The derived category of a GIT quotient

Daniel Halpern-Leistner

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What is geometric invariant theory (GIT)?

Let a reductive group $G$ act on a smooth quasiprojective (preferably projective-over-affine) variety $X$.

**Problem**

Often $X/G$ does not have a well-behaved quotient: e.g. $\mathbb{C}^N/\mathbb{C}^*$.  

**Grothendieck’s solution:** Consider the stack $X/G$, i.e. the *equivariant* geometry of $X$.

**Mumford’s solution:**

- the *Hilbert-Mumford numerical criterion* identifies unstable points in $X$, along with one parameter subgroups $\lambda$ which destabilize these points
- $X^{ss} = X - \{\text{unstable points}\}$, and (hopefully) $X^{ss}/G$ has a well-behaved quotient
Example: Grassmannian

The Grassmannian $G(2, N)$ is a GIT quotient of $V = \text{Hom}(\mathbb{C}^2, \mathbb{C}^N)$ by $GL_2$. The unstable locus breaks into *strata* $S_i$.

<table>
<thead>
<tr>
<th>Maximal Destabilizer (one param. subgroup)</th>
<th>Fixed Locus (column vectors)</th>
<th>Unstable Stratum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_0 = \begin{pmatrix} t &amp; 0 \ 0 &amp; t \end{pmatrix}$</td>
<td>$[0, 0]$</td>
<td>$S_0 = {\text{the 0 matrix}}$</td>
</tr>
<tr>
<td>$\lambda_1 = \begin{pmatrix} t &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$[0, \ast]$, $\ast \neq 0$</td>
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$\lambda_i$ chosen to maximize the numerical invariant $\langle \lambda_i, \det \rangle / |\lambda_i|$. 

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The topology of GIT quotients

**Theorem (M. Atiyah, R. Bott, F. Kirwan, L. Jefferey, and others)**

The restriction map in equivariant cohomology $H^*_G(X) \to H^*_G(X^{ss})$ is surjective, and the kernel can be described explicitly.

Can use this to compute Betti numbers in the form of the Poincare polynomial $P(X/G) = \sum t^i \dim H^i_G(X; \mathbb{Q})$.

**Theorem (Poincare polynomial additivity)**

Let $Z'((\lambda_i)) := Z(\lambda_i)/\lambda_i$ be the reduced centralizer, then

$$P(X/G) = P(X^{ss}/G) + \sum_i \frac{t^{\text{codim } S_{\alpha}}}{1 - t^2} P(\{\lambda_i - \text{fixed}\}/Z'(\lambda_i))$$

**Example:** Using the fact that $P(*/G) = \prod_e (1 - t^{2(e+1)})^{-1}$, where $e$ are the exponents of $G$,

$$\frac{1}{(1 - t^2)(1 - t^4)} = \frac{1}{1 - t^2} P(G(2, N)) + \frac{t^{N-1}}{1 - t^2} P(P^{N-1}) + \frac{t^{2N}}{(1 - t^2)(1 - t^4)}$$
What is the derived category?

My work categorifies this classical story...

**Definition**

Let $X$ be an algebraic variety, then $D^b(X)$ is a linear category
- objects: chain complexes of coherent sheaves on $X$
- morphisms: chain maps $F^* \to G^*$, and inverses for quasi-isomorphisms

For example, if $Y \subset X$ and $\cdots \to F^{-1} \to F^0 \to O_Y$ is a locally free resolution, then $O_Y \simeq F^*$ in $D^b(X)$

We will study the derived category of *equivariant* coherent sheaves $D^b(X/G)$ and $D^b(X^{ss}/G)$. When $X^{ss}/G$ is a variety, this is equivalent to the usual derived category.
Why you should care?

The derived category of $X$ is a “categorication” of the algebraic $K$-theory

$$K_0(X) = \{\text{objects of } D^b(X)\}/\text{isomorphism}.$$ 

It also remembers:

- sheaf cohomology $H^i(X, F) = \text{Hom}_{D^b(X)}(\mathcal{O}_X, F[i])$,
- intersection theory via the Chern character $ch : K^0(X) \to A^*(X) \otimes \mathbb{Q}$,
- rational cohomology $H^*(X; \mathbb{Q})$ via “periodic-cyclic homology” of $D^b(X)$, and
- Hodge-cohomology $H^p(X, \Omega^q_X)$ via the “Hochschild homology” of $D^b(X)$. 

