_j)^{-1} \otimes E_j \subset \mathcal{O}_X[t_1^{\pm}, \ldots t_p^{\pm}] \otimes E \\ &= \bigoplus_{m_1,\ldots,m_p \geq -1} \mathcal{O}_X \cdot t^{\underline{m}} \otimes E_{\max\{j|m_j=-1\}}\end{aligned}$$

By construction we have $\operatorname{colim} \tilde{E}_w \simeq E$. The associated graded in the k^{th} direction, $\mathrm{gr}^{(k)} E_\bullet$, corresponds to $\tilde{E} \otimes^L \mathcal{O}_{\{t_k=0\}}$, regarded as a $(p-1)$-multigraded $\mathcal{O}_X[t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_p]$-module. Note that multiplication by t_k is an isomorphism $\mathcal{O}_X \cdot t^{\underline{m}} \otimes E_{\max\{j|m_j=-1\}} \to \mathcal{O}_X \cdot t^{\underline{m}'} \otimes E_{\max\{j|m'_j=-1\}}$ unless $k = \max\{j|m_j = -1\}$, in which case it is injective, and thus multiplication by t_k is injective. Furthermore, none of the graded pieces are mapped to $\mathcal{O}_X \cdot t^{\underline{m}} \otimes E_{\max\{j|m_j=-1\}}$ where $m_k = -1$. Thus we have

$$\begin{aligned}\tilde{E}/t_k\tilde{E} \simeq &\bigoplus_{\substack{m_1,\ldots,m_p \geq -1 \\ m_k=-1}} \mathcal{O}_X \cdot t^{\underline{m}} \otimes E_{\underbrace{\max\{j|m_j = -1\}}_{\text{always } \geq k}} \\ &\oplus \bigoplus_{\substack{m_1,\ldots,m_{k-1} \geq -1 \\ m_k=0 \\ m_{k+1},\ldots,m_p \geq 0}} \mathcal{O}_X \cdot t^{\underline{m}} \otimes (E_{\max\{j|m_j=-1\}}/E_k)\end{aligned}$$

This object, $\mathrm{gr}^{(k)} E_\bullet$, is precisely the object in $\mathrm{D}_{qc}(\Theta^{p-1} \times X)$ which is constructed from the new filtration of length $p-1$

$$E_0/E_k \oplus E_k \supset E_1/E_k \oplus E_k \supset \cdots \supset E_{k-1}/E_k \oplus E_k \supset E_{k+1} \supset E_{k+2} \supset \cdots \supset E_p \supset 0.$$

It follows by induction that $\tilde{E}$ defines a morphism $f : \Theta^p \to \mathcal{M}^{\mathcal{C}}(v)$.

For any l, regarded as a character of $\mathbb{G}_m^p$, we can form the twist $\tilde{E}(l)$, which is the tensor product with the pullback of the corresponding invertible sheaf on Θ^p. This does not affect the fibers over points of Θ, so $\tilde{E}(l)$ defines a map $\Theta^p \to \mathcal{M}^{\mathcal{C}}(v)$ as well.

In order to check that this defines a simplex $\sigma(E_\bullet, l) : \Delta_{p-1} \to \mathcal{D}(\mathcal{M}^{\mathcal{C}}(v), [E])$, one must check that the torus which stabilizes each point in $\mathbb{A}^p$ acts nontrivially (with at least one non-zero weight) on the fiber over that point. For the points with one-dimensional stabilizers, this amounts to remembering the t_k-grading of $\tilde{E}/t_k\tilde{E}$ and observing that it is concentrated in two different degrees. The computation for points with larger stabilizers is similar.

In order to show that these simplices cover $\mathcal{D}(\mathcal{M}^{\mathcal{C}}(v))$, we show that for any rational simplex $\Delta_p \to \mathcal{D}(\mathcal{M}^{\mathcal{C}}(v), [E])$, one can find a smaller p-simplex linearly embedded in Δ_p such that the composition $\Delta_p \to \Delta_p \to \mathcal{D}(\mathcal{M}^{\mathcal{C}}(v), [E])$ factors through $\sigma(E_\bullet, l) : \Delta_{q-1} \to \mathcal{D}(\mathcal{M}^{\mathcal{C}}(v), [E])$ for some descending filtration of length $q > p$.

For multi-indices $\underline{m}_1, \underline{m}_2 \in \mathbb{Z}^{p+1}$, we say that $\underline{m}_1 \leq \underline{m}_2$ if the inequality holds for every entry. Any map $\Theta^p \to \mathcal{M}^{\mathcal{C}}(v)$ is classified by an $\mathcal{O}_X[t]$-module of the form $\tilde{E} = \sum \mathcal{O}_X[t_1, \ldots, t_p] \cdot t^{\underline{w}_i} \otimes E_i \subset \mathcal{O}_X[t^{\pm}] \otimes E$ for some collection of subobjects $E_i \subset E$ with the property that $E_i \subset E_j$ whenever $\underline{m}_i \leq \underline{m}_j$. A map $\phi : \Theta^q \to \Theta^p$ is encoded by an $p \times q$ matrix with nonnegative integer entries A, and we have

$$\phi^* \tilde{E} = \sum \mathcal{O}_X[t_1, \ldots, t_q] \cdot t^{A^\tau \underline{m}_i} \otimes E_i \subset \mathcal{O}_X[t_1^\pm, \ldots, t_q^\pm] \otimes E$$

where A^τ denotes the transpose matrix, and we are regarding $\underline{m}_i$ as a column vector.

If we choose any column vector $v = (v_1, \ldots, v_p)$ with nonnegative entries such that $v \cdot (\underline{m}_i - \underline{m}_j) \neq 0$ for $i \neq j$, then for k sufficiently large the matrix $A = I_{p \times p} + kv \cdot v^\tau$ has the property that the vectors $A^\tau \underline{m}_i$ are totally ordered with respect to the partial order defined above. More precisely, positive cone $(\mathbb{R}_{\geq 0})^p$ admits a subdivision into rational polyhedral cones, where codimension 1 walls are the hyperplanes $\{v | v \cdot (\underline{m}_1 - \underline{m}_2) = 0\}$. For any $\phi : \Theta^p \to \Theta^p$ corresponding to a $p \times p$ matrix such that $A \cdot (\mathbb{R}_{\geq 0})^p$ is contained in one of these sub-cones, one will have $\phi^* \tilde{E}$ of the form $\sum \mathcal{O}_X[t] \cdot \underline{m}_i \otimes E_i$ with $\underline{m}_1 \leq \underline{m}_2 \leq \cdots$. Thus any point in Δ_{p-1} is contained in the image of a map $\Delta_{p-1} \to \Delta_{p-1}$ such that the composition with $\Delta_{p-1} \to \mathcal{D}(\mathcal{M}^\mathcal{C}(v), [E])$ has this property.

Now assume $\Theta^p \to \mathcal{M}^\mathcal{C}(v)$ is classified by an object with $\underline{m}_1 \leq \underline{m}_2 \leq \cdots \leq E_q$ and $E_1 \subset E_2 \subset \cdots \subset E_q$. Then we can find a matrix with nonnegative entries such that

$$A^\tau(\underbrace{1, \ldots, 1}_{i-\text{times}}, 0, \ldots, 0) = \underline{m}_i.$$

The corresponding map $\Delta_{p-1} \to \Delta_{q-1}$ composed with $\sigma(E_\bullet, l) : \Delta_{q-1} \to \mathcal{D}(\mathcal{M}^\mathcal{C}(v), [E])$ yields the original rational simplex $\Delta_{p-1} \to \mathcal{D}(\mathcal{M}^\mathcal{C}(v), [E])$. □

Corollary 4.11. *Points of* $\mathcal{D}(\mathcal{M}^\mathcal{C}(v), [E])$ *correspond to descending filtrations* $E = E_1 \supsetneq \cdots \supsetneq E_p \supsetneq 0$ *along with a choice of real weights satisfying* $w_1 < \cdots < w_p$.

To a weighted descending filtration we associate the sequence of triples

$$\alpha = \left\{(r_j, d_j, w_j) = (\Im Z(\operatorname{gr}_j E_\bullet), -\Re Z(\operatorname{gr}_j E_\bullet), w_j)\right\}_{j=1,\ldots,p} \quad (12)$$

We refer to any sequence of this form with $r_j - id_j \in \mathbb{H} \cup \mathbb{R}_{<0}$ and $w_1 < \cdots < w_p$ as a *rank-degree-weight* sequence. Because $Z(E)$ only depends on the numerical class of E, α is a topological invariant in the sense that these numbers are locally constant on $[\Theta, \mathcal{M}(v)]$.

The fact that $\mathcal{M}^\mathcal{C}(v) \subset \mathcal{M}(v)$ is a stack of full subgroupoids implies that $\mathcal{D}(\mathcal{M}^\mathcal{C}(v), [E]) \subset \mathcal{D}(\mathcal{M}(v), [E])$ is an inclusion of spaces, so it will suffice to construct stability functions from cohomology classes on $\mathcal{M}(v)$. The Euler characteristic induces a perfect pairing $\mathcal{N}(X) \otimes \mathcal{N}^{perf}(X) \to \mathbb{R}$, and it extends to a perfect $\mathbb{C}$-linear pairing between $\mathcal{N}(X) \otimes \mathbb{C}$ and $\mathcal{N}^{perf} \otimes \mathbb{C}$. Therefore we can find a unique element $w_Z \in \mathcal{N}^{perf}(X) \otimes \mathbb{C}$ such that

$$Z(x) = \chi(w_Z \otimes x), \forall x \in \mathcal{N}(X) \otimes \mathbb{C}$$

For any map of stacks, ϕ, we will also implicitly extend the functors ϕ^* and, when it is defined ϕ_*, $\mathbb{C}$-linearly to $\mathcal{N} \otimes \mathbb{C}$.

Let $\mathcal{E}$ denote the universal object in the derived category of $\mathcal{M} \times X$, and we consider the diagram

$$\mathcal{M}(v) \xleftarrow{p_1} \mathcal{M}(v) \times X \xrightarrow{p_2} X$$

We define the cohomology classes

$$\begin{aligned} l &:= |Z(v)|^2 \operatorname{ch}_1 \left((p_1)_* \left(\mathcal{E} \otimes p_2^* \Im \left(\tfrac{-w_Z}{Z(v)} \right) \right) \right) \\ b &:= 2 \operatorname{ch}_2 \left((p_1)_* \left(\mathcal{E} \otimes p_2^* \Im (w_Z) \right) \right) \end{aligned} \tag{13}$$

Where $\Im : \mathcal{N}(X) \otimes \mathbb{C} \to \mathcal{N}(X) \otimes \mathbb{R}$ denotes the imaginary part.

Lemma 4.12. *Let $f : \Theta \to \mathcal{M}(v)$ correspond to a weighted descending filtration $E_\bullet$. Let $\{(r_j, d_j, w_j)\}$ be the rank-degree-weight sequence associated to $E_\bullet$ as in (12) and let $Z(v) = -D + iR$. Then we have*

$$\tfrac{1}{q} f^* l = \sum_{j=1}^{p} w_j (R d_j - D r_j), \quad \text{and} \quad \tfrac{1}{q^2} f^* b = \sum_{j=1}^{p} w_j^2 r_j \tag{14}$$

Proof. If $\pi : \Theta \times X \to \Theta$ and $\mathcal{E}_f$ is the object classified by f, then in K-theory, we have $\mathcal{E}_f \simeq \sum_j u^{-w_j} [\operatorname{gr}_j E_\bullet]$ under the decomposition $K_0(X \times \Theta) \simeq K_0(X) \otimes \mathbb{Z}[u^{\pm}]$. We can thus compute

$$\begin{aligned} f^* l &= |Z(v)|^2 \operatorname{ch}_1 \left(\pi_* (\mathcal{E}_f \otimes p_2^* \Im (\frac{-w_Z}{Z(v)})) \right) \\ &= |Z(v)|^2 \operatorname{ch}_1 \left(\sum u^{-w_j} \Im \left(\frac{-\chi(X, w_Z \otimes \operatorname{gr}_j E_\bullet)}{Z(v)} \right) \right) \\ &= \sum w_j q \Im \left(\chi(X, w_Z \otimes \operatorname{gr}_j E_\bullet) \cdot \overline{Z(v)} \right) \end{aligned}$$

By the defining property of w_Z, we have $\chi(w_Z \otimes \operatorname{gr}_j E_\bullet) = (-d_j + i r_j)$, and the claim follows. The computation for $f^* b$ is almost identical, so we omit it. ∎

4.1.2. *The Harder-Narasimhan property and Θ-stability.* For a nonzero $E \in \mathcal{A}$, the phase $\phi(E) \in (0, 1]$ is the unique number such that $Z(E) = |Z(E)| e^{2\pi i \phi(E)}$, and by convention $\phi = 1$ if $Z(E) = 0$. For any object $E \in \mathcal{A}$, we say that E is semistable if there are no sub-objects $F \subset E$ in the abelian category $\mathcal{A}$ such that $\phi(F) > \phi(E)$. We say that $(\mathcal{A}, Z)$ has the *Harder-Narasimhan* property if every object E admits a filtration by subobjects $E = E_1 \supset \cdots \supset E_p \supset E_{p+1} = 0$ in $\mathcal{A}$ with each subquotient $\operatorname{gr}_j E_\bullet$ semistable and $\phi(\operatorname{gr}_j E_\bullet)$ increasing in j.

As in the situation of Bridgeland stability conditions, we let $\mathcal{P}(\phi)$ be the full subcategory spanned by semistable objects of phase ϕ. In addition, for any subset $I \subset (0, 1]$, we let $\mathcal{P}(I)$ be the full subcategory generated under extensions by semistable objects whose phase lies in I.

Lemma 4.13. *$\mathcal{P}([\epsilon, 1])$ is right orthogonal to $\mathcal{P}((0, \epsilon))$. If $(\mathcal{A}, Z)$ has the Harder-Narasimhan property, then we have a torsion theory $(\mathcal{P}([\epsilon, 1]), \mathcal{P}((0, \epsilon)))$ on $\mathcal{A}$.*

Proof. For the first claim it suffices to show that $\operatorname{Hom}(E_1, E_2) = 0$ for semistable objects with $\phi(E_1) > \phi(E_2)$. Letting $f : E_1 \to E_2$, if $\operatorname{im}(f)$ is nonzero, then the semistability of E_1 and E_2 implies that $\phi(E_1) \leq \phi(\operatorname{im}(f)) \leq \phi(E_2)$. Hence if this inequality is violated we must have $f = 0$.

For the second claim, it suffices to show, once we have semiorthogonality, that every E admits a short exact sequence $0 \to \tau_{\geq \epsilon} E \to E \to \tau_{< \epsilon} E \to 0$ with $\tau_{\geq \epsilon} E \in \mathcal{P}([\epsilon, 1])$ and $\tau_{< \epsilon} E \in \mathcal{P}((0, \epsilon))$. Choosing a HN filtration $E_\bullet$ of E, we let $\tau_{\geq \epsilon} E = E_j$ and $\tau_{< \epsilon} E = E / E_j$ where j is the smallest index for which $\phi(\operatorname{gr}_j E_\bullet) \geq \epsilon$. ∎

It follows from this lemma that if $(\mathcal{A}, Z)$ has the HN property, then the HN filtration of any object is functorial, and in particular morphisms between objects of $\mathcal{A}$ are compatible with their HN filtrations. The main result of this section is that the Harder-Narasimhan property is equivalent to the optimal degeneration problem as we have formulated it.

Theorem 4.14. *Let $\mathcal{A} \subset \mathrm{D}^b(X)$ be a full abelian subcategory of the heart of a t-structure and let $Z : \mathcal{N}(X) \to \mathbb{C}$ a homomorphism mapping $\mathcal{A}$ to $\mathbb{H} \cup \{0\}$. For any $E \in \mathcal{A}$, let*

$$\mu : \mathcal{D}(\mathcal{M}(v), [E]) \to \mathbb{R} \cup \{\pm\infty\}$$

be the numerical invariant associated to the cohomology classes (13) via Construction 2.19, and let M^μ the corresponding stability function of Definition 3.1.

Then $E \in \mathcal{A}$ is semistable if and only if $M^\mu([E]) \leq 0$. Furthermore the following are equivalent:

(1) The pair $(\mathcal{A}, Z)$ has the Harder-Narasimhan property; and
(2) Every object in $\mathcal{A}$ has a maximal torsion subobject, and for every unstable $E \in \mathcal{F}$, the restriction of μ to $\mathcal{D}(\mathcal{M}^{\mathcal{F}}(v), [E])$ obtains a unique maximum (with $\mu < \infty$).

Remark 4.15. It follows from Lemma 4.18 below that if $Z(\mathcal{A}) \subset \mathbb{Q} + i\mathbb{Q}$, then the unique maximum in (2) occurs at a rational point of $\mathcal{D}(\mathcal{M}^{\mathcal{F}}(v), [E])$. When this is the case, the unique (up to ramified coverings) map $\Theta \to \mathcal{M}^{\mathcal{F}}(v)$ with $f(1) \simeq [E]$ which maximizes $\mu(f)$ classifies the Harder-Narasimhan filtration of E.

The proof of Theorem 4.14 bears a strong formal resemblance to the analysis of semistability for filtered isocrystals in p-adic Hodge theory as presented in [DOR]. For any sequence of complex numbers $\{z_j\} \subset \mathbb{H} \cup \mathbb{R}_{\leq 0}$, we associate the convex polyhedron

$$\mathrm{Pol}(\{z_j\}) = \left\{ \sum \lambda_j z_j + c \,\middle|\, \lambda_j \in [0,1] \text{ and } c \in \mathbb{R}_{\geq 0} \right\} \subset \mathbb{H} \cup \mathbb{R}_{\geq 0}$$

We will also overload this notation in several ways. For any filtration $E_\bullet$ in $\mathcal{A}$, we let $\mathrm{Pol}(E_\bullet) = \mathrm{Pol}(\{Z(\mathrm{gr}_j E_\bullet)\})$, and for any object $E \in \mathcal{A}$ we let $\mathrm{Pol}(E) := \mathrm{Pol}(E_\bullet)$, where $E_\bullet$ is the HN filtration of E.

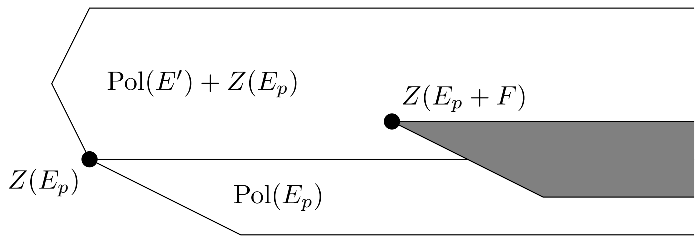
Lemma 4.16. *Let $(\mathcal{A}, Z)$ be an abelian category with central charge which has the Harder-Narasimhan property. Then for any E and any subobject $F \subset E$, one has $Z(F) \in \mathrm{Pol}(E)$*

Proof. We proceed by induction on the length of the HN filtration of E. The base case is when E is semistable, in which case the statement of the lemma is precisely the definition of semistability, combined with the fact that $\Im Z(F) \leq \Im Z(E)$ for any subobject.

Let $E_p \subset E$ be the first subobject in the HN filtration, so by hypothesis E_p is semistable, say with phase ϕ_p. Lemma 4.13 guarantees that any subobject of E must have phase $\leq \phi_p$. Let $E' = E/E_p$ and let F' be the image of $F \to E'$. Then can pullback the defining sequence for E' to F':

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_p & \longrightarrow & E_p + F & \longrightarrow & F' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_p & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

The length of the HN filtration of E' is one less than E, and $\mathrm{Pol}(E)$ decomposes as a union of $\mathrm{Pol}(E_p)$ and a shift of $\mathrm{Pol}(E')$ by $Z(E_p)$. Thus by the inductive hypothesis $Z(E_p + F) = Z(F') + Z(E_p)$ lies in $Z(E_p) + \mathrm{Pol}(E')$, which we illustrate graphically:



We also have the short exact sequence $0 \to E_p \cap F \to E_p \oplus F \to E_p + F \to 0$, which implies that

$$\begin{aligned} Z(F) &= Z(E_p + F) + Z(E_p \cap F) - Z(E_p) \\ &= Z(E_p + F) - Z(E_p / E_p \cap F) \end{aligned}$$

We have $\Im Z(E_p/E_p \cap F) \leq \Im Z(E_p)$, and the phase of $Z(E_p/E_p \cap F)$ is $\geq \phi_p$ because E_p is semistable. It follows that $Z(F)$ must lie in the shaded region above, which is a translate of $\mathrm{Pol}(E_p)$, and hence $Z(F) \in \mathrm{Pol}(E)$. NB: Even though we have drawn the diagram assuming that $\Im Z(E_p) > 0$, the argument works without modification in the case where E_p is torsion as well.

□

Let $\alpha = \{(r_j, d_j, w_j)\}$ be a rank-degree-weight sequence. For $j = 1, \ldots, p$, we define the j^{th} phase $\phi_j \in (0, 1]$ by the property that $ir_j - d_j \in \mathbb{R}_{>0} e^{i\phi_j}$. Note that by Lemma 4.12, the numerical invariant can be expressed formally in terms of the rank-degree-weight sequence $\mu = \mu(\alpha)$. Furthermore it extends continuously to rank-degree-weight sequences where the w_j are arbitrary real numbers rather than integers.

Lemma 4.17 (Insertion and deletion)**.** *Let α and α' be rank-degree-weight sequences such that α' is obtained from α by discarding the $(k+1)^{st}$ element and relabelling $r'_k = r_k + r_{k+1}$ and $d'_k = d_k + d_{k+1}$, and*

$$w'_j := \begin{cases} w_j, & \text{if } j < k \\ \frac{w_j r_j + w_{j+1} r_{j+1}}{r_j + r_{j+1}}, & \text{if } j = k \\ w_{j+1}, & \text{if } j > k \end{cases}$$

And assume that $r_j + r_{j+1} > 0$. We have:

- *If $\mu(\alpha) \geq 0$ and $\phi_k \geq \phi_{k+1}$, then $\mu(\alpha') \geq \mu(\alpha)$ with equality if and only if $\phi_k = \phi_{k+1}$ and $\mu(\alpha) = 0$.*
- *If $\mu(\alpha') \geq 0$ and $\phi_k < \phi_{k+1}$, then $\mu(\alpha) > \mu(\alpha')$.*

Proof. We denote $\mu = L/\sqrt{B}$. Substituting α' for α, the numerator and denominator change by

$$\begin{aligned}\Delta L &= w_k'(d_k + d_{k+1}) - w_k d_k - w_{k+1} d_{k+1} \\ &= \frac{w_{k+1} - w_k}{r_k + r_{k+1}} (d_k r_{k+1} - r_k d_{k+1}) \\ \Delta B &= (w_k')^2 (r_{k+1} + r_k) - w_k^2 r_k - w_{k+1}^2 r_{k+1} \\ &= -\frac{r_k r_{k+1}}{r_k + r_{k+1}} (w_k - w_{k+1})^2\end{aligned}$$

Note that $\Delta B \leq 0$. Also the sign of $d_k r_{k+1} - d_{k+1} r_k$, and hence the sign of ΔL, is the same as the sign of $\phi_k - \phi_{k+1}$, and $\Delta L \neq 0$ if $\phi_k \neq \phi_{k+1}$. The claim follows from these observations. □

On the level of decreasing weighted filtrations, the modification $\alpha \mapsto \alpha'$ in Lemma 4.17 corresponds to deleting the $(k+1)^{st}$ subobject from the filtration $E = E_1 \supset \cdots \supset E_p \supset E_{p+1} = 0$ and adjusting the weights appropriately. We also observe that if $\phi_k \geq \phi_{k+1}$ then this does not change $\operatorname{Pol}(E_\bullet)$. Hence Lemma 4.17 shows that from any decreasing weighted filtration $E_\bullet$, we can make a sequences of deletions to obtain a decreasing weighted filtration $E'_\bullet$ such that $\phi'_1 < \cdots \phi'_{p'}$, $\operatorname{Pol}(E_\bullet) = \operatorname{Pol}(E'_\bullet)$, and $\mu(E'_\bullet) \geq \mu(E_\bullet)$ with strict inequality if $E_\bullet$ was not convex to begin with.

Given a sequence $\{z_j\} \subset \mathbb{H} \cup \mathbb{R}_{\leq 0}$, we let $R = \sum r_j$ and define a continuous piecewise linear function $h_{\{z_j\}} : [0, R] \to \mathbb{R}$,

$$h_{\{z_j\}}(r) := \sup \{ x \,|\, ir - x \in \operatorname{Pol}(\{z_j\}) \}.$$

Note that this only depends on $\operatorname{Pol}(\{z_j\})$.

Lemma 4.18. *Let $ir_j - d_j \in \mathbb{H}$ be sequence of points such that $\phi_1 < \cdots < \phi_p < 1$. Let $\nu_j := d_j / r_j$, $D := \sum d_j$, $R := \sum r_j$, and $\nu := D/R$, and let $h(x) := h_{\{ir_j - d_j\}}(x)$. Then μ is maximized by assigning the weights $w_j \propto \nu_j - \nu$. The maximum is*

$$\frac{\mu}{R} = \sqrt{\left(\sum \nu_j^2 r_j\right) - \nu^2 R} = \sqrt{\int_0^R (h'(x))^2 dx - \nu^2 R}. \tag{15}$$

Proof. We can think of the numbers $ev_1, \cdots, r_p$ as defining an inner product $\vec{a} \cdot \vec{b} = \sum a_j b_j r_j$. Then given a choice of weights $\vec{w} = (w_1, \cdots, w_p)$, the numerical invariant can be expressed as

$$\mu = \frac{R}{|\vec{w}|} \vec{w} \cdot (\vec{\nu} - \nu \vec{1})$$

where $\vec{\nu} = (\nu_1, \cdots, \nu_p)$ and $\vec{1} = (1, \ldots, 1)$. From linear algebra we know that this quantity is maximized when $\vec{w} \propto \vec{\nu} - \nu\vec{1}$, and the maximum value is $R|\vec{\nu} - \nu\vec{1}|$. In the case when $\nu_1 < \cdots < \nu_p$ the assignment $\vec{w} \propto \vec{\nu} - \nu\vec{1}$ satisfies the constraints $w_1 < w_2 < \cdots < w_p$. The integral formula simply expresses $\nu_j^2 r_j$ as the integral of $h'(x)$ on an interval of length r_j along which it is constant of value ν_j. □

Proof of Theorem 4.14. First we show that $E \in \mathcal{A}$ is unstable if and only if $M^\mu([E]) > 0$. From the construction of μ the latter is equivalent to the existence of a map $f : \Theta \to \mathcal{M}^{\mathcal{A}}(v)$ with $f(1) \simeq E$ and $\frac{1}{q} f^* l > 0$. If we have a single subobject $F \subset E$ with $\phi(F) > \phi(E)$, we consider this as a two step filtration

$\operatorname{gr}_2 E_\bullet = F$ and $\operatorname{gr}_1 E_\bullet = E/F$ and $w_2 > w_1$ arbitrary. Then Lemma 4.12 shows that

$$\begin{aligned}\frac{1}{q} f^* l &= w_1 \Im\left((Z(E) - Z(F))\overline{Z(E)}\right) + w_2 \Im\left(Z(F)\overline{Z(E)}\right) \\ &= (w_2 - w_1)\Im\left(Z(F)\overline{Z(E)}\right) > 0\end{aligned}$$

Conversely, assume that $f : \Theta \to \mathcal{M}^{\mathcal{A}}(v)$ is such that $\mu(f) > 0$. Lemma 4.17 shows that if $E_\bullet$ is the decreasing weighted filtration corresponding to f, then by deleting steps in the filtration and appropriately adjusting the weights (we might have to perturb slightly and then scale so that the weights are integers) we obtain a new descending weighted filtration $E'_\bullet$ such that $\phi'_1 < \cdots < \phi'_{p'}$ and $\mu(E'_\bullet) > \mu(E_\bullet) > 0$. In particular this new filtration must be nontrivial, and $\phi'_{p'} > \phi(E)$, so $E'_{p'} \subset E$ is a destabilizing subobject.

Proof that (1) $\Rightarrow$ (2)*:*

If $(\mathcal{A}, Z)$ has the Harder-Narasimhan property, then the maximal torsion subsheaf of $E \in \mathcal{A}$ is the truncation $\tau_{\geq 1} E$ provided by Lemma 4.13. Lemmas 4.16, 4.17, and 4.18 essentially verify the principal (B) for the stability function μ on $\mathcal{D}(\mathcal{M}^{\mathcal{F}}(v), [E])$.

More precisely, Lemma 4.17 shows that starting from any decreasing weighted filtration with $\mu(\alpha) > 0$ and such that $\phi_k \geq \phi_{k+1}$, one can produce a new decreasing weighted filtration $E'_\bullet$ such that $\mu(E'_\bullet) > \mu(E_\bullet)$, $\phi'_1 < \cdots < \phi'_{p'}$, and $\operatorname{Pol}(E_\bullet) = \operatorname{Pol}(E'_\bullet)$. It thus suffices to maximize μ over the set of filtrations with this property. Lemma 4.16 shows that for any such $E_\bullet$ we have $\operatorname{Pol}(E_\bullet) \subset \operatorname{Pol}(E)$. Thus once we have shown that the expression (15) is strictly monotone increasing with respect to inclusion of polyhedra, it will follow that if $\mu(E_\bullet) > 0$ for some weighted descending filtration of E, then $\mu(E_\bullet)$ is strictly less than the value of μ obtained by assigning weights to the HN filtration as in Lemma 4.18.

To show that (15) is monotone increasing with respect to inclusion of polyhedra, we let $h_1, h_2 : [0, R] \to \mathbb{R}$ be two continuous piecewise linear functions with $h'_i(x)$ decreasing and with $h_1(x) \leq h_2(x)$ with equality at the endpoints of the interval. We must show that $\int_0^R (h'_1(x))^2 < \int_0^R (h'_2(x))^2$. First by suitable approximation with respect to a Sobolev norm it suffices to prove this when h_i are smooth functions with $h'' < 0$.[19] Then we can use integration by parts

$$\begin{aligned}\int_0^R (h'_2)^2 - (h'_1)^2 dx &= \int_0^R (h'_2 + h'_1)(h'_2 - h'_1) dx \\ &= (h'_2 - h'_1)(h_2 - h_1)|_0^R - \int_0^R (h_2 - h_1)(h''_2 + h''_1) dx\end{aligned}$$

The first term vanishes because $h_1 = h_2$ at the endpoints, and the second term is strictly positive unless $h_1 = h_2$.

Proof that (2) $\Rightarrow$ (1)*:*

First if any E admits a maximal torsion subobject $\tau_{\geq 1} E$, then, being torsion, the object $\tau_{\geq 1}$ is semistable, and it suffices to show that the torsion free quotient $\tau_{<1} E = E/\tau_{\geq 1}$ admits a HN filtrations with phases < 1.

[19] One can probably prove the inequality without appealing to analysis by using a discrete integration by parts argument.

Let $E \in \mathcal{F}$ be unstable, so μ obtains a positive maximum at a unique point of $\mathcal{D}(\mathcal{M}^{\mathcal{F}}(v), [E])$. Proposition 4.10 allows us to associate a weighted decreasing filtration (with real weights) to that point, and we let $\alpha = \{(r_j, d_j, w_j)\}$ be the associated rank-degree-weight sequence. Because α maximizes μ, Lemma 4.17 implies that α must have the property that $\phi_1 < \cdots \phi_p < 1$. It thus suffices to show that $\operatorname{gr}_k E_\bullet$ is semistable with respect to $(\mathcal{A}, Z)$.

Let $\operatorname{gr}_k E_\bullet \twoheadrightarrow F$ be a torsion-free quotient such that $\phi(F) < \phi(\operatorname{gr}_k E_\bullet)$. Then we can define a new weighted filtered object $E'_\bullet$ by "insertion," with

$$E'_j = \begin{cases} E_j, & \text{if } j \leq k \\ \ker(E_k \twoheadrightarrow F), & \text{if } j = k+1 \\ E_{j-1}, & \text{if } j > k+1 \end{cases}$$

and $w'_k < w'_{k+1}$ chosen appropriately so that $w_k = (r_k w'_k + r_{k+1} w'_{k+1})/(r_k + r_{k+1})$. It follows from the initial hypothesis on the phase of F and the short exact sequence

$$0 \to \operatorname{gr}_{k+1} E'_\bullet \to \operatorname{gr}_k E_\bullet = E'_k / E'_{k+2} \to F \simeq \operatorname{gr}_k E'_\bullet \to 0$$

that $\phi(\operatorname{gr}_{k+1} E'_\bullet) > \phi(\operatorname{gr}_k E'_\bullet)$. Lemma 4.17 now implies that, provided $\mu(E_\bullet) \geq 0$, we have $\mu(E'_\bullet) > \mu(E_\bullet)$. This would contradict the maximality of f, and so we see that $\operatorname{gr}_k E_\bullet$ must be semistable. □

4.1.3. *Θ-stratification of $\mathcal{M}^{\mathcal{F}}(v)$.* We now connect our analysis of $\mathcal{M}(v)$ with the results of Section 3. Recall that in the context of moduli problems, a subset, Σ, of isomorphism classes of $\mathfrak{X}(k)$ is said to be bounded if there is a finite type k-scheme T and a morphism $T \to \mathfrak{X}$ such that every point of Σ lies in the image of $T(k) \to \mathfrak{X}(k)$. We shall consider pre-stability conditions $(\mathcal{A}, Z)$ which have two additional properties

- *Openness*: given an object $E \in D^b(T \times X)$, the locus of $t \in T$ over which $E_t \in \mathcal{F}$ is open, and
- *Boundedness*: for any family $E_t \in \mathcal{A}$ parameterized by a finite type test scheme T, the family of objects in $\mathcal{F}$ with phase bounded from above and which admit a surjection from E_t for some $t \in T$ is bounded.

As for examples satisfying these hypotheses, the notable example is Example 4.3 (See [HL4, Lemma 1.7.9] for the boundedness property). These hypotheses are also known to hold in several examples for Bridgeland stability conditions.

Lemma 4.19 (Lemma 3.15 of [T1])**.** *Let $(Z, \mathcal{A})$ be an algebraic Bridgeland stability condition on $D^b(X)$. Assume that the functor $\mathcal{M}(v)^{ss}$ is bounded for every class $v \in \mathcal{N}(X)$. Then $(Z, \mathcal{A})$ satisfies the boundedness property above.*

In Section 4 of [T1], Toda shows that the openness property as well as the boundedness of $\mathcal{M}(v)^{ss}$, and hence the boundedness property above, hold for a class of Bridgeland stability conditions constructed in [B$^+$]. Thus the following Proposition provides a class of examples of Θ-stratifications which are not explicitly related to a geometric invariant theory problem.

Proposition 4.20. *Let X be a smooth projective variety, and let $(\mathcal{A}, Z)$ be a pre-stability condition on $D^b(X)$ satisfying the openness and boundedness properties above.*

(1) If $\mathcal{A}$ is Noetherian, then $(Z, \mathcal{A})$ has the Harder-Narasimhan property,

(2) if $(Z, \mathcal{A})$ has the Harder-Narasimhan property, then $\mathcal{M}^{\mathcal{F}}(v) \subset \mathcal{M}(v)$ is an open substack (hence algebraic), and the numerical invariant associated to the classes of Equation 13 defines a Θ-stratification of $\mathcal{M}^{\mathcal{F}}(v)$, and

(3) $\mathcal{M}^{ss}(v) \subset \mathcal{M}(v)$ is an open substack for all $v \in \mathcal{N}(X)$ with $\Im Z(v) > 0$.

Remark 4.21. The smoothness hypothesis only appears in our identification of the derived deformation theory of an object $E \in D^b(X) = \mathrm{Perf}(X)$ with $R\Gamma(E^* \otimes E[1])$. More generally, a similar analysis, with the exact same proof, holds for a singular X, replacing the tangent complex with $R\underline{\mathrm{Hom}}(E, E[1]) \in D^+(X)$. Unfortunately at this time, the author is not aware of a suitable treatment of the derived moduli stack of objects in $D^b(X)$, so we have restricted our focus slightly.

If we consider the usual t-structure on $D^b(X)$, then for any $\phi : S \to \mathcal{M}(v)$ classifying a flat family of coherent sheaves $\mathcal{E}$ on X, the algebraic space $[\Theta, \mathfrak{X}] \times_{\mathfrak{X}} S$ is a disjoint union of generalized Quot schemes for the family $\mathcal{E}$. Thus we see that for $\mathcal{M}(v)$ to be reductive in the sense of Definition 2.27 is equivalent to the properness of the Quot scheme. For a general t-structure on $D^b(X)$ satisfying the openness property above, the existence of an algebraic generalized Quot space follows from the algebraicity of $[\Theta, \mathcal{M}(v)]$. Our key observation is that this generalized Quot space satisfies the valuative criterion for properness for any t-structure on $D^b(X)$.

Lemma 4.22. *Let X be a projective scheme, and let $\mathcal{A}$ be the heart of a t-structure on $D^b(X)$. Then for any $v \in \mathcal{N}(X)$, the stack $\mathcal{M}(v)$ is weakly reductive.*

Proof. Let R be a discrete valuation ring over k, and K its field of fractions, and let $(\pi) \subset R$ be the maximal ideal. The valuative criterion amounts to the following question: if we have an object $N \in D^b(\mathcal{O}_X \otimes R\,-\mathrm{mod})^{\heartsuit}$ which defines a family over $\mathrm{Spec}\, R$ and a descending weighted filtration of $N \otimes_R K$, then is there a unique descending weighted filtration of N inducing the given one on $N \otimes_R K$ and such that all of the subquotients of N define families over R as well?

Because R is a DVR, for $N \in D^b(\mathcal{O}_X \otimes R\,-\mathrm{mod})^{\heartsuit}$ to define a family, it is necessary and sufficient that $N \otimes_R (R/(\pi)) \in D(X)^{\heartsuit}$, or equivalently that $\mathrm{Cone}(N \xrightarrow{\times\pi} N) \in D(X)^{\heartsuit}$. We claim that this is equivalent to the canonical homomorphism $N \to N \otimes_R K$ being injective. We can identify $N \otimes_R K$ with the colimit of the diagram $N \to N \to N \to \cdots$, where each arrow is multiplication by π. First, we have $\ker(\times\pi : N \to N) \subset \ker(N \to N \otimes_R K)$, which gives one direction of the claim. The other direction uses the fact that bounded coherent objects commute with filtered colimits in $D^+_{qc}(X \times \mathrm{Spec}\, R)$. If the composition $K \to N \to N \otimes_R K$ is 0, then the composition $K \to N \xrightarrow{\times \pi^m} N$ vanishes for some $m > 0$, which implies that the map $K \to N$ vanishes.

The observations above reduce the valuative criterion to the following question: if $M' \subset N \otimes_R K$ is an $\mathcal{O}_X \otimes K$-submodule, then is there a unique extension to a map of short exact sequences of $\mathcal{O}_X \otimes R$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & N \otimes_R K & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

which identifies $M' \simeq N' \otimes_R K$ and $M'' \simeq N'' \otimes_R K$ and such that both dotted arrows are injective in $D_{qc}(\mathcal{O}_X \otimes K)^\heartsuit$. It is a general fact, considering only the abelian category $D_{qc}(\mathcal{O}_X \otimes R)$, that such a diagram exists, with N' the equalizer and N'' identified with the image of the composition $N' \to N \otimes_R K \to M''$.

Finally we must show that the canonical map $N' \otimes_R K \to M'$ is an isomorphism (and thus so is $N'' \otimes_R K \to M''$). Note that because filtered colimits are t-exact in $D_{qc}(\mathcal{O}_X \otimes R)$, the homomorphism $N' \otimes_R K \to N \otimes_R K \simeq M$ is injective, and thus $N' \otimes_R K \to M'$ is injective. For surjectivity, it suffices to show that every map $P \to M'$ from $P \in D^b(\mathcal{O}_X \otimes R)^\heartsuit$ factors through $N' \otimes_R K$. Because $N \otimes_R K \simeq M$, for any map $P \to M$ with $P \in D^b(\mathcal{O}_X \otimes R)^\heartsuit$ fits into a diagram

$$\begin{array}{ccccc} & & & & N \\ & & & \nearrow & \downarrow \\ P & \xrightarrow[\times \pi^m]{} & P & \longrightarrow & M \end{array}$$

If the bottom arrow factors through M', then by definition of N' as a fiber product the top arrow factors through N', which shows that our initial map $P \to M'$ factors through $N' \otimes_R K$. $\square$

Lemma 4.23. *Let $(Z, \mathcal{A})$ be a pre-stability condition such that $\mathcal{A}$ is Noetherian, and for any $E \in \mathcal{A}$ and any $c \in (0, 1]$, the set*

$$\{\phi(F) \,|\, \exists \text{ a surjection } E \twoheadrightarrow F\} \subset \mathbf{H} \cup \mathbb{R}_{\leq 0}$$

is finite, the $(Z, \mathcal{A})$ satisfies the HN property.

Proof. This follow immediately from [B, Proposition 2.4]. $\square$

Proof of Proposition 4.20. The claim (1) follows immediately from the previous lemma. Thus let us assume that $(Z, \mathcal{A})$ satisfies the Harder-Narasimhan property for the remainder of the proof. In order to verify that $\mathcal{M}^{\mathcal{F}}(v) \subset \mathcal{M}(v)$ is open, let $T \to \mathcal{M}(v)$ classify a flat family of objects $E_t \in \mathcal{A}$ of class v over a k-scheme of finite type, T. The locus of t for which $E_t \in \mathcal{F}$ is the complement of the image under ev_1 of a substack $\mathfrak{Y} \subset [\Theta, \mathcal{M}(f)]$, defined as the union of the connected components classifying weighted descending filtrations of the form $\cdots = 0 \subset F_1 \subset F_0 = \cdots$ where F_0 is of class v and F_0 is of torsion class. The boundedness hypothesis implies that the objects E_t which are not torsion free lie in the image of finitely many connected components of $\mathfrak{Y} \times_{[\Theta, \mathcal{M}(v)]} S$, hence Lemma 4.22 implies that the set of such $t \in T$ is closed.

Principle (B+) for the numerical invariant μ is guaranteed by the boundedness hypothesis above. Indeed for any finite type T with a map $\phi : T \to \mathcal{M}^{\mathcal{F}}(v)$, recalling the correspondence between finite filtrations and rational simplices of Proposition 4.10, one can choose rational simplices of $\mathcal{C}omp(\mathfrak{X}, \phi)$ corresponding to finite filtrations which are convex. The boundedness hypothesis guarantees that this set of simplices is finite, and Theorem 4.14 guarantees that for every point of T, the HN-filtration corresponds to a point on one of these simplices.

Next we verify the lifting property of Equation 11. Let us base-change to $\bar{k}$ so that we may drop it from our notation. Specifying a map $\Theta \to \left[B\mathbb{G}_m, \mathcal{M}^{\mathcal{F}}(v)\right]$ is equivalent to specifying a map $\Theta \times \mathbb{G}_m \to \mathcal{M}^{\mathcal{F}}(v)$, which is equivalent (See Lemma 4.8) to specifying a graded object $E = \bigoplus_{v \in \mathbb{Z}} E_v$ such that $E_v \in \mathcal{F}$ for all $v \in \mathbb{Z}$, along with a descending weighted filtration of each E_v such that $\operatorname{gr}_w E_v \in \mathcal{F}$

for all $v, w \in \mathbb{Z}$. Specifying the commutative square (11) amounts to specifying an object $E \in \mathcal{F}$ along with a descending weighted filtration such that $\mathrm{gr}_v E = E_v$ for all v. For each v we let $w_{v;1} > \cdots > w_{v;n_v}$ be the set of weights for which $\mathrm{gr}_w E_v \neq 0$. If we choose an $N > 2 \max_{v,i} |w_{v;i}|$, then we can define a filtration of E such that $\mathrm{gr}_{Nv+w_{v;i}} E_\bullet \simeq \mathrm{gr}_{w_{v;i}} E_v$. The lift in 11 corresponds to the unique map $\Theta^2 \to \mathcal{M}$ connected this filtration with the original filtration of E with $\mathrm{gr}_v E_\bullet = E_v$.

The remainder of the proof is almost a direct application of Theorem 3.16. The boundedness hypothesis implies that the morphism ev_1 is quasi-compact when restricted to the connected components of $[\Theta, \mathcal{M}(v)]$ corresponding to filtrations $E_\bullet$ with a fixed choice of classes $[\mathrm{gr}_j E_\bullet] \in \mathcal{N}(X)$, and this combined with Lemma 4.22 implies that $\mathcal{M}(v)$ is reductive. However, because $\mathcal{M}^{\mathcal{F}}(v)$ itself is not reductive, we must make some modifications to the proof of Lemma 3.22.

As in that proof, consider a morphism $T \to \mathcal{M}^{\mathcal{F}}(v) \subset \mathcal{M}$ with T finite type, and let $T' \subset T$ denote the locally closed subscheme on which $c = M^\mu$ is maximal. Let $\mathfrak{Y} \subset [\Theta, \mathcal{M}(v)]$ be a connected component containing a point which is the unique maximizer of μ for some point in T', and let $\mathfrak{Y}^{\mathcal{F}} \subset \left[\Theta, \mathcal{M}^{\mathcal{F}}(v)\right]$ be the corresponding connected component (or collection of connected components). Then we have a diagram

$$\begin{array}{ccccc} & & \mathfrak{Y}^{\mathcal{F}} & \hookleftarrow & \mathfrak{Y} \\ & & \downarrow{\scriptstyle ev_1} & & \downarrow{\scriptstyle ev_1} \\ T & \longrightarrow & \mathcal{M}^{\mathcal{F}}(v) & \hookleftarrow & \mathcal{M}(v) \end{array}$$

Where both horizontal arrows in the square are open immersions. By Proposition 1.4, the open substack $\mathfrak{Y}^{\mathcal{F}} \subset \mathfrak{Y}$ consists of those points whose image under the composition $\mathfrak{Y} \to [B\mathbb{G}_m, \mathcal{M}(v)] \to \mathcal{M}(v)$ lies in $\mathcal{M}^{\mathcal{F}}(v)$. On the other hand, we know from Theorem 4.14 that for a point in $f \in \left[\Theta, \mathcal{M}^{\mathcal{F}}(v)\right]$ such that $\mu(f) = M^\mu(f(1))$, the point $f(0)$, corresponding to the associated graded of the HN filtration, also lands in $\mathcal{M}^{\mathcal{F}}(v)$. It follows that $\mathfrak{Y}^{\mathcal{F}} = \mathfrak{Y}$ after base change to T, because by the maximality of c any point in the fiber product $\mathcal{Y} \times_{\mathcal{M}(v)} T$ is the optimal destabilizer of its image point. In particular $\mathfrak{Y}^{\mathcal{F}} \times_{\mathcal{M}^{\mathcal{F}}(v)} T \to T$ is proper, and hence T' is closed, which was the key inductive step in the proof of Lemma 3.22. Thus the conclusion of Lemma 3.22 holds, and the remainder of the proof of Theorem 3.16 applies verbatim, so μ defines a weak Θ-stratification.

In order to show that we have a Θ-stratification, it suffices, by Lemma 3.5, to show that for any geometric point $f \in \mathfrak{S}_{\mu=c}$ in the image of $\sigma : \left[B\mathbb{G}_m, \mathcal{M}^{\mathcal{F}}(v)\right] \to \left[\Theta, \mathcal{M}^{\mathcal{F}}(v)\right]$, corresponding to a split Harder-Narsimhan filtration, the map on tangent spaces $T_f [\Theta, \mathcal{M}(v)] \to T_{f(1)}\mathcal{M}(v)$ is injective. We shall prove this using a bit of derived algebraic geometry because it simplifies things, although presumably it is not necessary for the proof.

The tangent complex of the derived stack $\mathrm{Perf}(X)$, representing the moduli of families of perfect complexes on X, is $\mathcal{E}^* \otimes \mathcal{E}[1]$, where $\mathcal{E}$ is the universal perfect complex on $\mathrm{Perf}(X)$. Note that there is an open substack of "universally gluable" morphisms such that $\bigsqcup_v \mathcal{M}(v)$ is an open substack of this stack. It follows that the mapping stack has a canonical derived structure such that given a map $S \to \mathrm{Perf}(X)$ classifying a perfect complex E_S on $X \times S$, the tangent complex at this S point is $T_S \mathrm{Perf}(X) = R(\pi_S)_*(E_S^* \otimes E_S[1])$, where $\pi_S : X \times S \to S$ is the projection.

It follows that the tangent complex of $[\Theta, \mathrm{Perf}(X)]$ at a point is the weight 0 piece of the filtered complex $E_S^* \otimes E_S[1]$. Recall (Proposition 1.6) that the category of quasi-coherent complexes on $X \times \Theta$ is equivalent to the category of graded complexes $\{E_i\}$ with morphisms $E_i \to E_{i+1}$, and the tensor product of $E_\bullet$ with $F_\bullet$ is the collection

$$(E \otimes F)_\bullet = \mathrm{Cone}\left(\bigoplus_{i+j=0} E_i \otimes F_{j+\bullet} \xrightarrow{x-y} \bigoplus_{i+j=0} E_i \otimes F_{j+\bullet}\right)$$

where x is the morphism induced by the structure maps $E_i \to E_{i-1}$ and y is induced by the structure maps $F_j \to F_{j-1}$. It follows that the tangent complex of $[\Theta, \mathrm{Perf}(X)]$ at a map $S \times \Theta \to \mathrm{Perf}(X)$, classifying a collection $(E_S)_\bullet$ of perfect complexes on S, is the complex

$$T_S\,[\Theta, \mathrm{Perf}(X)] \simeq \mathrm{Cone}\left(\bigoplus_i E_i^* \otimes E_i[1] \xrightarrow{x-y} \bigoplus_i E_i^* \otimes E_i[1]\right)$$

In the case where the generalized filtration $E_\bullet$ is split, so that $E_i \simeq \bigoplus_{j \geq i} Q_j$ and the maps $E_i \to E_{i-1}$ are the inclusion of summands, one can calculate that for a filtered object $E_\bullet$ with $E = \mathrm{colim}\, E_i \in \mathrm{Perf}(X)$ we have a comutative diagram

$$\begin{array}{ccccc} T_{E_\bullet}\,[\Theta, \mathrm{Perf}(X)] & \xrightarrow{Dev_1} & T_E\,\mathrm{Perf}(X) & & \\ \downarrow \simeq & & \downarrow \simeq & & \\ \bigoplus_{j \geq i} Q_i^* \otimes Q_j[1] & \longrightarrow & \bigoplus_{j,i} Q_i^* \otimes Q_j[1] & \longrightarrow & \bigoplus_{j<i} Q_i^* \otimes Q_j[1] \end{array}$$

where the bottom row is an exact triangle. Thuse injectivity of the map on tangent spaces induced by ev_1 is guaranteed by the vanishing of $H_0 R\Gamma(X, \bigoplus_{j<i} Q_i^* \otimes Q_j)$. In the case of a HN-filtration, where each Q_i is semistable of phase ϕ_i with $\phi_j < \phi_i$ for $j < i$, this vanishing is guaranteed. Thus we have completed the proof. □

Remark 4.24. One can extend this to a Θ-stratification of all of $\mathcal{M}(v)$. The unsatisfying way to do this is simply to choose, for every Harder-Narasimhan polytope whose last point is torsion, a weight for the maximal torsion subobject which is greater than the weights appearing in the HN filtration for the torsion free quotient. One can then use this to identify connected components of $[\Theta, \mathcal{M}(v)]$ which lead to a Θ-stratification.

Another way to accomplish this might be to slightly enlarge the class of numerical invariants we consider. Rather than considering numerical invariants $\mu : U \subset \mathcal{C}omp(\mathfrak{X}, \mathrm{id}_{\mathfrak{X}}) \to \mathbb{R}$, one should consider numerical invariants valued in a totally ordered real vector space, such as the space of polynomials of degree $\leq d$. It is likely that the discussion of Section 3 extends to this context essentially without modification, and that one can reformulate the notion of Gieseker stability in this framework.

Remark 4.25. It seems that a pre-stability condition as in Proposition 4.20 actually defines a Θ-stratification of the full (derived) stack $D^b(X)$, in the sense that any $E \in D^b(X)$ extends canonically (up to choices of weights) to a map $\Theta \to D^b(X)$ classifying the canonical "filtration" of E whose subquotients are semistable objects of increasing phase. The phases appearing in this filtration will not lie in the interval $(0, 1]$ if $E \notin D^b(X)^\heartsuit$. As the stack $D^b(X)$ is not a 1-stack, the algebraicity of the mapping stack $\left[\Theta, D^b(X)\right]$ is not on entirely rigorous footing (although see [HP] for a sketch of one possible approach), and so a treatment of this is beyond the scope of this paper. Still the degeneration space $\mathcal{D}(\mathrm{Perf}(X), [E])$ is well defined, and one could ask to what extent the stratification of $\mathrm{Perf}(X)$ is controlled by a continuous function induced by Z.

Remark 4.26. Let Σ be a smooth curve, and let G be a reductive group. Let $\mathfrak{X} = \mathrm{Bun}_G(\Sigma)$, then a geometric point of $[\Theta, \mathfrak{X}]$, i.e. a morphism $f : \Theta \to \mathrm{Bun}_G(\Sigma)$ is equivalent to either of the following data

(1) an equivariant G-bundle on $\mathbb{A}^1 \times \Sigma$, where $\mathbb{G}_m^*$ acts on the first factor
(2) a 1PS $\lambda : \mathbb{G}_m \to G$, and a principal P_λ-bundle, E over Σ

and under the second identification the point $f(1) \in \mathrm{Bun}_G(\Sigma)$ is the extension of structure group from P_λ to G. The second identification works in families as well, hence we have

$$[\Theta, \mathfrak{X}] = \bigsqcup_{[\lambda]} \mathrm{Bun}_{P_\lambda}(\Sigma)$$

The first description follows from descent on the action groupoid of $\mathbb{G}_m$ on $\mathbb{A}^1 \times \Sigma$ and the fact that the functor $\mathrm{Bun}_G(X)$ defines a stack for the fppf topology (see Diagram (1)). The second description follows from the formal observation

$$\underline{\mathrm{Hom}}(\Theta, \underline{\mathrm{Hom}}(\Sigma, */G)) \simeq \underline{\mathrm{Hom}}(\Sigma \times \Theta, */G) \simeq \underline{\mathrm{Hom}}(\Sigma, \underline{\mathrm{Hom}}(\Theta, */G))$$

and the description of $\mathrm{Hom}(\Theta, */G) = [\Theta, */G]$ from Corollary 1.12. Thus the main result of [GN] can be summarized as saying that $\mathrm{Bun}_G(\Sigma)$ admits a Θ-stratification by the type of the canonical parabolic reduction.

Cohomology classes on $\mathrm{Bun}_G(\Sigma)$ can be constructed geometrically via "transgression" along the universal diagram

$$\begin{array}{ccccc} T \times \Sigma & \longrightarrow & \mathrm{Bun}_G(\Sigma) \times \Sigma & \longrightarrow & */G \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi} & & \\ T & \xrightarrow{f} & \mathrm{Bun}_G(\Sigma) & & \end{array} \tag{16}$$

If we choose a coherent sheaf F on Σ and a representation V of G, then we have the cohomology class[20]

$$\mathrm{ch}\, R\pi_*(F \otimes V_{E_{univ}}) \in H^{even}(\mathrm{Bun}_G(\Sigma); \mathbb{Q})$$

where ch denotes the Chern character and $V_{E_{univ}} = E_{univ} \times_G V$ is the locally free sheaf on $\mathrm{Bun}_G(\Sigma) \times \Sigma$ associated to the representation V by the universal G-bundle E_{univ}. In this way, one constructs tautological classes with which to form candidate numerical invariants on $\mathrm{Bun}_G(\Sigma)$. One might expect that the Θ-stratification described in [GN] is controlled by a numerical invariant constructed in this manner. This holds for $G = \mathrm{GL}_N$ and $G = \mathrm{SL}_N$ by the above results for $\mathcal{M}^v(\Sigma)$, where we regard $\mathrm{Bun}_G(\Sigma) \simeq \bigsqcup_v \mathcal{M}^{\mathcal{F}}(v)$ for the usual t-structure on $D^b(\Sigma)$.

[20] Using Grothendieck-Riemann-Roch this can also be expressed as the cohomological pushforward of cohomology classes on $\mathrm{Bun}_G(\Sigma) \times \Sigma$, namely we can write this cohomology class as $[\Sigma] \cap ((1 + \frac{1}{2}c_1(K)) \cdot \mathrm{ch}(F) \cdot \mathrm{ch}(V))$.

4.2. **K-stability of polarized varieties.** In this section we formulate K-stability for polarized projective varieties in the framework of Θ-stability, following [D]. We work over a field k of characteristic 0. Let $\mathcal{V}ar$ denote the moduli stack of polarized varieties. By definition we have

$$\mathcal{V}ar_d(T) = \left\{ \begin{array}{c} \text{flat projective morphisms } X \to T \text{ of rel. dim. } r, \\ \text{with a relatively ample invertible sheaf } L \text{ on } X \text{ of degree } d \end{array} \right\}$$

There is a universal projective variety $\mathcal{X} \to \mathcal{V}ar_d$, where $\mathcal{X}$ classifies families of polarized varieties just like $\mathcal{V}ar_d$, but with the additional data of a section of $X \to T$.

Lemma 4.27. *The stack $\mathcal{V}ar_d$ is a local quotient stack. A morphism $\Theta \to \mathcal{V}ar_d$ corresponds to a $\mathbb{G}_m$-equivariant flat family $X \to \mathbb{A}^1$ along with a relatively ample equivariant invertible sheaf – this is known as a test-configuration in the K-stability literature.*

Let $\pi : X \to T$ be a family of projective varieties of dimension r polarized by L. Grothendieck-Riemann-Roch implies that the i^{th} Chern class of $R\pi_*(L^n)$ is a polynomial of degree $r+i$ in n. Therefore the coefficients of these polynomials provide cohomology classes $H^{2i}(\mathcal{V}ar_d)$, which are referred to as tautological cohomology classes. We introduce the notation[21]

$$\begin{aligned} \mathrm{ch}_0(R\pi_*(L^n)) &= \frac{n^r}{r!}a_0 + \frac{n^{r-1}}{(r-1)!}a_1 + \cdots \\ \mathrm{ch}_1(R\pi_*(L^n)) &= \frac{n^{r+1}}{(r+1)!}d_0 + \frac{n^r}{r!}d_1 + \cdots \\ 2\,\mathrm{ch}_2(R\pi_*(L^n)) &= \frac{n^{r+2}}{(r+2)!}q_0 + \frac{n^{r+1}}{(r+1)!}q_1 + \cdots \end{aligned}$$

The tautological classes are algebraic, because they can be expressed as linear combinations of successive finite differences of the Chern classes of the vector bundles $R\pi_*(L^n)$ for $n \gg 0$. For instance, by [KM] there are canonically defined invertible sheaves M_i over T along with canonical and functorial isomorphisms

$$\det R\pi_*(L^n) \simeq M_{r+1} \otimes (M_r)^{\binom{n}{1}} \otimes \cdots \otimes (M_0)^{\binom{n}{r+1}} \text{ for all } n.$$

The canonical and functorial invertible sheaves $D_i = D_i(L)$ with $c_1(D_i(L)) = d_i$ can be written as combinations of these M_i.

Observe that if M is a line bundle on T, we can define a polarized family $(X, \tilde{L})$, where $\tilde{L} := L \otimes \pi^* M$. Formally, this is an action of the group stack $B\mathbb{G}_m$ on $\mathcal{V}ar_d$. By the base change formula, $\mathrm{ch}(R\pi_*(\tilde{L}^n)) = e^{nc_1(M)}\,\mathrm{ch}(R\pi_*(L^n))$. Unwinding this formula, we have the following effect on the tautological classes:

$$\begin{aligned} \tilde{a}_i &= a_i \\ \tilde{d}_i &= d_i + c_1(M)a_i \\ \tilde{q}_i &= q_i + 2c_1(M)d_i + c_1(M)^2 a_i \end{aligned}$$

A similar pattern continues for the coefficients of $k!\,\mathrm{ch}_k(R\pi_*\tilde{L}^n)$.

If one wants to define a non-vacuous notion of stability using some combination of the classes $d_i \in H^2(\mathcal{V}ar_d)$, then one must use a class $l \in H^2$ which is $B\mathbb{G}_m$-invariant. Otherwise, any $f : \Theta \to \mathcal{V}ar_d$ could be twisted by a sufficiently large line bundle

[21]Note that the integer a_0 is by definition the degree of L.

on Θ so that $f^*l > 0$, and every point would be unstable. We define the following cohomology classes, which one can check are invariant under the transformation $L \mapsto L \otimes \pi^*(M)$:

$$\begin{aligned} l &= a_1 d_0 - a_0 d_1 \quad \in H^2(\mathcal{V}ar_d) \\ b &= a_0^2 q_0 - a_0 d_0^2 \quad \in H^4(\mathcal{V}ar_d) \end{aligned} \tag{17}$$

The the numerical invariant induced by l and b via Construction 2.19 is the normalized Futaki invariant $\hat{\Psi}$ introduced in [D], up to a constant multiple.[22] Whereas in [D] the a_i,d_i, and q_i are treated as numbers, we have shown that in the context of algebraic geometry, they can be interpreted as cohomology classes on the stack $\mathcal{V}ar_d$.

Definition 4.28. A polarized variety (X, L) representing a point $x \in \mathcal{V}ar_d$ is said to be K-semistable if for all $k > 0$, the point $(X, L^k) \in \mathcal{V}ar_{k^r d}$ is semistable with respect to the class l in (17).

In order to formulate this notion of semistability in terms of a numerical invariant on a stack, we will introduced a new moduli problem $\mathcal{V}ar'_{\mathbb{Q}}$. First let us observe the following.

Lemma 4.29. *Let $f : \Theta \to \mathcal{V}ar_d$ correspond to a test configuration $(\tilde{X}, L)$ over $\mathbb{A}^1/\mathbb{G}_m$. We can compose with a suitable covering map $\Theta \to \Theta$ and tensor by a pullback of a line bundle from Θ to get a new test configuration $\tilde{f}$ with a canonical isomorphism $f(1) \simeq \tilde{f}(1)$ such that $\tilde{f}^* D_0 = 0$.*

With this modified test configuration, the classes l and b are, up to a constant multiple depending on the Hilbert polynomial of X, the leading coefficients in ch_1 and $2\,\mathrm{ch}_2$ of $R\pi_* \tilde{L}$. This suggests the following modification of the moduli problem

$$\mathcal{V}ar'_d(T) := \{f : T \to \mathcal{V}ar_d \text{ along with an isomorphism } f^* D_0 \simeq \mathcal{O}_T\}$$

Forgetting the trivialization of $f^* D_0$ defines a morphism $p : \mathcal{V}ar'_d \to \mathcal{V}ar_d$, identifying $\mathcal{V}ar'_d$ with the total space of the $\mathbb{G}_m$-torsor over $\mathcal{V}ar_d$ corresponding to D_0. In particular $\mathcal{V}ar'_d$ is a local quotient stack as well. The lemma above implies a bijection

$$[\Theta, \mathcal{V}ar'_d]_x \xrightarrow{\simeq} [\Theta, \mathcal{V}ar_d]_x \,/\, \mathrm{Pic}(\Theta)$$

for each $x \in \mathcal{V}ar_d(k)$. Because the numerical invariant on $\mathcal{V}ar$ defined by l and b was invariant under the action of $\mathrm{Pic}(\Theta) = \mathbb{Z}$, the question of maximizing μ on $\mathcal{D}(\mathcal{V}ar_d, x)$ is essentially equivalent to the question on $\mathcal{D}(\mathcal{V}ar'_d, x)$.

Lemma 4.30. *Scalar multiplication by α on L acts on $D_0(L)$ as scalar multiplication by $\alpha^{(r+1)d}$.*

Proof. Scalar multiplication acts on $R\pi_*(L^n)$ as multiplication by α^n, and thus it acts on $\det R\pi_*(L^n)$ as multiplication by $\alpha^{n \,\mathrm{ch}_0 R\pi_*(L^n)}$. The claim follows by examining the action of this scalar multiplication on the $(r+1)^{st}$ iterated sequential difference of $\det R\pi_*(L^n)$. □

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[22] In fact, in [D] Donaldson begins with the classes $l = -d_1$ and $b = q_0$ which are not $B\mathbb{G}_m$-invariant, and then arrives at (17) by maximizing the resulting numerical invariant over all possible twists of a test configuration by a line bundle on Θ. It is then automatic that the resulting expression is $B\mathbb{G}_m$-invariant.

This lemma implies that $\mathcal{V}ar_d$ and $\mathcal{V}ar'_d$ have the same set of geometric points. Indeed over a field $D_0(L)$ can always be trivialized, and over an algebraically closed field any two trivializations are related by scalar multiplication on L. This means that from the perspective of coarse moduli problems, the stack $\mathcal{V}ar'_d$ is as good as $\mathcal{V}ar_d$.

Remark 4.31. An alternative moduli functor one might consider would be the quotient of $\mathcal{V}ar_d$ by the action of $B\mathbb{G}_m$, which parametrizes polarized families of varieties up to twists of the polarization by a line bundle on the base. The classes l and b of (17), as they are $B\mathbb{G}_m$-invariant, can be regarded as classes on this quotient stack. A similar situation occurs on the moduli of GL_N bundles on a smooth curve, where one can either work with a $B\mathbb{G}_m$-invariant class in $H^2(\mathrm{Bun}_{\mathrm{GL}_N})$ or pass to the moduli of SL_N bundles. The stack $\mathcal{V}ar'_d$, which is a rigidification of $\mathcal{V}ar_d$, is analogous to the stack of SL_N bundles.

Remark 4.32. Inspecting the definition (17), on sees that $p^*l = p^*(a_0 d_1)$ and $p^*b = p^*(a_0^2 q_0)$ in $H^*(\mathcal{V}ar'_d)$. Given $f : \Theta \to \mathcal{V}ar'_d$, corresponding to a test configuration (X, L) over Θ for which $d_0 \simeq 0$, we have

$$\begin{aligned} \mu(f) &= -\frac{f^*l}{\sqrt{f^*b}} = -\frac{d_1}{\sqrt{q_0}} \\ &\propto -\lim_{n\to\infty} \frac{\sqrt{n}^{2-r}\,\mathrm{ch}_1(H^0(X_0, L_0^n))}{\sqrt{2\,\mathrm{ch}_2(H^0(X_0, L_0^n))}} \end{aligned}$$

Regarding $H^0(X, L_0^n)$ as a representation of $\mathbb{G}_m$, ch_1 denotes the sum of the weights and $2\,\mathrm{ch}_2$ denotes the sum of the squares of the weights.

For any polarized family (X, L) over T, we have a canonical isomorphism $D_0(L^k) \simeq D_0(L)^{k^{r+1}}$. Thus a trivialization of $D_0(L)$ canonically induces a trivialization of $D_0(L^k)$. This allows us to define a morphism $\mathcal{V}ar'_d \to \mathcal{V}ar'_{k^r d}$ for $k > 0$ given by the map $(X, L) \mapsto (X, L^k)$.

Definition 4.33. For any quasi-compact quasi-separated scheme T, we define the groupoid $\mathcal{V}ar'_{\mathbb{Q}}(T)$ as a filtered colimit,

$$\mathcal{V}ar'_{\mathbb{Q}}(T) := \operatorname*{colim}_{d\in\mathbf{N}} \mathcal{V}ar'_d(T).$$

This forms a presheaf of groupoids on qc.qs. schemes.

First we note that while all points of $\mathcal{V}ar'_d$ have a canonical $\mu_{(r+1)d}$ in their stabilizer groups coming from scalar multiplication on the relatively ample bundle, points of $\mathcal{V}ar'_{\mathbb{Q}}$ have no automorphisms coming from scalar multiplication.

Lemma 4.34. *For any qc-qs scheme T, the groupoid $\mathcal{V}ar'_{\mathbb{Q}}(T)$ can be described as*

obj: polarized families $(X, L)/T$ with an isomorphism $\mathcal{O}_T \simeq D_0(L)$.

mor: isomorphisms $f : X_1 \simeq X_2$ over T such that $\exists \varphi : L_1^{k_1} \xrightarrow{\simeq} f^* L_2^{k_2}$ for some $k_1, k_2 > 0$ s.t. the composition $\mathcal{O}_T \simeq D_0(L_1^{k_1}) \simeq D_0(L_2^{k_2}) \simeq \mathcal{O}_T$ is the identity.

Proof. The class of objects is clear, so the content of the lemma is the claim that if there is an isomorphism $\varphi : L_1^{k_1} \simeq f^* L_2^{k_2}$, then this corresponds to a unique isomorphism in $\mathcal{V}ar'_{\mathbb{Q}}(T)$. Lemma 4.30 implies that over each connected component of T, any two isomorphisms as above differ by scalar multiplication by a root of unity. Thus any two such isomorphisms agree after replacing $L_i^{k_i}$ with $L_i^{k_i p}$ and φ with φ^p for $p = (r+1)d$. □

Lemma 4.35. *$\mathcal{V}ar'_{\mathbb{Q}}$ is a stack for the fppf topology.*

Proof. First we show that for any two families, $\mathcal{X}_i = (X_i, L_i, \mathcal{O}_T \simeq D_0(L_i))$, $i = 1, 2$, over T, the functor $\underline{\mathrm{Iso}}(\mathcal{X}_1, \mathcal{X}_2)$ is an fppf sheaf of sets on T. Let $U \to T$ be an fppf cover, and let $p_1, p_2 : U \times_T U \to U$ be the two projections, and let $f \in \mathrm{Iso}(\mathcal{X}_1|_U, \mathcal{X}_2|_U)$ be such that $p_1^* f = p_2^* f$. By fppf descent for maps of schemes, this uniquely gives an isomorphism $f' : X_1 \to X_2$ over T. We also have an isomorphism $\phi : L_1^{k_1}|_U \to f^* L_2^{k_2}|_U$, respecting the trivializations of $D_0(L_i|_U)$, such that after raising to some further power, the restrictions $p_1^*\phi^l = p_2^*\phi^l$. It follows that there is a unique isomorphism $\phi' : L_1^{lk_1} \to (f')^* L_2^{lk_2}$ respecting the trivializations of $D_0(L_i)$. This shows that $\underline{\mathrm{Iso}}(\mathcal{X}_1, \mathcal{X}_2)$ is an fppf sheaf over T.

Next we verify that every descent datum is effective: Let $\mathcal{X}_1$ and $\mathcal{X}_2$ are families over U, and let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be an isomorphism of families over U satisfying the cocycle condition on $U \times_T U$. This means that the isomorphism $f : X_1 \to X_2$ satisfies the cocycle condition over U, and hence defines an algebraic space X over T. Also there is an isomorphism $\phi : L_1^{k_1} \to f^* L_2^{k_2}$ which respects the trivializations of $D_0(L_i)$ and satisfies the cocyle condition on $U \times_T U$ after raising to some power $\phi^l : L_1^{lk_1} \to f^* L_2^{lk_2}$. Thus the invertible sheaf descends to X, along with the trivialization $D_0(L)$, and X/T is projective. □

Consider the tautological classes a_i, d_i, and q_i in $H^{even}(\mathcal{V}ar'_d)$. If we replace L with L^p, then $a_i \mapsto p^{n-i} a_i$, $d_i \mapsto p^{n+1-i} d_i$, and $q_i \mapsto p^{n+2-i} q_i$. It follows from these homogeneity properties that the following formulas give well-defined cohomology classes on $\mathcal{V}ar'_{\mathbb{Q}}$:

$$l = \frac{-1}{a_0} d_1 \in H^2(\mathcal{V}ar'_{\mathbb{Q}}; \mathbb{R}), \qquad b = a_0^{-\frac{n+2}{n}} r_0 \in H^4(\mathcal{V}ar'_{\mathbb{Q}}; \mathbb{R}) \tag{18}$$

In light of Definition 4.28 and Lemma 4.29, the notion of stability for points in $\mathcal{V}ar'_{\mathbb{Q}}$ induced by the class l of (18) is exactly the notion of K-stability, and the numerical invariant μ on $\mathcal{V}ar'_{\mathbb{Q}}$ induced by these classes is precisely the normalized Futaki invariant of [D]. In later work, we intend to return to the existence and uniqueness question for this numerical invariant on $\mathcal{V}ar'_{\mathbb{Q}}$.

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