The derived category of a GIT quotient

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Table of contents

What is geometric invariant theory (GIT)?

Let a reductive group G act on a smooth quasiprojective (preferably projective-over-affine) variety X.

Problem

Often X/G does not have a well-behaved quotient: e.g. $\mathbf{C}^N/\mathbf{C}^*$.

Grothendieck's solution: Consider the stack X/G, i.e. the *equivariant* geometry of X.

Mumford's solution:

- the *Hilbert-Mumford numerical criterion* identifies unstable points in X, along with one parameter subgroups λ which *destabilize* these points
- $X^{ss} = X - \{\text{unstable points}\}$, and (hopefully) X^{ss}/G has a well-behaved quotient

Example: Grassmannian

The Grassmannian $\mathbf{G}(2, N)$ is a GIT quotient of $V = \mathrm{Hom}(\mathbf{C}^2, \mathbf{C}^N)$ by GL_2. The unstable locus breaks into *strata* S_i.

Maximal Destabilizer (one param. subgroup)	Fixed Locus (column vectors)	Unstable Stratum
$\lambda_0 = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$	$[0, 0]$	$S_0 = \{\text{the 0 matrix}\}$
$\lambda_1 = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$	$[0, *]$, $* \neq 0$	$S_1 = \{\text{rank 1 matrices}\}$

λ_i chosen to maximize the numerical invariant $\langle \lambda_i, \det \rangle / |\lambda_i|$.

The topology of GIT quotients

Theorem (M. Atiyah, R. Bott, F. Kirwan, L. Jefferey, and others)

The restriction map in equivariant cohomology $H_G^*(X) \to H_G^*(X^{ss})$ is surjective, and the kernel can be described explicitly.

Can use this to compute *Betti numbers* in the form of the Poincare polynomial $P(X/G) = \sum t^i \dim H_G^i(X; \mathbf{Q})$.

Theorem (Poincare polynomial additivity)

Let $Z'(\lambda_i) := Z(\lambda_i)/\lambda_i$ be the reduced centralizer, then

$$P(X/G) = P(X^{ss}/G) + \sum_i \frac{t^{\mathrm{codim}\, S_\alpha}}{1-t^2} P\left(\{\lambda_i - \mathit{fixed}\}/Z'(\lambda_i)\right)$$

Example: Using the fact that $P(*/G) = \prod_e (1 - t^{2(e+1)})^{-1}$, where e are the *exponents* of G,

$$\frac{1}{(1-t^2)(1-t^4)} = \frac{1}{1-t^2} P(\mathbf{G}(2,N)) + \frac{t^{N-1}}{1-t^2} P(\mathbf{P}^{N-1}) + \frac{t^{2N}}{(1-t^2)(1-t^4)}$$

What is the derived category?

My work *categorifies* this classical story...

Definition

Let X be an algebraic variety, then $D^b(X)$ is a linear category

- objects: chain complexes of coherent sheaves on X
- morphisms: chain maps $F^\bullet \to G^\bullet$, and inverses for quasi-isomorphisms

For example, if $Y \subset X$ and $\cdots \to F^{-1} \to F^0 \to \mathcal{O}_Y$ is a locally free resolution, then $\mathcal{O}_Y \simeq F^\bullet$ in $D^b(X)$

We will study the derived category of *equivariant* coherent sheaves $D^b(X/G)$ and $D^b(X^{ss}/G)$. When X^{ss}/G is a variety, this is equivalent to the usual derived category.

Why you should care?

The derived category of X is a "categorication" of the algebraic K-theory

$$K_0(X) = \{\text{objects of } D^b(X)\}/\text{isomorphism}.$$

It also remembers:

- sheaf cohomology $H^i(X, F) = \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, F[i])$,
- intersection theory via the Chern character $ch : K^0(X) \to A^*(X) \otimes \mathbf{Q}$,
- rational cohomology $H^*(X; \mathbf{Q})$ via "periodic-cyclic homology" of $D^b(X)$, and
- Hodge-cohomology $H^p(X, \Omega^q_X)$ via the "Hochschild homology" of $D^b(X)$.