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There is a natural restriction functor $r : D^b(X/G) \rightarrow D^b(X^{ss}/G)$

**Theorem (HL; Ballard, Favero, Katzarkov)**

Let $G_w \subset D^b(X/G)$ be the full subcategory consisting of complexes such that

$$\mathcal{H}^*(F^\bullet|_{\{\lambda_i \text{ fixed}\}}) \text{ has weights in } [w_i, w_i + \eta_i) \text{ for all } i$$

where $\eta_i$ is the weight of $\lambda_i$ on the conormal bundle $\det N_{S_i}^\vee X$.

Then $r : G_w \rightarrow D^b(X^{ss}/G)$ is an equivalence of categories.

Furthermore, the kernel of the restriction functor can be described explicitly.

This implies that for any invariant which depends functorially on the derived category, the restriction functor is surjective.

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Example: Grassmannian $G(2, N)$

<table>
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<tr>
<th>Destabilizer</th>
<th>Fixed Locus</th>
<th>Stratum</th>
<th>Window width</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_0 = \begin{pmatrix} t &amp; 0 \ 0 &amp; t \end{pmatrix}$</td>
<td>$[0, 0]$</td>
<td>$S_0 = {\text{the 0 matrix}}$</td>
<td>$\eta_0 = 2N$</td>
</tr>
<tr>
<td>$\lambda_1 = \begin{pmatrix} t &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$[0, \ast], \ast \neq 0$</td>
<td>$S_1 = {\text{rank 1 matrices}}$</td>
<td>$\eta_1 = N - 1$</td>
</tr>
</tbody>
</table>

Weight windows for $G(2, 4)$ on the character lattice of $GL_2$:

The only representations which fit in the weight windows $0 \leq \lambda_0 \cdot \chi < 8 \quad 0 \leq \lambda_1 \cdot \chi < 3$ are those in the Kapranov exceptional collection: $\mathcal{C}$, $\det$, $\det^2$, $\mathcal{C}^2$, $\mathcal{C}^2 \otimes \det$, and $S^2\mathcal{C}^2$. 
Equivalences of derived categories

Say $X/G$ has two GIT quotients $X_{ss}^+/G$ and $X_{ss}^-/G$. Under certain conditions one can verify

**Ansatz**

One can choose $w, w'$ such that $G^+_w = G^-_{w'} \subset D^b(X/G)$. This gives an equivalence $\Phi_w : D^b(X_{ss}^+/G) \to D^b(X_{ss}^-/G)$.

In a large class of examples, called balanced wall crossings, the Ansatz holds whenever $\omega_X|_{\{\lambda_i\text{-fixed}\}}$ has weight 0 w.r.t. $\lambda_i$.

**Example:** $X/G = \text{Hom}(\mathbb{C}^2, \mathbb{C}^N) \times \text{Hom}(\mathbb{C}^N, \mathbb{C}^2)/\text{GL}_2$

There are two possible GIT quotients $X_{ss}^+ = \{(a, b)| a \text{ injective}\}$ and $X_{ss}^- = \{(a, b)| b \text{ surjective}\}$. The Kapranov collection generates both $G^+_{0,0}$ and $G^-_{1-2N,2-N}$.
Autoequivalences of derived categories

When the Ansatz holds for multiple \( w \), \( \Phi_{w+1}^{-1} \circ \Phi_w \) is a nontrivial autoequivalence, which is \textit{not geometric in origin}.

**Example**

Given positive numbers \( \{a_i\} \) and \( \{b_j\} \) with \( \sum a_i = \sum b_j \), the space \( \bigoplus_i \mathbb{C} a_i \oplus \bigoplus_j \mathbb{C} -b_j / \mathbb{C}^* \) is a balanced wall crossing. The autoequivalence is a \textit{Seidel-Thomas spherical twist}.

For other balanced wall crossings, \( \Phi_{w+1}^{-1} \Phi_w \) can be described as a composition of spherical twists. This supports a prediction of \textit{homological mirror symmetry} that \( \Phi_{w+1}^{-1} \Phi_w \) is “monodromy” on a “Kähler moduli space” as shown:

\[ \Phi_{w+1}^{-1} \circ \Phi_w \]