Derived Kirwan surjectivity

There is a natural restriction functor $r : D^b(X/G) \to D^b(X^{ss}/G)$

Theorem (HL; Ballard, Favero, Katzarkov)

Let $G_w \subset D^b(X/G)$ be the full subcategory consisting of complexes such that

$$\mathcal{H}^*(F^\bullet|_{\{\lambda_i \text{ fixed}\}}) \text{ has weights in } [w_i, w_i + \eta_i) \text{ for all } i$$

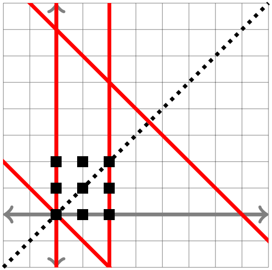
where η_i is the weight of λ_i on the conormal bundle $\det N^\vee_{S_i} X$. Then $r : G_w \to D^b(X^{ss}/G)$ is an equivalence of categories. Furthermore, the kernel of the restriction functor can be described explicitly.

This implies that for any invariant which depends functorially on the derived category, the restriction functor is surjective.

Example: Grassmannian $G(2, N)$

Destabilizer	Fixed Locus	Stratum	Window width
$\lambda_0 = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$	$[0, 0]$	$S_0 = \{\text{the 0 matrix}\}$	$\eta_0 = 2N$
$\lambda_1 = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$	$[0, *], * \neq 0$	$S_1 = \{\text{rank 1 matrices}\}$	$\eta_1 = N - 1$

Weight windows for $\mathbf{G}(2, 4)$ on the character lattice of GL_2:



The only representations which fit in the weight windows $0 \leq \lambda_0 \cdot \chi < 8 \quad 0 \leq \lambda_1 \cdot \chi < 3$ are those in the *Kapranov exceptional collection*: $\mathbf{C}$, det, $\det^2$, $\mathbf{C}^2$, $\mathbf{C}^2 \otimes \det$, and $S^2\mathbf{C}^2$.

Equivalences of derived categories

Say X/G has two GIT quotients X_+^{ss}/G and X_-^{ss}/G. Under certain conditions one can verify

Ansatz

One can choose w, w' such that $\mathcal{G}_w^+ = \mathcal{G}_{w'}^- \subset D^b(X/G)$. This gives an equivalence $\Phi_w : D^b(X_+^{ss}/G) \to D^b(X_-^{ss}/G)$.

In a large class of examples, called *balanced* wall crossings, the Ansatz holds whenever $\omega_X|_{\{\lambda_i\text{-fixed}\}}$ has weight 0 w.r.t. λ_i.

Example: $X/G = \mathrm{Hom}(\mathbf{C}^2, \mathbf{C}^N) \times \mathrm{Hom}(\mathbf{C}^N, \mathbf{C}^2)/GL_2$

There are two possible GIT quotients $X_+^{ss} = \{(a, b) | a \text{ injective}\}$ and $X_-^{ss} = \{(a, b) | b \text{ surjective}\}$. The Kapranov collection generates both $\mathcal{G}_{0,0}^+$ and $\mathcal{G}_{1-2N,2-N}^-$.

Autoequivalences of derived categories

When the Ansatz holds for multiple w, $\Phi_{w+1}^{-1} \circ \Phi_w$ is a nontrivial *autoequivalence*, which is *not geometric in origin*.

Example

Given positive numbers $\{a_i\}$ and $\{b_j\}$ with $\sum a_i = \sum b_j$, the space $\bigoplus_i \mathbf{C}_{a_i} \oplus \bigoplus_j \mathbf{C}_{-b_j} / \mathbf{C}^*$ is a balanced wall crossing. The autoequivalence is a *Seidel-Thomas spherical twist*.

For other balanced wall crossings, $\Phi_{w+1}^{-1}\Phi_w$ can be described as a composition of spherical twists. This supports a prediction of *homological mirror symmetry* that $\Phi_{w+1}^{-1}\Phi_w$ is "monodromy" on a "Kähler moduli space" as shown: