EQUIVARIANT HODGE THEORY AND NON-COMMUTATIVE GEOMETRY

DANIEL HALPERN-LEISTNER AND DANIEL POMERLEANO

Abstract. We develop a version of Hodge theory for a large class of smooth cohomologically proper quotient stacks $X/G$ analogous to Hodge theory for smooth projective schemes. We show that the noncommutative Hodge-de Rham sequence for the category of equivariant coherent sheaves degenerates. This spectral sequence converges to the periodic cyclic homology, which we canonically identify with the topological equivariant $K$-theory of $X$ with respect to a maximal compact subgroup $M \subset G$. The result is a natural pure Hodge structure of weight $n$ on $K^M_n(X^\text{an})$. We also treat categories of matrix factorizations for equivariant Landau-Ginzburg models.

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If $X$ is a smooth projective variety over $\mathbb{C}$, then the cohomology groups $H^n(X;\mathbb{C})$ can be equipped with a pure Hodge structure of weight $n$. The theory of Hodge structures then allows one to “linearize” many important problems in algebraic geometry. Our goal is to develop such a linearization for the equivariant algebraic geometry of a quasi-projective variety $X$ along with the action of a compact Lie group, $M$. Note that the complexification $G$ of $M$, a reductive algebraic group, acts on $X$ as well, and it is natural to ask for a Hodge theory associated intrinsically to the algebraic stack $\mathfrak{X} := X/G$.

One such linearization follow from the results of [D], which establishes a canonical mixed Hodge structure on the cohomology of any smooth simplicial scheme and in particular on the equivariant cohomology, $H^*_G(X)$, which is the cohomology of the simplicial nerve of the action groupoid of $G$
on $X$. Building on these ideas, one can even associate a motive to the stack $X/G$ as a colimit of motives of schemes as in [MV, Section 4.2].

The present paper diverges from the classical approach to Hodge theory for (simplicial) schemes in two notable respects. The first is the class of stacks which inherit pure Hodge structures. We will consider the class of $M$-quasiprojective schemes which admit a complete KN stratification (See Definition 2.1), which includes two commonly studied examples:

1. $X$ which are projective-over-affine, such that $\dim \Gamma(X, \mathcal{O}_X)^M < \infty$;
2. $X$ such that $X/G$ admits a projective good quotient;$^1$

The idea of regarding equivariant geometries (1) and (2) as “proper” is very intuitive and shows up in diverse contexts. Focusing on stacks which admit a complete KN stratification builds on work of C. Teleman [T2], where it was shown that a version of the Hodge-de Rham spectral sequence for $H^*_G(X)$ degenerates for such $G$-schemes and that the (a priori mixed) Hodge structure on $H^*_G(X)$ is pure in this case.

The paper [HLP] studies these properness phenomena systematically by introducing the class of cohomologically proper stacks, which includes all of the above quotient stacks as important examples. That paper showed that cohomological proper stacks behave in many respects like proper schemes, for example the mapping stack $\text{Map}(X, Y)$ is algebraic whenever the source $X$ is cohomologically proper and flat and $Y$ is locally finitely presented with quasi-affine diagonal. The Hodge theoretic results in this paper can be regarded as further evidence that the notion of cohomological properness precisely captures the properness phenomena appearing in several places in equivariant algebraic geometry.

The second important theme of this paper is that we make systematic use of noncommutative algebraic geometry, which views dg-categories $A$ over a field $k$ as “noncommutative spaces”. In noncommutative algebraic geometry, there are two natural invariants attached to $A$— the Hochschild chain complex, $C^\bullet(A)$, which plays the role of noncommutative Hodge cohomology and $C^{\text{per}}(A)$ which is a dg-module over $k((u))$ where $u$ has homological degree $-2$ which behaves like noncommutative Betti-cohomology. There is a canonical Hodge filtration of the complex $C^{\text{per}}(A)$ whose associated graded is $C_\bullet(A) \otimes k((u))$ (We will recall some of the details below). In our study, the category $A$ will be the derived category of equivariant coherent sheaves, $\text{Perf}(X/G)$.

We show that one can recover the equivariant topological $K$-theory $K_M(X)$, as defined in [AS+, S2], from the dg-category $\text{Perf}(X/G)$. The first ingredient is the recent construction by A. Blanc of a topological $K$-theory spectrum $K^{\text{top}}(A)$ for any dg-category $A$ over $\mathbb{C}$ [B3]. Blanc constructs a Chern character natural transformation $\text{ch} : K^{\text{top}}(A) \to HP(A)$, conjectures that $\text{ch} \otimes \mathbb{C}$ is an equivalence for any smooth and proper dg-category $A$, and proves this property for smooth schemes. We show that $\text{ch} \otimes \mathbb{C}$ is an isomorphism for all categories of the form $\text{Perf}(Y)$, where $Y$ is a smooth DM stack or a smooth quotient stack. In fact, we expect that this “lattice conjecture” should hold for a much larger class of dg-categories, such as the categories $D^b(X)$ for any finite type $\mathbb{C}$-stack and $\text{Perf}(X/G)$ for any quotient stack.

Following some ideas of Thomason in [T5], we next construct a natural “topologization” map $\rho_{G,X} : K^{\text{top}}(\text{Perf}(X/G)) \to K_M(X)$ for any smooth $G$-quasiprojective scheme $X$ which is an equivalence. In fact, for an arbitrary $G$-quasiprojective scheme we have a comparison isomorphism $\rho_{G,X} : K^{\text{top}}(\text{Perf}(X/G)) \to K_M^G(X)$, where the latter denotes the $M$-equivariant Spanier-Whitehead dual of the spectrum $K_M(X)$ (this will be discussed in more detail below).

$^1$Recall that $X/G$ admits a good quotient if there is an algebraic space $Y$ and a $G$-invariant map $\pi : X \to Y$ such that $\pi_* : \text{QCoh}(X/G) \to \text{QCoh}(Y)$ is exact and $(\pi_* \mathcal{O}_X)^G \simeq \mathcal{O}_Y$. 

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Theorem A (See Theorem 3.9 and Theorem 3.20). For any smooth quasi-projective $G$-scheme $X$, the topologization map and the Chern character provide equivalences \(^2\)

\[ K^*_M(X^{an}) \otimes \mathbb{C} \xrightarrow{\rho_{G,X}} \pi_nK^{top}(\text{Perf}(X/G)) \otimes \mathbb{C} \xrightarrow{\text{ch}} H_\bullet C^{der}_\bullet(\text{Perf}(X/G)) \]

Note that the groups $K^*_M(X)$ are modules over $\text{Rep}(M)$, the representation ring of $M$. We say that a $\text{Rep}(M)$-linear Hodge structure of weight $n$ is a finite $\text{Rep}(M)$-module $E$ along with a finite filtration of the finite $\text{Rep}(M)$-module $E \otimes \mathbb{C}$ inducing a Hodge structure of weight $n$ on the underlying abelian group $E$. Using the previous identification $K^*_M(X) \otimes \mathbb{C} \simeq H_{-n}C^{der}_\bullet(\text{Perf}(X/G))$, we will show

Theorem B (See Theorem 3.22). For any smooth $M$-quasiprojective scheme admitting a complete KN stratification, the non-commutative Hodge-de Rham sequence for $K^*_M(X) \otimes \mathbb{C}$ degenerates on the first page, \(^3\) equipping $K^*_M(X)$ with a pure $\text{Rep}(M)$-linear Hodge structure of weight $n$, functorial in $X$. There is a canonical isomorphism

\[ \text{gr}^p K^*_M(X) \simeq H^{n-2p}(R\Gamma(I^{der}_X, \mathcal{O}_{I^{der}_X})). \]

In this theorem, $I^{der}_X$ denotes the derived inertia stack, sometimes referred to as the “derived loop stack.” As we will see in Lemma 4.7 below, we can express this more concretely as

\[ R\Gamma(I^{der}_X, \mathcal{O}_{I^{der}_X}) \simeq R\Gamma(G \times X \times X, \mathcal{O}_\Gamma \otimes^L \mathcal{O}_\Delta)^G, \]

where $G$ acts on $G \times X \times X$ by $g \cdot (h, x, y) = (ghg^{-1}, gx, gy)$ and the two $G$-equivariant closed subschemes of $G \times X \times X$ are defined as $\Gamma = \{(g, x, gx)\}$ and $\Delta = \{(g, x, x)\}$ respectively.

Example. Along the way, show that the lattice conjecture holds for an arbitrary smooth DM stack, and explicitly compute the Hochschild invariants of $\text{Perf}(\mathcal{X})$. For a smooth and proper DM stack, we construct an isomorphism of Hodge structures

\[ \pi_nK^{top}(\text{Perf}(\mathcal{X})) \otimes \mathbb{Q} \simeq \bigoplus_k H^{2k-n}_{\text{Betti}}(I^{cl}_X, \mathbb{Q}(k)). \]

The key observation in establishing the degeneration property for $\text{Perf}(X/G)$ is that the formation of the Hochschild complex takes semiorthogonal decompositions of dg-categories to direct sums, and its formation commutes with filtered colimits. Thus if $\mathcal{A}$ is a retract of a dg-category which can be built from the derived category of smooth and proper DM stacks via an infinite semiorthogonal decomposition, then the degeneration property holds for $\mathcal{A}$.

Example. This simplest example is the quotient stack $\mathbb{A}^n/\mathbb{G}_m$, where $\mathbb{G}_m$ acts with positive weights. Then the objects $\mathcal{O}_{\mathbb{A}^n}\{w\} \in \text{Perf}(\mathbb{A}^n/\mathbb{G}_m)$, which denote the twist of the structure sheaf by a character of $\mathbb{G}_m$, form an infinite full exceptional collection. Therefore the Hochschild complex of $\text{Perf}(\mathbb{A}^n/\mathbb{G}_m)$ is quasi-isomorphic to a countable direct sum of copies of $C_\bullet(\text{Perf}(\text{Spec}(k)))$, and the degeneration property follows.

We can formulate this most cleanly in terms of $G$. Tabuada’s universal additive invariant $\mathcal{U}_k : \{\text{dgCat}_k\} \to \mathcal{M}_k$ [T1, BGT]. Here $\mathcal{M}_k$ is the $\infty$-category which is the localization of the $\infty$-category of small dg-categories which formally splits all semiorthogonal decompositions into direct sums, and $\mathcal{U}_k$ is the localization map.

Theorem C (See Theorem 2.7). Let $X/G$ be a smooth quotient stack admitting a semi-complete KN stratification. Then there is a smooth DM quotient stack $\mathcal{Y}$ such that $\mathcal{U}_k(X/G)$ is a direct summand of $\mathcal{U}_k(D^b(\mathcal{Y}))^{\mathbb{N}}$ in $\mathcal{M}_k$. $\mathcal{Y}$ is proper when $\text{Perf}(X/G)$ is a proper dg-category.

\(^2\)We will see that these homology level equivalences are induced by suitable chain maps

\(^3\)By the Lefschetz principle, versions of this step work over an arbitrary field $k$ of characteristic $0$. 

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In fact, this theorem plays a key role in establishing both Theorems A and B. It should be noted that these motivic decompositions, while very useful for proving abstract theorems, do not capture many of the essential properties of the equivariant Hodge theory. For example, they do not respect the Rep(M) structure.

**Additional results.** We extend and complement the results above in several ways. Most substantially, we prove an analogue of Theorem C for many categories of singularities (or equivalently categories of matrix factorizations). The motivic decomposition in this case is substantially more subtle than the case of Perf(X/G). The generalization makes extensive use of notions from derived algebraic geometry together with a careful study of the desingularization procedure of [K3]. The (noncommutative) Hodge theory of matrix factorization categories is a subject which is still in its infancy. In particular, it is not known whether the degeneration property holds for categories MF(X,W), where X is a smooth DM stack and Crit(W) is proper.

We reduce this question to a purely commutative statement, Assumption 2.26, which essentially states that the work of A. Ogus and V. Vologodsky [OV] can be generalized to DM stacks. We hope to upgrade this assumption to a theorem in follow-up work. Contingent upon Assumption 2.26, we show in Corollary 2.27 that the degeneration property holds for matrix factorization categories MF(X'/G',W) when the quotient stack Crit(W)/G is cohomologically proper. We also show that our results imply the degeneration property for certain graded matrix factorization categories as well (under the same assumption).

In addition, we spend some time discussing more explicit models for the Hochschild homology and periodic cyclic homology for quotient stacks in Section 4. We present different descriptions of the Hochschild complex for Perf(X/G) and MF(X,G,W) in the situation where X is a linear representation of G, a smooth affine G-scheme, or a smooth general quasi-projective G-scheme. For example, we show that when X is smooth and affine, there is an explicit bar-type complex computing the Hochschild homology of Perf(X/G). As an application of Theorem B, we prove an HKR type theorem for the completion of this bar complex at various points of Spec(Rep(G)) when X/G is cohomologically proper. A corollary of this theorem is a description of the completed Hochschild homology modules equipped with the Connes operator in terms of differential forms equipped with the de Rham differential.

**Further questions.** As noted above, we prove the degeneration property for an important class of cohomologically proper stacks. At the same time, our counterexamples to the non-commutative degeneration property, stacks of the form BU for unipotent U, are also important counterexamples in the theory of cohomologically proper stacks. They are examples of stacks on which coherent sheaves have finite dimensional (higher) cohomology, yet the mapping stack Map(BU,Y) often fails to be algebraic. In particular, it raises the natural question.

**Question 0.1.** Do there exist examples of perfect, smooth, and cohomologically proper k stacks X for which the Hodge-de Rham sequence associated to Perf(X) does not degenerate?

In addition to this question, we believe that our main theorem for Hodge structures on $K^M_*(X)$ raises many questions for further inquiry into the role of Hodge theory in equivariant algebraic geometry. For example, it is plausible that the results above could be extended to construct mixed Hodge structures on some version of $K$-theory for arbitrary finite type stacks. In a different direction, one of the central notions in Hodge theory is that of a variation of Hodge structure. For simplicity, let S be an affine scheme and suppose further that $\pi : X/G \to S$ is a smooth equivariant family over S such that all of the fibers $X_s/G$ admit complete KN stratifications. Most of the techniques that we have developed work in families, which allows one to establish the existence of suitable Hodge filtrations on the quasi-coherent sheaf $H^*_C S^\per(\text{Perf}(X/G))$. We therefore believe it is quite likely that one can develop a theory of equivariant period maps.
Finally, although we make use of non-commutative algebraic geometry, all of the differential graded categories in this paper are of commutative origin. It is interesting to try to formulate in non-commutative terms a criterion for the Hodge-de Rham spectral sequence to degenerate. Theorem B suggests the following concrete question: Let $A$ be a proper dg-category which is a module over $\text{Perf}(BG)$. Suppose that $A \otimes_{\text{Perf}(BG)} k \cong \text{Perf}(R)$, where $R$ is a dg-algebra which is homotopically finitely presented, homologically bounded and such that $H_*(R)$ is a finitely generated module over $HH_0(R)$.

**Question 0.2.** Does the Hodge-de Rham spectral sequence always degenerate for such $A$?

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1. **Preliminaries**

Throughout this work, unless explicitly stated otherwise, we work over a fixed subfield $k \subset \mathbb{C}$. All of our functors are understood to be derived, so we write $i_*$ for $Ri_*$, $i^*$ for $Li^*$, $\text{Hom}$ for $R\text{Hom}$, etc. We will work with stacks over the étale site of $k$-schemes. By convention, unless otherwise indicated the term *quotient stack* will denote a quotient of a quasi-projective $k$-scheme by a linearizable action of an algebraic $k$-group $G$, and we denote it $X/G$.

Our stacks will be classical whenever we are studying the derived category of coherent sheaves $D^b \text{Coh}(X/G)$ and its relatives ($\text{QC}(X), \text{Perf}(X)$, etc.), but when we discuss categories of matrix factorizations $\text{MF}(X, W)$ and its relatives ($\text{IndCoh}(X), \text{PreMF}(X, W), \text{PreMF}^\infty(X, W)$, etc.), it will be convenient to work with derived stacks. We reassure the reader who is mostly interested in establishing a Hodge structure on equivariant $K$-theory that the relevant sections only make use of classical quotient stacks and are independent of our discussion of the categories $\text{MF}(X, W)$.

We will work with $k$-linear dg-categories. For some of the more abstract arguments involving homotopy limits and colimits and symmetric monoidal structures, it will be more convenient to replace them with equivalent stable (i.e. pre-triangulated) dg-categories in the Morita model structure on dg-categories (for instance a fibrant replacement will suffice), then to regard them as $k$-linear stable $\infty$-categories (for instance via the equivalence of $[C]$). We permit ourselves a bit of fluidity on this point, so that we may refer both to the literature on dg-categories and stable $\infty$-categories as needed for constructions which evidently make sense in either context.

1.1. **$A$-modules and the noncommutative Hodge-de Rham sequence.** Let us recall the negative cyclic and periodic cyclic homology of a small $k$-linear dg-category, $A$. We let $C_\bullet(A) \in D(\Lambda)$ denote the (mixed) Hochschild complex of $A$, regarded as a dg-module over $\Lambda = k[B]/B^2$ where $B$ has homological degree 1 and acts on $C_\bullet(A)$ by the Connes differential. We have

$$C^{(n)}_\bullet(A) := C_\bullet(A) \otimes k[u]/(u^n)$$

$$C^-_\bullet(A) := \lim_{\leftarrow n} C^{(n)}_\bullet$$

$$C^{per}_\bullet(A) := C^-_\bullet(A) \otimes_{k[[u]]} k((u))$$

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4This is sometimes referred to as a $G$-quasi-projective scheme.
where \( u \) is a variable of homological degree \(-2\). The differential on each complex is given by \( d + uB \), where \( d \) is the differential on \( C_\bullet(A) \). In fact, these constructions make sense for any \( \Lambda \)-module \( M \).

We sometimes denote the negative cyclic construction \( M^{S1} \) and the periodic cyclic construction \( M^{Tate} \). See Lemma 1.6 below.

**Definition 1.1 ([KS]).** The category \( A \) is said to have the degeneration property if \( H_\ast(C^{(n)}_\bullet(A)) \) is a flat \( k[[u]] / (u^n) \)-module for all \( n \geq 1 \).

It is immediate from the definitions that the degeneration property is preserved by filtered colimits of \( \text{dg-categories} \). It is also known that the degeneration property holds for categories of the form \( A = \text{Perf}(\mathcal{R}) \), where \( \mathcal{R} \) is a smooth and proper \( \text{dg-algebra} \) concentrated in homologically non-positive degrees [K1]. In particular, this holds when \( A = \text{Perf}(\mathcal{X}) \), where \( \mathcal{X} \) is a smooth and proper Deligne-Mumford stack over \( k \) [HR], although a more direct argument in this case follows from Proposition 3.16 below. If \( A \) satisfies the degeneration property, then \( H_\ast(C^{\ast}_\bullet(A)) \) is a flat \( k[[u]] \)-module. (See [KS, Corollary 9.1.3]).

The degeneration property owes its name to its relationship with the noncommutative Hodge-de Rham spectral sequence. This is the spectral sequence associated to the filtration\(^5\) of the complex

\[
F^p C^{\text{per}}_\ast(A) = u^p \cdot C^{\ast}_\bullet(A) \subset C^{\text{per}}_\ast(A).
\]

The \( E_1 \) page of the spectral sequence is \( \text{gr } C^{\text{per}}_\ast(A) \simeq C_\ast(A) \otimes k((u)) \). The degeneration property implies that this spectral sequence degenerates on the first page, so the associated graded of the resulting filtration on \( H_\ast(C^{\text{per}}_\bullet(A)) \) is isomorphic to \( H_\ast(C_\ast(A)) \otimes k((u)) \). Under the assumption that \( A \) is suitably bounded, we can say something more precise:

**Lemma 1.2.** Let \( A \) be a \( \text{dg-category} \) such that \( H_\ast(C_\ast(A)) \) is homologically bounded above and which satisfies the degeneration property. Then there exists an (non-canonical) isomorphism \( H_\ast(C^{\ast}_\bullet(A)) \simeq H_\ast(C_\ast(A)) \otimes k[[u]] \).

**Proof.** This follows from the remark before Theorem 4.14 of [KKP]. \( \square \)

The hypothesis of Lemma 1.2, that \( H_\ast(C_\ast(A)) \) is homologically bounded above, will apply to \( \text{Perf}(\mathcal{X}) \) for all smooth \( k \)-stacks \( \mathcal{X} \) of finite cohomological dimension such that \( \text{QC}(\mathcal{X}) \) is compactly generated.

**Remark 1.3.** The \( \Lambda \)-module \( C_\ast(A) \) is functorial in \( A \). When \( A \) is a symmetric monoidal \( k \)-linear \( \infty \)-category, exterior tensor product followed by the symmetric monoidal product gives a natural map

\[
C^{\ast}_\bullet(A) \otimes C^{\ast}_\bullet(A) \to C^{\ast}_\bullet(A \otimes A) \to C^{\ast}_\bullet(A)
\]

and likewise for \( C^{\text{per}}_\ast(A) \). On the level of homology, this gives \( H_\ast C^{\text{per}}_\ast(A) \) the structure of a commutative \( k((u)) \)-algebra, and for any symmetric monoidal functor between symmetric monoidal \( \infty \)-categories \( A \to \mathcal{B} \), the resulting map \( H_\ast C^{\text{per}}_\ast(A) \to H_\ast C^{\text{per}}_\ast(\mathcal{B}) \) is a map of commutative \( k((u)) \)-algebras.

We will also discuss \( \text{dg-categories} \) which are linear over the ring \( k((\beta)) \), where \( \beta \) is variable of homological degree \(-2\) [P]. We may form the \( k((\beta)) \)-linear Hochschild complex \( C^{k((\beta))}_\ast(A) \), which is a module over the \( \text{CDGA} \) \( A((\beta)) = k((\beta))[B]/(B^2) \). We may therefore form the associated complexes \( C^{k((\beta))}_\ast(A), C^{k((\beta))}_\ast(A), C^{k((\beta))}_\ast(A) \), and \( C^{k((\beta))}_\ast(\mathcal{B}) \).

\(^5\)Note that the filtration is not a filtration of \( k((u)) \)-modules, as \( u : F^p \subset F^{p+1} \). As explained to us by D. Kaledin, this can be understood by thinking of \( u \) as the Tate motive. In other words when \( k \subset C \), rather than regarding \( k[[u]] \) simply as a complex (where \( u \) has cohomological degree 2), we regard it as \( H^*(F^\infty) \) with its Hodge structure, which places \( u \) in \( F^1C[[u]] \).
Definition 1.4. A \(k((\beta))\)-linear dg-category \(A\) is said to have the \(k((\beta))\)-linear degeneration property if \(H_*(C^\bullet_k((\beta))\cdot (n)(A))\) is a flat \(k[u]/(u^n)\)-module for all \(n \geq 1\).

Much less is known about the \(k((\beta))\)-linear degeneration property than about its \(\mathbb{Z}\)-graded analogue. However, we will establish the \(k((\beta))\)-linear degeneration property for some categories of singularities of quotient stacks, contingent on Assumption 2.26 below.

1.2. Categories of singularities on stacks. In this section we will explain some results on categories of singularities, or equivalently categories of matrix factorizations, on suitably nice stacks. The following definitions and lemmas follow those in \([P]\) with only minor adjustments, so we will be somewhat brief.

Definition 1.5. A Landau-Ginzburg (LG) model is a pair \(\mathcal{X}, W\), where \(\mathcal{X}\) is a smooth \(k\)-stack such that the automorphism groups of its geometric points are affine and \(W : \mathcal{X} \to \mathbb{A}^1\).

In particular, \(\mathcal{X}\) is a QCA stack over \(k\) in the sense of \([DG]\). Our primary examples of interest will be quotient stacks \(\mathcal{X} := \mathcal{X}/G\) over a field \(k\) of characteristic zero. Throughout this paper, for notational simplicity we will assume that \(\text{Crit}(W)\) is contained in \(\mathcal{X}_0 := \mathcal{X} \times_{\mathbb{A}^1} \{0\}\). For any locally finitely presented algebraic stack, \(\mathcal{X}\), one can define the category

\[
\text{IndCoh}(\mathcal{X}) := \lim_{\leftarrow S \in \text{Aff}/\mathcal{X}} \text{Ind}(D^b\text{Coh}(S)),
\]

where \(\text{Aff}/\mathcal{X}\) denotes the \(\infty\)-category of finite type commutative differential graded algebras over \(\mathcal{X}\), regarded as a derived stack, and \(\text{Ind}(D^b\text{Coh}(\cdot))\) is regarded as functor \(\text{Aff}/\mathcal{X} \to \text{dgCat}\) via shriek pullback. \(^6\)

In proving the Thom-Sebastiani theorem below, we will use the fact that for QCA stacks \(\mathcal{X}\) and \(\mathcal{X}'\), the canonical functors

\[
\text{Ind}(D^b\text{Coh}(\mathcal{X})) \to \text{IndCoh}(\mathcal{X}), \quad \text{and}
\]

\[
\text{Ind}(D^b\text{Coh}(\mathcal{X})) \otimes \text{Ind}(D^b\text{Coh}(\mathcal{X}')) \to \text{Ind}(D^b\text{Coh}(\mathcal{X} \times \mathcal{X}'))
\]

are equivalences \([DG]\).

**Note:** In what follows below, we will assume that all of our dg-categories are idempotent complete. Thus, any dg-category \(\mathcal{C}\) which is not idempotent complete will be tacitly replaced with its idempotent completion.

We now equip the bounded derived category of coherent sheaves on the zero fiber, \(D^b\text{Coh}(\mathcal{X}_0)\), with a \(k[[\beta]]\)-linear structure, where \(\beta\) is a variable of homological degree -2. This arises from a homological \(S^1\)-action on the category \(D^b\text{Coh}(\mathcal{X}_0)\), in the terminology of \([P]\), which concretely refers to a natural action of \(H_*(S^1; k) \simeq \Lambda\) on the Hom-complexes of the category. The formal variable \(\beta\) arises via the same construction which leads to the formal variable \(u\) acting on \(C^\bullet_k(A)\), but we use different variable names to avoid confusion between these two \(S^1\)-actions, especially when we discuss the \(k((\beta))\)-linear negative cyclic homology below.

Notice that \(\text{Spec}(\Lambda)\) admits the structure of a derived group scheme, so the \(\infty\)-category \(\text{IndCoh}(\Lambda)\) admits a symmetric monoidal structure given by convolution: Given \(F, G \in D^b\text{Coh}(\Lambda)\), \(F \circ G := m_*(F \boxtimes G)\), where \(m : \text{Spec}(\Lambda) \times \text{Spec}(\Lambda) \to \text{Spec}(\Lambda)\) is the group multiplication. The following is an enrichment of standard Koszul duality results.

\(^6\)We refer the reader to \([G2,\ Section\ 10]\) for a foundational discussion of this construction.
Lemma 1.6 ([P, Proposition 3.1.4]). The functor
\[ D^b \text{Coh}(\Lambda) \to \text{Perf}(k[[\beta]]) \]
\[ V \mapsto V^{S_1} := \text{Hom}_\Lambda(k, V) \]
extends to a symmetric monoidal equivalence, leading to a symmetric monoidal equivalence
\[ \text{IndCoh}(\Lambda)^\otimes \cong (k[[\beta]] - \text{Mod})^\otimes \]

This proposition is based upon the following observation which we flag for later use: Let \((V, d)\) be a complex with a \(\Lambda\)-action. There is a quasi-isomorphism of complexes
\[ V^{S_1} \cong (V[[\beta]], d + \beta B). \]
This functor is manifestly not monoidal for arbitrary complexes. However, we have that:

Lemma 1.7. The natural inclusion of complexes
\[ (V[[\beta]], d + \beta B) \to (V[[\beta]], d + \beta B) \]
is a quasi-isomorphism whenever \((V, d)\) is homologically-bounded above.

In particular, the functor \(V \to V^{S_1}\) is monoidal when restricted to \(D^b \text{Coh}(\Lambda)\) and thus on the colimit completions. The stack \(X_0\) admits an action by the derived group scheme \(\text{Spec}(\Lambda)\) which defines the upper horizontal arrow in the cartesian square:

\[
\begin{array}{ccc}
X_0 \times \text{Spec}(\Lambda) & \longrightarrow & X_0 \\
p_1 \downarrow & & \downarrow i \\
X_0 & \longrightarrow & X
\end{array}
\]

Thus, \(D^b \text{Coh}(X_0)\) can be seen to admit the structure of a module category over \(D^b \text{Coh}(\Lambda)^\otimes\) and by the above lemma, a \(k[[\beta]]\)-linear structure. We now explain a way to think of this action in somewhat more concrete terms. Observe that \(O_{X_0} \cong A := (\mathcal{O}_X[e], de = W)\). Given two complexes of bounded coherent sheaves \(M, N\) over \(A\) their pushforward\(^7\) Hom-complex \(\text{Hom}_X(i_*M, i_*N)\) inherits a \(\Lambda\)-module structure given by
\[ B : \phi \mapsto \epsilon \circ \phi + \phi \circ \epsilon \]

Remark 1.8. In this setting, for any object \(F \in D^b \text{Coh}(X_0)\),
\[ F[1] \to i^*i_*F \to F \]
and thus there is a distinguished natural transformation \(\text{id}[-2] \to \text{id}\), which is \(\beta\) above.

Lemma 1.9 ([P, Proposition 3.2.1]). Given objects \(M, N \in D^b \text{Coh}(X_0)\), we have a \(k[[\beta]]\)-linear equivalence \(\text{Hom}_X(i_*M, i_*N)^{S_1} \cong \text{Hom}_{X_0}(M, N)\).

Proof. For any object \(M \in D^b \text{Coh}(X_0)\) we have a functorial equivalence\(^8\)
\[ \text{Tot}^\otimes \{ \cdots \to i^*i_*M[2] \to i^*i_*M[1] \to i^*i_*M \} \cong M \]
where the maps
\[ i^*i_*M[1] \to i^*i_*M \]

\(^7\)With this presentation of \(O_{X_0}\) the pushforward functor simply forgets the \(A\)-module structure.

\(^8\)The notation \(\text{Tot}^\otimes\) refers to the functor which takes diagrams of complexes of the form \(\cdots \to M_1 \to M_0\) and forms the direct sum totalization of the corresponding double complex. Equivalently, \(\text{Tot}^\otimes\) denotes the geometric realization of simplicial object in the \(\infty\)-category of complexes corresponding to \(\cdots \to M_1 \to M_0\) under the Dold-Kan correspondence.
are induced by $B$. We have that
\[
\text{Hom}_{\mathcal{X}_0}(M, N) \cong \text{Tot}\{\text{Hom}_{\mathcal{X}_0}(i^*i_* M, N)\} = \text{Hom}\_\mathcal{X}(i_* M, i_* N)^{S^1}.
\]
One calculates as on page 20 of [P] that the induced action is the one quoted above.

\[
\square
\]

Observe that by our assumption that $\mathcal{X}$ is QCA, the complex $\text{Hom}_{\mathcal{X}}(i_* M, i_* N)$ is homologically (totally) bounded [DG], so we are in the situation of Lemma 1.7.

**Definition 1.10.** We define $\text{PreMF}(\mathcal{X}, W) := \text{D}^b\text{Coh}(\mathcal{X}_0)$ with the additional $k[[\beta]]$-linear structure given by setting
\[
\text{Hom}_{\text{PreMF}(\mathcal{X}, W)}(M, N) := \text{Hom}_{\mathcal{X}}(i_* M, i_* N)^{S^1}.
\]

Suppose $\mathcal{Z}$ is a closed substack of $\mathcal{X}_0$. We will denote by $\text{PreMF}_\mathcal{Z}(\mathcal{X}, W)$ the natural generalization of the above construction applied to the category with supports $\text{D}^b\text{Coh}_\mathcal{Z}(\mathcal{X}_0)$.

**Definition 1.11.** Finally, we define the category $\text{MF}(\mathcal{X}, W)$ to be
\[
\text{MF}(\mathcal{X}, W) := \text{PreMF}(\mathcal{X}, W) \otimes_{k[[\beta]]} k((\beta))
\]

There are also Ind-complete versions $\text{PreMF}^\infty(\mathcal{X}, W)$ and $\text{MF}^\infty(\mathcal{X}, W)$.

**Lemma 1.12 (P, Proposition 3.4.1).** $\text{MF}(\mathcal{X}, W)$ is a $dg$-enhancement of the category $\text{D}^b\text{Coh}(\mathcal{X}_0)/\text{Perf}(\mathcal{X}_0)$.

**Proof.** Let $M \in \text{D}^b\text{Coh}(\mathcal{X}_0)$. The essential point is that $M \in \text{Perf}(\mathcal{X}_0)$ iff $\beta^n = 0 \in \text{Hom}(M, M)$ for large enough $n$. We have that a null-homotopy of $\beta^n$ is equivalent to realizing $M$ as a homotopy retract of
\[
\text{Tot}^\oplus \{i^*i_* M[n] \to i^*i_* M[n - 1] \to \cdots \to i^*i_* M\}.
\]

Such a totalization of a finite diagram of perfect complexes is perfect. Conversely, if $M$ is perfect, then $M$ is compact (again this follows easily from the fact that $\mathcal{X}_0$ is QCA), and the identity morphism factors through a finite piece of the complex
\[
\text{Tot}^\oplus \{\cdots \to i^*i_* M[2] \to i^*i_* M[1] \to i^*i_* M\} \cong M,
\]
which is a homotopy colimit of its finite truncations. This proves the lemma.

\[
\square
\]

**Remark 1.13.** By [P, Proposition 4.1.6], if $\text{Crit}(\mathcal{W}) \subset \mathcal{Z} \subset \mathcal{X}_0$, $\text{MF}_\mathcal{Z}(\mathcal{X}, W) \cong \text{MF}(\mathcal{X}, W)$. The proof of this proposition applies here, because it only depends on the fact that for $\mathcal{U} := \mathcal{X}_0 \setminus \mathcal{Z}$, $\text{D}^b\text{Coh}(\mathcal{U})/\text{Perf}(\mathcal{U}) = 0$.

Given two LG-models $(\mathcal{X}, W_1)$ and $(\mathcal{X}', W_2)$, we will consider the pair $(\mathcal{X} \times \mathcal{X}', \pi_1^* W_1 + \pi_2^* W_2)$. We denote by $\ell$ the natural inclusion
\[
\ell : \mathcal{X}_0 \times \mathcal{X}_0' \rightarrow (\mathcal{X} \times \mathcal{X}')_0
\]

**Theorem 1.14 (P, Theorem 4.1.3).** There is a $k[[\beta]]$-linear (coming from the diagonal $S^1$ action on the left-hand side) equivalence
\[
\ell_*^{k[[\beta]]}(\ast \boxplus \ast) : \text{PreMF}(\mathcal{X}, W_1) \otimes_{k[[\beta]]} \text{PreMF}(\mathcal{X}', W_2) \rightarrow \text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0'}(\mathcal{X} \times \mathcal{X}', \pi_1^* W_1 + \pi_2^* W_2).
\]

**Proof.** We define
\[
\text{PreMF}(\mathcal{X} \times \mathcal{X}', W_1, W_2) := \text{D}^b\text{Coh}(\mathcal{X}_0 \times \mathcal{X}_0'),
\]
equipped with its natural $k[[\beta_{W_1}, \beta_{W_2}]]$-linear structure. Observe that by [DG], we have a $k[[\beta_{W_1}, \beta_{W_2}]]$-linear equivalence
\[
\ast \boxplus \ast : \text{PreMF}(\mathcal{X}, W_1) \otimes_k \text{PreMF}(\mathcal{X}', W_2) \rightarrow \text{PreMF}(\mathcal{X} \times \mathcal{X}', W_1, W_2).
\]
This gives rise to an equivalence
\[
\text{PreMF}(\mathcal{X}, W_1) \otimes_{k[[\beta]]} \text{PreMF}(\mathcal{X}', W_2) \cong \text{PreMF}(\mathcal{X} \times \mathcal{X}', W_1, W_2) \otimes_{k[[\beta_{W_1}, \beta_{W_2}]!]} k[[\beta]]
\]
In [P, Lemma 4.1.2], Preygel checks that pushforward along \( \ell \) induces an equivalence
\[
\text{PreMF}(\mathcal{X} \times \mathcal{X}', W_1, W_2) \otimes_{k[[\beta_{W_1}, \beta_{W_2}]!]} k[[\beta]] \cong \text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0'}(\mathcal{X} \times \mathcal{X}', \pi_1^*W_1 + \pi_2^*W_2)
\]
This uses that \( \mathcal{X} \) is Noetherian. Denote the composition of these two equivalences by \( \ell_* (\otimes \otimes \ast) \).
Preygel enhances this functor to a \( k[[\beta]] \)-linear functor \( \ell_*^{k[[\beta]]}(\ast \otimes \ast) \). This \( k[[\beta]] \)-linear functor is automatically an equivalence because its underlying \( k \)-linear functor is an equivalence. \( \square \)

**Theorem 1.15.** Grothendieck duality lifts to \( k[[\beta]] \)-linear equivalence
\[
\text{PreMF}(\mathcal{X}, W) \to \text{PreMF}(\mathcal{X}, -W)^{\text{op}}
\]

**Proof.** This is proven exactly as in [P, Theorem 4.2.2]. \( \square \)

**Theorem 1.16.** Box product and Grothendieck duality give a \( k[[\beta]] \)-linear equivalence
\[
\text{PreMF}^{\infty}_{\mathcal{X}_0 \times \mathcal{X}_0'}(\mathcal{X} \times \mathcal{X}', -\pi_1^*W_1 + \pi_2^*W_2) \to \text{Fun}^{\text{cont}}_{k[[\beta]]}(\text{PreMF}^{\infty}(\mathcal{X}, W_1), \text{PreMF}^{\infty}(\mathcal{X}', W_2))
\]
which gives an equivalence
\[
\text{MF}^{\infty}(\mathcal{X} \times \mathcal{X}', -\pi_1^*W_1 + \pi_2^*W_2) \to \text{Fun}^{\text{cont}}_{k[[\beta]]}(\text{MF}^{\infty}(\mathcal{X}, W_1), \text{MF}^{\infty}(\mathcal{X}', W_2))
\]
after tensoring with \( k((\beta)) \). In the special case when \( \mathcal{X} = \mathcal{X}' \) and \( W_1 = W_2 = W \), let \( \Delta : X \to (X \times X)_0 \) be the natural map. The identity functor corresponds to \( \Delta_*\omega_X \) under this latter equivalence.

**Proof.** The dg-category \( \text{PreMF}^{\infty}(\mathcal{X}, W_1) \) is dualizable, with dual \( \text{Ind}(\text{PreMF}(\mathcal{X}, W_1)^{\text{op}}) \). By Theorem 1.15 we therefore have an identification \( \text{PreMF}^{\infty}(\mathcal{X}, W_1)^{\vee} \cong \text{PreMF}^{\infty}(\mathcal{X}, -W_1) \). By Morita theory, see e.g. Lemma 4.3.1 of [G1], we have a natural equivalence
\[
\text{PreMF}(\mathcal{X}, -W_1) \otimes_{k[[\beta]]} \text{PreMF}(\mathcal{X}', W_2) \to \text{Fun}^{\text{cont}}_{k[[\beta]]}(\text{PreMF}^{\infty}(\mathcal{X}, W_1), \text{PreMF}^{\infty}(\mathcal{X}', W_2))
\]
The result now follows from Theorem 1.14. The identification of the identity functor follows exactly as in Preygel. \( \square \)

1.3. Graded Landau-Ginzburg models.

**Definition 1.17.** A graded LG-model is a non-constant map \( W : \mathcal{X} \to \mathbb{A}^1/G_m \), where \( \mathcal{X} \) is a smooth algebraic \( k \)-stack whose automorphism groups at geometric points are affine, and \( G_m \) acts on \( \mathbb{A}^1 \) with weight one.

Let \( L \) denote the invertible sheaf classified by the composition \( \mathcal{X} \to \mathbb{A}^1/G_m \to BG_m \). Denote by \( \phi : \mathcal{X}' \to \mathcal{X} \) the associated \( G_m \)-torsor over \( \mathcal{X} \). To any graded LG-model, we use the term **associated LG-model** to denote the pair \( (\mathcal{X}', \phi^*W) \). We will see below that, in a precise sense, the graded LG-model is a refinement of its associated LG-model.\(^9\)

**Example 1.18.** The following example has been analyzed in detail by [I]. Let \( Y \) be a smooth variety over \( k \), \( E \) a vector bundle over \( Y \) and let \( s \in \Gamma(\mathcal{E}) \) be a regular section. We have an action of \( G_m \) on \( Q := \text{Tot}(\mathcal{E}^\vee) \) by scaling in the fibers. The function \( s \) therefore determines a mapping \( W_s : \mathcal{X} = \text{Tot}(\mathcal{E}^\vee)/G_m \to \mathbb{A}^1/G_m \).

\(^9\)Note also that given an LG-model \( (\mathcal{X}, W) \), we can forget the data of the trivialization of \( L \) to obtain a graded LG-model. This will correspond to forgetting the \( k((\beta)) \)-linear structure on the category \( \text{MF}(\mathcal{X}, W) \).
The map on morphism spaces is given by identifying
Commuting colimits and reshuffling indices, this is isomorphic to
\( k \), and tracing through the equivalences below on the
Pullback along
Proof of Proposition 1.20.

IndCoh
generate
\( Y \) suffices to show this when
\( \pi \) is conservative implies that objects of the form
\( \pi \) assuming only that
\( D \)

\( Z \)
Proposition 1.20.
Let
\( LG \)-model and its associated
\( LG \)-model.
We define
\( D \) objects in
\( D \)
Definition 1.19. We define
\( \text{and we obtain a natural transformation } - \otimes L[-2] \rightarrow \text{id which we invert below.} \)

**Definition 1.19.** We define \( D^b_{\text{sing}}(X, W) \) to be the idempotent completion of the dg-category with objects in \( D^b_{\text{Coh}}(X_0) \) and morphisms between \( M, N \) given by
\[
\text{Hom}_{D^b_{\text{sing}}(X, W)}(M, N) = \text{hocolim}_p \text{Hom}_{D^b_{\text{Coh}}(X_0)}(M, N \otimes L^{-p})[2p].
\]

For a \( \mathbb{Z} \)-graded \( k \)-linear dg-category \( \mathcal{C} \), we may tensor with \( k(\beta) \), thereby collapsing the grading on \( \text{Hom}(E, F) \) to a \( \mathbb{Z}/2 \)-grading. The following lemma describes the relationship between a graded
\( \mathbb{LG} \)-model and its associated \( \mathbb{LG} \)-model.

**Proposition 1.20.** Let \( W : X \rightarrow \mathbb{A}^1/G_m \) be a graded \( \mathbb{LG} \)-model, and let \( W' : X' \rightarrow \mathbb{A}^1 \) be the associated \( \mathbb{LG} \)-model. Then we have a canonical equivalence of \( \mathbb{Z}/2 \)-graded dg-categories
\( D^b_{\text{sing}}(X, W) \otimes_k k(\beta) \cong \text{MF}(X', W') \).

**Lemma 1.21.** Let \( \pi : X \rightarrow Y \) be a smooth affine morphism of QCA stacks. Then every object of
\( D^b_{\text{Coh}}(X) \) is a retract of \( \pi^* F \) for some \( F \in D^b_{\text{Coh}}(Y) \).

**Proof.** First note that the analogous claim holds for perfect stacks using \( \text{Perf} \) instead of \( D^b_{\text{Coh}} \) and assuming only that \( \pi \) is affine. Indeed, the fact that the pushforward functor \( \pi_* : \text{QC}(X) \rightarrow \text{QC}(Y) \) is conservative implies that objects of the form \( \pi^* F \) with \( F \in \text{Perf}(Y) \) generate \( \text{QC} \).

In order to conclude the same for \( D^b_{\text{Coh}} \), we must imitate this argument for the categories \( \text{IndCoh} \). The pushforward \( \pi_{\text{IndCoh},*} \) again has a left adjoint \( \pi^*_{\text{IndCoh}} \) which preserves \( D^b_{\text{Coh}} \) and agrees with the usual pullback functor there. Because \( \pi_{\text{IndCoh},*} \) satisfies base change with respect to the shriek pullback \( [G2, 5.2.5] \), and \( \text{IndCoh} \) satisfies fpqc descent with respect to shriek pullback, it suffices to show this when \( Y = Y \) is a scheme and hence \( X = X \) is as well.

Because \( \pi \) is smooth, \( [G2, \text{Proposition 4.5.3}] \) implies that the canonical functor \( \text{QC}(X) \otimes \text{QC}(Y) \) \( \text{IndCoh}(Y) \rightarrow \text{IndCoh}(X) \), induced by the pullback \( \pi_{\text{IndCoh},*} \) and the action of \( \text{QC}(X), \otimes \) on \( \text{IndCoh}(X) \), is an equivalence. In particular, this implies that objects of the form \( E \otimes \pi_{\text{IndCoh},*}(F) \) generate \( \text{IndCoh}(X) \). By the observation that objects of the form \( \pi^* F \) generate \( \text{QC}(X) \), it follows that objects of the form \( \pi_{\text{IndCoh},*}(F) \) generate \( \text{IndCoh}(X) \). Thus the morphism \( \pi_{\text{IndCoh},*} \) is conservative.

**Proof of Proposition 1.20.** Pullback along \( \phi : X' \rightarrow X \) defines a functor
\[
\phi_{k(\beta)} : D^b_{\text{sing}}(X, W) \otimes_k k(\beta) \rightarrow \text{MF}(X', W')
\]
by \( k(\beta) \)-linear extension of a certain \( k \)-linear functor which on the level of objects sends \( F \rightarrow \phi^*(F) \).

The map on morphism spaces is given by identifying
\[
\text{Hom}_{D^b_{\text{sing}}(X, W)}(M, N) = \text{hocolim}_p \text{Hom}_{D^b_{\text{Coh}}(X_0)}(M, N \otimes L^{-p})[2p]
\]
and tracing through the equivalences below on the \( q = 0 \) piece. By Lemma 1.21, the functor \( \phi_{k(\beta)}^* \) is essentially surjective and it will follow from its construction that it is a \( k \)-linear equivalence. Commuting colimits and reshuffling indices, this is isomorphic to
\[
\cong \text{hocolim}_p \bigoplus_q \text{Hom}_{D^b_{\text{Coh}}(X_0)}(M, L^q \otimes N \otimes L^{-p})[2p]
\]
In this presentation, the operator $\beta$ corresponds to the
isomorphism between $\text{Hom}_{D^b\text{Coh}(X_0)}(M, L^q \otimes N \otimes L^{-p})$ and $\text{Hom}_{D^b\text{Coh}(X_0)}(M, L^{p+1} \otimes N \otimes L^{-(q+1)})$. The morphism space is in turn isomorphic to:

$$\cong \text{hocolim}_p \text{Hom}_{\text{QC}(X_0)}(M, A \otimes N \otimes L^{-p})[2p]$$

where this latter isomorphism uses the fact that the object in the first argument is coherent and the object in the second argument is homologically bounded above. Using the adjunction for sheaves of algebras over $X_0$, we finally obtain that this is

$$\cong \text{hocolim}_p \text{Hom}_{D^b\text{Coh}(X_0')}(\phi^*(M), \phi^*(N) \otimes \phi^*(L)^{-p})[2p]$$

the cohomological operator $\beta$ now corresponds the to canonical isomorphisms:

$$\text{Hom}_{D^b\text{Coh}(X_0')}(\phi^*(M), \phi^*(N) \otimes \phi^*(L)^{-p}) \to \text{Hom}_{D^b\text{Coh}(X_0')}(\phi^*(M), \phi^*(N) \otimes \phi^*(L)^{-(p+1)})$$

which arise from the canonical trivialization $O_{X_0'} \to \phi^*(L^{-1})$. This operator is identified with the operator $\beta$ in Definition 1.10 and therefore our morphism space agrees with $\text{Hom}_{\text{MF}(X,W')}(\phi^*(M), \phi^*(N))$ as required.

$\square$

In the next section, we will establish the $k((\beta))$-linear degeneration property for LG-models $(X, W)$ such that $X$ admits a semi-complete KN stratification and $\text{Crit}(W)$ is cohomologically proper. The following observation is useful for establishing the $k$-linear degeneration property for $D^b_{\text{sing}}(X, W)$ when $W : X \to \mathbb{A}^1/G_m$ is a graded LG-model whose associated LG-model is of that form.

**Lemma 1.22.** Let $\mathcal{C}$ be a $\mathbb{Z}$-graded dg-category. Then the degeneration property for $\mathcal{C}$ is equivalent to the $k((\beta))$-linear degeneration property for $\mathcal{C} \otimes_k k((\beta))$.

**Proof.** We use the canonical model for the $k((\beta))$-linear Hochschild complex of a small $k((\beta))$-linear category $\mathcal{D}$,

$$C^k((\beta))_{\bullet}(\mathcal{D}) := \bigoplus_{o_1, o_2, \ldots, o_n} \text{Hom}(o_1, o_2) \otimes_{k((\beta))} \text{Hom}(o_2, o_3) \otimes_{k((\beta))} \cdots \otimes_{k((\beta))} \text{Hom}(o_n, o_1),$$

where $o_i$ are objects of $\mathcal{D}$. For the case when $\mathcal{D} = \mathcal{C} \otimes k((\beta))$, it follows from this formula that

$$C^k((\beta))_{\bullet}(\mathcal{D}) \cong C^k_{\bullet}(\mathcal{C}) \otimes k((\beta))$$

canonical on the level $\text{dg-}\Lambda((\beta))$-modules. We therefore have that

$$C^k((\beta), (n))_{\bullet}(\mathcal{D}) \cong C^k_{\bullet}(\mathcal{C})(\mathcal{C}) \otimes k((\beta))$$

on the level of chain complexes as well. The result follows since the homology $H_*(C^k_{\bullet}(\mathcal{C})(\mathcal{C}) \otimes k((\beta))) \cong H_*(C^k_{\bullet}(\mathcal{C})(\mathcal{C})) \otimes k((\beta))$ will be flat over $k[u]/u^n$ if and only if $H_*(C^k_{\bullet}(\mathcal{C}))$ is flat over the same ring. $\square$

2. THE NON-COMMUTATIVE MOTIVE OF A QUOTIENT STACK

In this section, we show that the noncommutative Hodge-de Rham spectral sequence degenerates for $\text{Perf}(X)$ for a large class of smooth quotient stacks subject to a properness condition, and we show degeneration for $\text{MF}(X, W)$ for a large class of Landau-Ginzburg models with smooth $X$ subject to a properness condition on $\text{Crit}(W)$, subject to the assumption that one already knows the degeneration property for LG-models on smooth orbifolds.

Our method for establishing the degeneration property will be to systematically realize the derived category of a smooth quotient stack as being “glued together” from (typically infinitely many) copies of the derived category of smooth Deligne-Mumford stacks. This method will be used several times throughout this paper, so we formulate our main result in a way that can be applied directly in different contexts.
We work with the category $\mathcal{M}_k$ of $k$-linear additive motives in the sense of [T1] (see also [BGT] for a construction using the framework of $\infty$-categories). This is the $\infty$-category obtained as the left Bousfield localization of the $\infty$-category of small $k$-linear dg-categories localized at the class of morphisms $\mathcal{C} \to \mathcal{A} \oplus \mathcal{B}$ coming from split exact sequences of small dg-categories $\mathcal{A} \to \mathcal{C} \to \mathcal{B}$. In other words, objects of $\mathcal{M}_k$ are dg-categories $[\mathcal{C}]$, where we have formally adjoined the relation $[\mathcal{C}] = [\mathcal{A}] \oplus [\mathcal{B}]$ whenever we have a semiorthogonal decomposition $\mathcal{C} = (\mathcal{A}, \mathcal{B})$. We also work with the $\infty$-category $\mathcal{M}_{k((\beta))}$ of $k((\beta))$-linear dg-categories, or equivalently $\mathbb{Z}/2\mathbb{Z}$-graded dg-categories. For $R = k$ or $k((\beta))$, we denote the localization functor $\mathcal{U}_R : \text{LinCat}_R \to \mathcal{M}_R$.

2.1. **Recollections on KN-stratifications.** Our primary geometric tool will be a “KN stratification” of a quotient stack, as defined in [T2, (1.1)] or [HL, Definition 2.2]. This is a decomposition of $X$ as a union of $G$-equivariant, smooth, locally closed subschemes

$$X/G = X^{ss}/G \cup \bigcup_i S_i/G. \tag{1}$$

For instance, when $X$ is projective-over-affine and $G$ is reductive, a KN-stratification of $X/G$ is induced by a choice of $G$-linearized ample line bundle $L$ and a Weyl-invariant inner product on the cocharacter lattice of $G$. Throughout our discussion, we will assume that we have fixed a choice of inner product on the cocharacter lattice of $G$, and we will refer to the KN-stratification induced by $L$ as the $L$-stratification.

For each $i$ there is a distinguished one parameter subgroup $\lambda_i$ of $G$. If we let $L_i$ be the centralizer of $\lambda_i$, then there is a smooth open subvariety $Z_i \subset X^{\lambda_i}$ which is $L_i$-invariant. Then by definition we have

$$S_i := G \cdot \left\{ x \in X \mid \lim_{t \to 0} \lambda_i(t) \cdot x \in Z_i \right\}$$

When the KN stratification arises from GIT, then in fact $Z_i$ is the semistable locus for the action of $L_i' = L_i/\lambda_i(G_m)$ on the closure of $Z_i$.

The main object of study in this paper will be quotients stacks admitting a KN stratification of the following form:

**Definition 2.1.** A KN-stratification of a quotient stack $X/G$ is semi-complete if $X^{ss}/G$ and $Z_i/L_i'$ all admit good quotients which are projective-over-affine. We say that the KN-stratification is complete if all of the good quotients are projective.

**Remark 2.2.** Given a KN-stratification of a $G$-scheme $X$, if $X^{ss}/G$ and $Z_i/L_i'$ all admit semi-complete (resp. complete) KN stratifications, then the stratification of $X$ can be refined to a semi-complete (resp. complete) KN stratification by replacing each stratum with the preimage of the strata of $Z_i/L_i'$ under the projection $S_i/G \to Z_i/L_i'$ and taking the distinguishing one-parameter subgroup of each of these new strata to be $\lambda_i$ plus a very small rational multiple of the distinguished one-parameter subgroup of the corresponding stratum in $Z_i/L_i'$ (which can be lifted to $L$ rationally).

In a sense the main theorem of GIT is the following:

**Theorem 2.3.** Given a reductive $G$ and any $G$-ample bundle on a projective-over-affine $G$-scheme $X$, the $L$-stratification is semi-complete.

Semi-complete KN stratifications are important because they lead to direct sum decompositions of noncommutative motives.

**Lemma 2.4.** If $X$ is a smooth $G$-scheme with KN stratification, we have an equivalence in $\mathcal{M}_k$

$$\mathcal{U}_k(\text{Perf}(X/G)) \simeq \mathcal{U}_k(\text{Perf}(X^{ss}/G)) \oplus \bigoplus_i \mathcal{U}_k(\text{Perf}(Z_i/L_i)).$$
Furthermore, if $W : X/G \to \mathbb{A}^1$ is an LG-model, then $\cup_{k(\beta)}(\text{MF}(X/G, W)) \in \mathcal{M}_{k(\beta)}$ is equivalent to the direct sum

$$\cup_{k(\beta)}(\text{MF}(X^{ss}/G, W)) \oplus \bigoplus_i \cup_{k(\beta)}(\text{MF}(Z_i/L_i, W|Z_i/L_i)).$$

Proof. The main theorem of [HL] provides an infinite semiorthogonal decomposition of $\text{Perf}(X/G)$ under these hypotheses. One factor of the semiorthogonal decomposition is equivalent to $\text{Perf}(X^{ss}/G)$, and the rest are of the form $\text{Perf}(Z_i/L_i)_{w}$, where the subscript denotes the full subcategory of objects whose homology sheaves are concentrated in weight $w$ with respect to $\lambda$. The fact that $\cup_k$ commutes with filtered colimits implies that the infinite semiorthogonal decomposition maps to an infinite direct sum decomposition of $\cup_k(\text{Perf}(X/G)) \in \mathcal{M}_k$. On the other hand, the category $\text{Perf}(Z_i/L_i)$ decomposes as a direct sum of the subcategories $\text{Perf}(Z_i/L_i)_{w}$ over all $w \in \mathbb{Z}$, so $\bigoplus_w \cup_k(\text{Perf}(Z_i/L_i)_{w}) \simeq \cup_k(\text{Perf}(Z_i/L_i)) \in \mathcal{M}_k$. The main semiorthogonal decomposition of [HL] extends to categories of singularities, and hence the argument above applies to $\text{MF}(X/G, W)$. \qed

We will also use KN stratifications to compare properness of the dg-category $\text{Perf}(X/G)$ to properness of the dg-category $\text{Perf}(X^{ss}/G)$ and $\text{Perf}(Z_i/L_i)$ for all $i$.

Lemma 2.5. Let $\mathcal{X}$ be a perfect derived $k$-stack of finite cohomological dimension. Then the following are equivalent

1. $H_i \Gamma(\mathcal{X}, F)$ is finite dimensional for all $i$ and all $F \in D^{-} \text{Coh}(\mathcal{X})$,
2. $R^i \Gamma(\mathcal{X}, F)$ is finite dimensional for all $F \in \text{Coh}(\mathcal{X})$,
3. for any stack $\mathcal{Y}$ with $\mathcal{Y}_{cl, \text{red}} \simeq \mathcal{X}_{cl, \text{red}}$ and with $\mathcal{O}_{\mathcal{Y}}$ eventually co-connective, $\text{Perf}(\mathcal{Y})$ is a proper dg-category.  

Furthermore, if $\mathcal{X}$ is a separated DM stack then this is equivalent to $\mathcal{X}$ being proper.

Proof. Finite cohomological dimension implies that for any $F \in D^{-} \text{Coh}(\mathcal{X})$ and all $i \in \mathbb{Z}$, there is a sufficiently high $n$ such that $H_i \Gamma(\mathcal{X}, \tau_{\leq n} F) \simeq H_i \Gamma(\mathcal{X}, F)$, so (2) $\Rightarrow$ (1). Also, (1) $\Rightarrow$ (2) because $\text{Coh}(\mathcal{X}) \subset D^{-} \text{Coh}(\mathcal{X})$ and $H_i \Gamma(F)$ vanishes in all but finitely many degrees. It is clear that (2) can be checked on $\mathcal{X}_{cl, \text{red}}$ because every $F \in \text{Coh}(\mathcal{X})$ is pushed forward from $\mathcal{X}_{cl}$, and any $F \in \text{Coh}(\mathcal{X}_{cl})$ has a finite filtration whose associated graded is pushed forward from $\mathcal{X}_{cl, \text{red}}$.

To show that (2) $\Leftrightarrow$ (3), it thus suffices to show that (2) is equivalent to $\text{Perf}(\mathcal{X})$ being a proper dg-category in the case when $\mathcal{X}$ is eventually co-connective. Because $\mathcal{X}$ is perfect, for any $F \in D^{b} \text{Coh}(\mathcal{X})$ and any $n$ we can find a perfect complex $P$ such that $F$ is a retract of $\tau_{\leq n} P$, so choosing $n$ large enough shows that $H_i \Gamma(\mathcal{X}, F)$ is a retract of $H_i \Gamma(\mathcal{X}, P)$, which is finite if $\text{Perf}(\mathcal{X})$ is a proper dg-category. On the other hand, $\text{Perf}(\mathcal{X}) \subset D^{b} \text{Coh}(\mathcal{X})$ if $\mathcal{X}$ is eventually co-connective, so $\text{Hom}_{\mathcal{X}}(E, F) = R\Gamma(E^{\vee} \otimes F)$ is finite dimensional for perfect complexes $E$ and $F$.

For the further claim, it suffices to assume that $\mathcal{X}$ is classical. In this case if $\mathcal{X}$ is a separated DM stack, one may find a proper surjection from a quasi-projective scheme $X \to \mathcal{X} [O]$, and then deduce that $X$ is proper from property (2), and hence $\mathcal{X}$ is proper. \qed

Lemma 2.6. Let $X/G$ be a quotient stack with a KN stratification. Then $\text{Perf}(X/G)$ is a proper dg-category if and only if $\text{Perf}(X^{ss}/G)$ and $\text{Perf}(Z_i/L_i)$ are proper dg-categories for all $i$.

Proof. It suffices to consider the case of a single closed stratum $S \subset X$ with center $Z \subset S$ and with open complement $U$.

First assume that $\text{Perf}(X/G)$ is a proper dg-category. [H, Theorem 2.1] a fully faithful embedding $\text{Perf}(U/G) \subset D^{-} \text{Coh}(X/G)$ (in fact one for each choice of $w \in \mathbb{Z}$), and to prove the lemma it will

\[10\] We will need to consider the derived critical locus of $W$ at one point in the proof, which is why we have introduced derived stacks here. If $\mathcal{X}$ is classical, then there is no need to replace $\mathcal{X}$ by an eventually co-connective approximation in (3).
suffice by Lemma 2.5 to show that this embedding preserves $R\Gamma$. We will adopt the notation of [H]: this amounts to showing that we can choose a $w$ such that for $F \in \mathcal{S}^w \subset D^- \operatorname{Coh}(\mathcal{X})$, which is identified with $D^- \operatorname{Coh}(\mathcal{X}^{ss})$ under restriction, we have $R\Gamma(\mathcal{X}, F) \simeq R\Gamma(\mathcal{X}^{ss}, F)$. This holds for $w = 0$ by [H, Lemma 2.8]

Regarding $X$ as a derived stack, we may define the derived fixed locus $\tilde{Z}/L$, whose underlying classical stack is $Z/L$. Then [H, Theorem 2.1] shows that the functor

$$i_* \pi^* : D^- \operatorname{Coh}(\tilde{Z}/L') \simeq D^- \operatorname{Coh}(\tilde{Z}/L)^0 \to D^- \operatorname{Coh}(X/G)$$

is fully faithful. By Lemma 2.5 the dg-category $\operatorname{Perf}(\tilde{Z}/L')$ is proper, and thus so is $\operatorname{Perf}(Z/L')$.

Conversely, assume that $\operatorname{Perf}(Z/L')$ and $\operatorname{Perf}(U/G)$ are both proper dg-categories. We will show that $\operatorname{Perf}(X/G)$ is proper by invoking Lemma 2.5 and showing that $H_n \operatorname{R}\Gamma(X, F)^G$ is finite dimensional for any $n$ and any coherent sheaf $F$. Again by [H, Theorem 2.1], we can functorially write $F$ as a finite extension of an object $F' \in \mathcal{S}^0$ and two objects supported on the unstable stratum $S = S/G$, one in $D^- \operatorname{Coh}_S(\mathcal{X})^{>0}$ and one in $D^- \operatorname{Coh}_S(\mathcal{X})^{<0}$. In particular as noted above we have $R\Gamma(X, F') \simeq R\Gamma(U/G, F')$, which has finite dimensional homology.

Thus it suffices to show that $R\Gamma(X, F')$ has finite dimensional homology for any $F'' \in D^- \operatorname{Coh}(\mathcal{X})$ which is set theoretically supported on $S$. Because $X$ has finite cohomological dimension, we may truncate $F''$ so that it lies in $D^b \operatorname{Coh}(\mathcal{X})$, and then in can be built out of a sequence of extensions of shifts of objects of the form $i_* E$ for $E \in \operatorname{Coh}(S/G)$. Thus it suffices to show that $\operatorname{Perf}(S/G)$ is proper. A similar filtration argument using the baric decomposition of [H, Lemma 2.2] can be used to deduce that $\operatorname{Perf}(S/G)$ is proper because $\operatorname{Perf}(Z/L)$ is proper. Finally, the projection $Z/L \to Z'/L'$ is a $\mathbb{G}_m$-gerbe, so the pushforward preserves perfect complexes, and thus $\operatorname{Perf}(Z/L)$ is proper if $\operatorname{Perf}(Z'/L')$ is proper. □

2.2. The chop-it-up method. We will consider the class of stacks which have semi-complete KN stratifications as in Definition 2.1. We use the notation $\mathcal{C} \oplus \mathbb{N}$ to denote the direct sum of countably many copies of the dg-category $\mathcal{C}$.

**Theorem 2.7.** Let $G$ be an algebraic group. Let $X$ be a smooth $G$-quasiprojective $\mathbb{C}$-scheme with a semi-complete KN stratification, and let $W : X/G \to \mathbb{A}^1$ be a map. Then there is a smooth Deligne-Mumford quotient stack $\mathcal{Y}$ with a map $\mathcal{Y} \to \mathbb{A}^1$ such that $\mathcal{U}_k(\operatorname{Perf}(X/G))$ is a direct summand of $\mathcal{U}_k(\operatorname{Perf}(\mathcal{Y})) \oplus \mathbb{N}$ in $\mathcal{M}_k$, and $\mathcal{U}_{k(\langle \beta \rangle)}(\operatorname{MF}(X/G, W))$ is a direct summand of $\mathcal{U}_{k(\langle \beta \rangle)}(\operatorname{MF}(\mathcal{Y}, W)) \oplus \mathbb{N}$ in $\mathcal{M}_{k(\langle \beta \rangle)}$. Furthermore

1. If $\operatorname{Perf}(X/G)$ is a proper dg-category, then $\mathcal{Y}$ can be chosen to be proper.
2. If $\operatorname{Perf}(\operatorname{Crit}(W)/G)$ is a proper dg-category, then the $W$ can be chosen so that $\operatorname{Crit}(W)$ is proper.

**Remark 2.8.** The proof is constructive, and actually produces something a bit stronger: if $\mathcal{C}$ is the $\infty$-category of small dg-categories, then $\operatorname{Perf}(X/G)$ lies in the smallest subcategory containing $\operatorname{Perf}(\mathcal{Y})$ and closed under countable semi-orthogonal gluings and passage to semi-orthogonal factors. The same holds for $\operatorname{MF}(X/G, W)$ in the $\infty$-category of $k(\langle \beta \rangle)$-linear dg-categories.

**Remark 2.9.** A recent result in preparation of Lunts et. al., following work of D. Bergh [B2], actually shows that for any smooth and proper DM stack $\mathcal{Y}$, the category $\operatorname{Perf}(\mathcal{Y})$ is geometric, i.e. is a semiorthogonal summand of Perf of a smooth proper scheme $Y$. $Y$ will be projective in our case. It follows that $\mathcal{U}_k(\operatorname{Perf}(\mathcal{Y}))$ is a retract of $\mathcal{U}_k(\operatorname{Perf}(Y))$, and hence in the situation (1) above, we can even assume that $Y$ is a smooth projective scheme.

**Example 2.10.** If $X$ is projective-over-affine with a linearizable $G$-action, then the condition that $\operatorname{Perf}(X/G)$ is a proper dg-category is equivalent to the condition that $H_0 R\Gamma(X, 0_X)^G$ is finite dimensional, by [HLP, Proposition 2.24]. The same applies to $\operatorname{Crit}(W)$ for a function $W : X/G \to \mathbb{A}^1$. 15
Example 2.11. We can write and algebraic $k$-group $G$ as a semidirect product $G = U \times L$, where $U$ is its unipotent radical and $L$ its reductive quotient. Assume that there is a one-parameter subgroup $\lambda : \mathbb{G}_m \to L$ which is central in $L$ and acts with positive weights on $\text{Lie}(U)$ in the adjoint representation of $G$. Then this one-parameter subgroup defines a single KN stratum $S = X = \{\ast\}$, and $Z/L = \ast/L' \to \ast$ is a good quotient. Thus Theorem 2.7 applies to a large class of categories of the form $\text{Perf}(BG)$, including when $G$ is a parabolic subgroup of a reductive group.

Example 2.12. If $G$ is as in the previous example, and $X$ is a smooth projective-over-affine $G$-scheme, then one can consider the Bialynicki-Birula stratification of $X$ under the action of $\lambda(\mathbb{G}_m)$, which is a KN stratification. If this is exhaustive, and $\Gamma(X^L, \mathcal{O}_{X^L})^L$ is finite dimensional, then the Bialynicki-Birula stratification can be refined to a complete KN stratification of $X$ as in Remark 2.2.

Our proof of Theorem 2.7 will proceed by a delicate inductive argument. One of the key tools is the following:

**Lemma 2.13.** Let $\pi : Y \to X$ be a rational morphism of finite-type $k$-stacks, meaning $R\pi_*\mathcal{O}_Y \simeq \mathcal{O}_X$. Then $\mathcal{U}_k(\text{Perf}(\mathcal{X}))$ is a summand of $\mathcal{U}_k(\text{Perf}(\mathcal{Y}))$ in $\mathcal{M}_k$. Likewise for any function $W : X \to \mathbb{A}^1$, $\text{MF}(X,W)$ is a summand of $\text{MF}(Y,W)$ in $\mathcal{M}_k(\beta)$.

**Proof.** First consider the categories $\text{Perf}(\mathcal{Y})$ and $\text{Perf}(\mathcal{X})$. The unit of adjunction $\mathcal{U}_k(\text{Perf}(\mathcal{X}))$ is fully faithful and admits a right adjoint. Hence $\text{Perf}(\mathcal{X})$ is a semiorthogonal factor of $\text{Perf}(\mathcal{Y})$. For any map $W : X \to \mathbb{A}^1$, these functors are $\text{Perf}(\mathbb{A}^1)$-linear, and it follows that this semiorthogonal decomposition descends to $\text{D}^b\text{Coh}$ and $\text{Perf}$ of the zero fiber of $W$ and hence to $\text{MF}(\mathcal{Y},W)$.

We will apply Lemma 2.13 in three different situations.

**Example 2.14.** If $\pi : Y \to X$ is a flat morphism of algebraic stacks such that for every $k$-point of $X$ the fiber $Y_p$ satisfies $\Gamma(Y_p, \mathcal{O}_{Y_p}) \simeq k$, then $\pi$ is rational. If $\pi$ is not flat, then the same is true if we take $Y_p$ to refer to the derived fiber.

**Example 2.15.** Any representable birational morphism of smooth $k$-stacks is rational. Indeed we can reduce this to the case for schemes, as birational morphisms are preserved by flat base change and the property of a morphism being rational is fpqc-local on the base.

**Example 2.16.** Let $G \to H \to K$ be an extension of linearly reductive groups, and let $K$ act on a scheme, $X$. Then the morphism $p : X/H \to X/K$ is a $G$-gerbe – after base change to $X$ this morphism becomes the projection $X \times BG \to X$. Thus because $G$ is linearly reductive $Rp_*\mathcal{O}_{X/H} \simeq \mathcal{O}_{X/K}$.

Let $\pi : X' \to X$ be a projective morphism of smooth projective-over-affine varieties which is equivariant with respect to the action of a reductive group, $G$. For a $G$-ample invertible sheaf $L$ on $X$ and a relatively $G$-ample invertible sheaf $M$ on $X'$, we consider the fractional polarization $L_\epsilon = L + \epsilon M$ for $\epsilon \in \mathbb{Q}$. We will need the following:

**Lemma 2.17.** [T2, Lemma 1.2] For any small positive $\epsilon \in \mathbb{Q}$, the $L_\epsilon$-stratification of $X'$ refines the preimage of the $L$-stratification of $X$.

Finally, we need another GIT lemma:

**Lemma 2.18.** Let $X$ be a $G$-quasi-projective scheme which admits a good quotient $\pi : X \to Y$ such that $Y$ is projective-over-affine. Then $X = X^{ss}$ for some linearized projective-over-affine $G$-scheme $X$, which can be chosen to be smooth if $X$ is smooth.
Proof. The proof of [T2, Lemma 6.1] applies verbatim: one constructs a relative $G$-compactification for $X \to Y$ by choosing a sufficiently large coherent $F \subset \pi_i \mathcal{O}_X$ so that $X$ embeds in the projectivization of $\text{Spec}_Y \text{Sym}(F)$. The closure of $X$ is projective over $Y$, and hence projective-over-affine, and it has a linearization for which $\tilde{X}^{ss} = X$ by the cited argument. Furthermore, one can equivariantly resolve any singularities occurring in $\tilde{X} \setminus X$ if $X$ is smooth. \hfill \Box

Proof of Theorem 2.7. Over the course of the proof, we will actually construct a finite set of smooth DM stacks $y_1, \ldots, y_N$ such that $\mathcal{U}_k(\text{Perf}(X/G))$ is a retract of $\mathcal{U}_k(\text{Perf}(y_1))^\oplus N \oplus \cdots \oplus \mathcal{U}_k(\text{Perf}(y_N))^\oplus N$, and likewise for the category of matrix factorizations, and then we may take $y = y_1 \sqcup \cdots \sqcup y_N$ at the end. We shall prove the theorem by induction on the rank of $G$.

Note that by Lemma 2.4 and the definition of a semi-complete $\text{KN}$-stratification, it suffices to prove the claim for quotient stacks which have projective-over-affine good quotients. For our purposes, it will be more convenient to consider smooth $G$-schemes which are projective-over-affine, and by Lemma 2.18 and Lemma 2.4 it suffices to prove the claim for open unions of $\text{KN}$ strata in a quotient stack of this form. We fix a $G$-ample bundle $L$ on $X$ and consider the $L$-stratification as in Equation 1.

Case $X^{ss} = \emptyset$:

By Lemma 2.4 we must the claims for $\mathcal{U}_k(\text{Perf}(Z_i/L_i))$ and $\mathcal{U}_k(\text{Perf}(Z_i/L_i, W))$ for all $i$ for which $Z_i \subset U$. First assume that the inclusion $\lambda(G_m) \subset L_i$ admits a splitting $L_i \to G_m$, so that $L_i \cong G_m \times L_i'$ where the left factor is $\lambda(G_m)$. Then $Z_i/L_i \cong B G_m \times Z_i/L_i'$, so $\mathcal{U}_k(\text{Perf}(Z_i/L_i))$ is a direct sum of copies of $\mathcal{U}_k(\text{Perf}(Z_i/L_i'))$. The same applies to $\text{MF}(Z_i/L_i, W)$, which also admits a direct sum decomposition as $k((\beta))$-linear categories by the weights of $G_m$, with each factor isomorphic to $\text{MF}(Z_i/L_i', W)$. This is the only point of the proof at which an infinite direct sum enters, and it is an infinite direct sum of copies of the same category, hence throughout the proof we will only encounter a finite set of distinct DM stacks.

If $\lambda(G_m) \subset L_i$ is not split, then we can choose a surjective homomorphism $\tilde{L} \to L_i$ with finite kernel, where $\tilde{L} \cong G_m \times L'$ and $G_m \times \{1\} \to L_i$ factors through $\lambda(G_m)$. The morphism $p: Z_i/\tilde{L} \to Z_i/L_i$ is rational, hence Lemma 2.13 reduces the problem to showing the claim for $(Z_i/\tilde{L}, W)$. By the argument of the previous paragraph it again suffices to prove the claims for $(Z_i/L_i', W)$.

Let $Z_i$ be the closure of $Z_i$, which is a connected component of $X^\lambda_i$ and hence smooth and projective-over-affine. Then $Z_i$ is the semistable locus for the action of $L_i'$ on $Z_i$, and $L_i'$ has lower rank than $L_i$, so the first two claims of the theorem follow from the inductive hypothesis. We may also apply the inductive hypothesis to claims (1) and (2) of the theorem once we establish that $\text{Perf}(Z_i/L_i')$ is a proper dg-category if $\text{Perf}(U/G)$ is and, respectively, $\text{Perf}(\text{Crit}(W|Z_i)/L_i')$ is a proper dg-category if $\text{Perf}(\text{Crit}(W|U)/G)$ is. This follows from Lemma 2.6.

Case $X^s = X^{ss} \neq \emptyset$:

The argument in the case where $X^{ss} = \emptyset$ applies here as well, so the inductive hypothesis implies that the conclusion of the theorem holds for $\text{Perf}(Z_i/L_i)$ and $\text{MF}(Z_i/L_i, W)$ for all $Z_i \subset U$. By Lemma 2.4 it suffices to show that the claims hold for $(X^{ss}/G, W)$. In this case $X^{ss}/G$ is a smooth separated Deligne-Mumford stack. Furthermore if $\text{Perf}(U/G)$ is a proper dg-category then so is $\text{Perf}(X^{ss}/G)$ by Lemma 2.6, and hence $X^{ss}/G$ is a proper DM stack by Lemma 2.5. Likewise those lemmas imply that $\text{Crit}(W|X^{ss})/G$ is a proper DM stack if $\text{Perf}(\text{Crit}(W|U)/G)$ is a proper dg-category.

Case $X^s \neq \emptyset$ but $X^{ss} \neq X^{ss}$:

As in the previous case, it suffices to show the claims for $(X^{ss}/G, W)$. Here we use the main result of [K3], which says that there is a birational morphism $\pi: X' \to X$ such that $(X')^{ss}(L_c) = (X')^s(L_c)$,
where $L_{\epsilon} = \pi^*L + \epsilon M$ for a suitable relatively $G$-ample $M$. By Lemma 2.17 the open subset $U' := \pi^{-1}(X^{ss}(L))$ is a union of KN strata, and $\pi: U' \to X^{ss}(L)$ is rational, so by Lemma 2.13 we may reduce the main statements of the theorem for $\text{Perf}(U/G)$ and $\text{MF}(U/G, W)$ to the corresponding claims for $(U'/G, W|_{U'})$, which fall under the previous case. In order to prove the further claim (1), note that the fact that $U'/G \to U/G$ is proper implies that $\text{Perf}(U'/G)$ is a proper dg-category, so again we may reduce to the previous case.

Proving claim (2) amounts to showing that $\text{Perf}(\text{Crit}(W|_{U'})/G)$ is a proper dg-category when $\text{Perf}(\text{Crit}(W|_{U'})/G)$ is. This is a bit more subtle, and requires us to revisit the construction of $X'$ from [K3] more carefully: $X'$ is obtained from $X$ by blowing up along a sequence of closed $G$-equivariant subvarieties which are described as the closures of certain explicit subvarieties of $X^{ss}(L)$. Inside $X^{ss}(L)$, the locus of each blow up is a smooth closed subvariety of the form $G \cdot V$, where $V$ is the fixed locus of a positive dimensional reductive subgroup $R \subset G$. Thus $U'$ is obtained from $X^{ss}(L)$ by blowing up this sequence of smooth subvarieties.

To finish the proof of claim (2), we use the this description of $U'$ to show that $\text{Crit}(W|_{U'}) \to \text{Crit}(W|_{X^{ss}})$ is a proper map. For any point $x \in V$, the $G$-invariance of $W$ implies that $(dW)_x \in (\Omega^2_{X,x})^R \subset \Omega^1_{X,x}$, which can be canonically identified with $\Omega^1_{V,x}$. The latter maps injectively to $\Omega^1_{\text{Bl}_{G \cdot V}X^{ss}, y}$ for any $y$ in the fiber of $x$ under $p: \text{Bl}_{G \cdot V}X^{ss} \to X^{ss}$. It follows from this observation and $G$-equivariance that $\text{Crit}(W|_{\text{Bl}_{G \cdot V}X^{ss}}) = p^{-1}\text{Crit}(W)$. By iterating this we see that $\text{Crit}(W|_{U'}) = \pi^{-1}\text{Crit}(W|_{X^{ss}})$ and is thus proper over $\text{Crit}(W|_{X^{ss}})$.

Case $X^{ss} \neq \emptyset$ but $X^s = \emptyset$:

Let $Y$ be a smooth projective variety with a linearized $G$-action such that $R\Gamma(Y, O_Y) \simeq k$ and $Y^s \neq \emptyset$. For instance, $Y$ could be a suitable product of flag varieties, or a large projective space with a suitable linear $G$ action. By Lemma 2.17, the open subvariety $U \times Y \subset X \times Y$ is a union of KN strata for the $L_{\epsilon}$ stratification. The projection $U \times Y/G \to U/G$ is rational, and so by Lemma 2.13 it suffices to prove the claims for $(U \times Y/G, W)$. Note that $\text{Perf}(U \times Y/G)$ is a proper dg-category of $\text{Perf}(U/G)$ is. Also $\text{Crit}(W|_{U \times Y}) = \text{Crit}(W|_U) \times Y$ set theoretically, so $\text{Perf}((\text{Crit}(W|_{U \times Y}))$ is still a proper dg-category if $\text{Perf}(\text{Crit}(W|_{U})/G)$ is. Furthermore, $X^{ss} \times Y^{ss} \subset (X \times Y)^{ss}$, so there are points in $(X \times Y)^{ss}$ with finite stabilizer groups, i.e. $(X \times Y)^s \neq \emptyset$. We have thus reduced the claim to the previous case.

\[ \square \]

2.3. The degeneration property for quotient stacks. In [T2, Theorem 7.3], Teleman establishes the degeneration of a commutative Hodge-de Rham sequence, which converges to the equivariant Betti-cohomology $H^*_G(X)$, for a smooth quotient stack $X/G$ with a complete KN stratification. The argument in [T2] makes use of the KN stratification and has a similar flavor to the proof of Theorem 2.7. However the proof in the commutative case is substantially simpler. In the noncommutative situation, we are not aware of an argument to reduce the proof of degeneration to the case of the quotient of a smooth projective scheme by the action of a reductive group, as was done in [T2]. In addition, the motivic statement of Theorem 2.7 leads to the degeneration property for many categories of matrix factorizations.

However, using the motivic statement of Theorem 2.7, we can immediately deduce noncommutative HdR degeneration. The main observation is the following

**Lemma 2.19.** The degeneration property is closed under direct summands and arbitrary direct sums in $\mathcal{M}$. Likewise the $k((\beta))$-linear degeneration property is closed under direct summands and arbitrary direct sums in $\mathcal{M}_{k((\beta))}$.

**Proof.** The formation of the Hochschild complex $C_\bullet(-)$ is an additive invariant of dg-categories, hence factors through $\mathcal{U}_k$ uniquely up to contractible choices. The claim follows from the fact that
the operation \( D(\Lambda) \to D(k) \) taking mapping \((M,d,B) \mapsto (M \otimes k[u]/(u^n), d + uB) \) commutes with filtered colimits and in particular infinite direct sums, and the fact that an infinite direct sum of \( k[u]/(u^n) \) modules is flat if and only if every summand is flat. The same argument applies verbatim to the \( k((\beta)) \)-linear degeneration property.

**Corollary 2.20.** Let \( G \) be a reductive group and let \( X \) be a smooth \( G \)-quasiprojective scheme which admits a complete KN stratification. If \( \text{Perf}(X/G) \) is a proper dg-category, then \( \text{Perf}(X/G) \) has the degeneration property.

**Example 2.21.** As a counterexample, consider \( \text{Perf}(BG_a) \). This category is Morita equivalent to the category \( \text{Perf}(k[e]/(e^2)) \) where \( e \) has degree \(-1\). By the (graded-commutative) HKR theorem, proposition 5.4.6 of [L], we have that \( H_\bullet C_\bullet(\text{Perf}(k[e]/(e^2))) \cong k[e]/(e^2) \otimes \text{Sym}(\delta e) \), where \( \delta e \) has degree 0. By theorem 5.4.7 of the same book, the Connes operator goes to the de Rham differential which sends \( e \to \delta e \) and so the spectral sequence does not degenerate.

We also observe, somewhat surprisingly, that the derived category of coherent sheaves on certain singular quotient stacks also has the degeneration property. We will consider the following geometric set up

- \( X/G = X^{ss}/G \cup \bigcup_i S_i/G \) is a complete KN stratification (Definition 2.1) of a smooth quotient stack,
- \( V \) is a \( G \)-equivariant locally free sheaf on \( X \) such that \( V|_{Z_i} \) has \( \lambda_i \)-weights \( \leq 0 \) for all \( i \), and
- \( \sigma \in \Gamma(X,V)^G \) is an invariant section.

Note that the quantization-commutes-with-reduction theorem [T2] implies that if the \( \lambda_i \)-weights of \( V|_{Z_i} \) are strictly negative, then \( \Gamma(X,V)^G \cong \Gamma(X^{ss}(L),V)^G \) (this is referred to as adapted in [T2]). Using the methods of [HL] one can show that \( \dim \Gamma(X,V)^G < \infty \) even when the \( \lambda_i \) weight of \( V|_{Z_i} \) vanishes for some \( i \).

**Amplification 2.22.** In the set up above, if

1. \( \sigma \) is regular on \( X^{ss} \) with smooth vanishing locus, and
2. for all \( i \) the restriction of \( \sigma \) to \( (V|_Z)^{\lambda=0} \), the summand of \( V|_{Z_i} \) which is fixed by \( \lambda(\mathbb{G}_m) \), is regular with smooth vanishing locus,

then there is a smooth and proper DM quotient stack \( Y \) such that \( \bigcup_k (\text{D}^b \text{Coh}(X_{\sigma}/G)) \) is a retract of \( \bigcup_k (\text{Perf}(Y))^{\otimes N} \).

**Proof.** We apply the structure theorem for the derived zero locus \( X_\sigma \) in [H, Theorem 3.2], whose derived category is just the derived category of the sheaf of cdga’s over \( X/G \) given by the Koszul algebra

\[
A = (\text{Sym}(V^\vee[1]), d\phi = \phi(s)).
\]

The structure theorem constructs an infinite semiorthogonal decomposition which generalizes the main structure theorem of [HL]. One factor is isomorphic to \( \text{D}^b \text{Coh}(X^{ss}/G) \), and the remaining factors are isomorphic to \( \text{D}^b \text{Coh}(Z_i'/L_i)^w \), where \( Z_i' \) denotes the derived zero locus of \( \sigma \) restricted to \( (V|_Z)^{\lambda=0} \), and the superscript \( w \) denotes the full subcategory of \( \text{D}^b \text{Coh}(Z_i'/L_i) \) consisting of complexes whose homology is concentrated in weight \( w \).

In order to apply this theorem, we must check that after restricting the cotangent complex \( L_{X_{\sigma}/G} \) to \( Z_i'/L_i \) and looking at the summand with \( \lambda \)-weights \( < 0 \), there is no fiber homology in homological degree 1. Because \( X_\sigma \) is a derived zero section, we have

\[
(L_{X_{\sigma}/G|_{Z_i'}})^{\lambda<0} \cong \left[ (V^\vee|_{Z_i'})^{\lambda<0} \to (\Omega X|_{Z_i'})^{\lambda<0} \to 0 \right]^{\otimes (g^\vee)^{\lambda<0}}.
\]

So the weight hypotheses on \( V|_{Z_i} \) imply that this is a two term complex of locally free sheaves in homological degrees 0 and \(-1\), and hence has no fiber homology in homological degree 1.
Given the structure theorem for $\text{D}^b\text{Coh}(X_\sigma/G)$, the proof of Lemma 2.4 now applies verbatim to give a finite direct sum decomposition

$$\mathcal{U}_k(\text{D}^b\text{Coh}(X_\sigma/G)) = \mathcal{U}_k(\text{D}^b\text{Coh}(X^a_\sigma/G)) \oplus \bigoplus_i \mathcal{U}_k(\text{D}^b\text{Coh}(Z_i'/L_i)).$$

Under the hypotheses of the amplification, each factor in this direct sum decomposition is $\text{D}^b\text{Coh}$ of a smooth quotient stack satisfying the hypotheses of Theorem 2.7, and the result follows. \hfill \Box

**Remark 2.23.** Note that when $V$ is strictly adapted to the KN stratification, then the condition (2) in the previous amplification is vacuous.

**Corollary 2.24.** In the set up of Amplification 2.22, the category $\text{D}^b\text{Coh}(X_\sigma/G)$ has the degeneration property.

**Remark 2.25.** There are at least two additional ways to prove the degeneration property in the previous corollary. One can construct a graded LG-model $W : \text{Tot}(V^\vee)/G \times \mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$ such that $\text{D}^b_{\text{sing}}(\text{Tot}(V^\vee)/G \times \mathbb{G}_m, W) \simeq \text{D}^b\text{Coh}(X_\sigma/G)$ as in [I]. Then one can use a semiorthogonal decomposition of $\text{D}^b_{\text{sing}}(\text{Tot}(V^\vee)/G \times \mathbb{G}_m, W)$ analogous to those of Lemma 2.4 to deduce the motivic decomposition $\mathcal{U}_k(\text{D}^b\text{Coh}(X_\sigma/G)) = \mathcal{U}_k(\text{D}^b\text{Coh}(X^a_\sigma/G)) \oplus \bigoplus_i \mathcal{U}_k(\text{D}^b\text{Coh}(Z_i'/L_i))$ used in the proof of the previous proposition.

Alternatively, if one is only interested in the degeneration property, then one can observe that under the hypotheses of Amplification 2.22, $\text{Perf}(\text{Crit}(W)/G)$ is a proper dg-category in the underlying LG-model $\text{Tot}(V^\vee)/G \to \mathbb{A}^1$, by Lemma 2.6. Then one can invoke Proposition 1.20.

In order to make a similar claim for LG-models, we must assume a degeneration statement for LG-models on orbifolds. For any affine LG-model $W : X \to \mathbb{A}^1$, we have a $k((\beta)) \otimes \Lambda$-module $(\Omega^*_X, -dW \wedge -)$, where $B$ acts via the de Rham differential.

**Assumption 2.26.** For any smooth LG-model $W : X \to \mathbb{A}^1$ with $X$ a smooth DM stack and $\text{Crit}(W)$ proper, the $k((\beta)) \otimes \Lambda$-module $R\Gamma(X_{\text{ss}}, (\Omega^*_\tau, -dW \wedge -))$ has the degeneration property.

This is a slight generalization of the results of [OV], although we are not aware of anywhere where this is shown in the literature. We will see in Corollary 4.6 that this assumption is equivalent to assuming that the degeneration property holds for all Deligne-Mumford LG-models $W : X \to \mathbb{A}^1$. Combining this observation with Theorem 2.7 provides the following:

**Corollary 2.27.** Suppose that Assumption 2.26 holds. Let $W : X/G \to \mathbb{A}^1$ be an LG-model, where $X$ is a smooth quasi-projective $G$-scheme which admits a semi-complete KN stratification. If $\text{Perf}(\text{Crit}(W)/G)$ is a proper dg-category, then the $k((\beta))$-linear degeneration property holds for $\text{MF}(X/G, W)$.

As noted above, if $X$ is projective-over-affine and $\Gamma(X, \mathcal{O}_{\text{Crit}(W)})^G$ is finite dimensional, then $\text{Perf}(\text{Crit}(W)/G)$ is a proper dg-category.

### 3. Hodge structures on equivariant $K$-theory

In this section we consider the action of a reductive group $G$ on a smooth quasi-projective $\mathbb{C}$-scheme $X$. Our goal is to identify the periodic cyclic homology $C^\text{per}(\text{D}^b\text{Coh}(X/G))$ with the complexification of the equivariant topological $K$-theory with respect to a maximal compact subgroup $M \subset G$, $K_M(X^{an})$ as defined in [S2]. Our final result, Theorem 3.22, will allow us to define a pure Hodge structure of weight $n$ on $K^n_M(X^{an})$ in the case where $X$ admits a complete KN stratification.

Rather than construct a direct isomorphism, we study an intermediate object, the topological $K$-theory of the dg-category $K^{\text{top}}(\text{D}^b\text{Coh}(X/G))$, as defined in [B3], which admits natural comparison isomorphisms with each of these theories.
In Blanc’s construction, $K^{\text{top}}(\mathcal{C})$ is constructed from the geometric realization of the presheaf of spectra on the category, $\text{Aff}$, of affine $\mathbb{C}$-schemes of finite type, $K(\mathcal{C}) : A \mapsto K(A \otimes \mathbb{C} \mathcal{C})$. The geometric realization of a presheaf, $|\cdot|$, is defined to be the left Kan extension of the functor $A \mapsto \Sigma^\infty(\text{Spec } A)_{+}^{an}$, regarded as functor with values in spectra, along the Yoneda embedding of the category of finite type $\mathbb{C}$-schemes into presheaves of spectra, $\text{Aff} \to \text{Sp}(\text{Aff})$. The geometric realization functor $|\cdot| : \text{Sp}(\text{Aff}) \to \text{Sp}$ admits a right adjoint, which assigns $M \in \text{Sp}$ to the presheaf of spectra $H_{B}(M) := \text{Hom}_{\text{Sp}}(\Sigma^\infty(\cdot)_{+}^{an}, M)$. The semi-topological $K$-theory is the geometric realization $K^{st}(\mathcal{C}) := |K(\mathcal{C})|$, regarded as a $K^{st}(\mathbb{C})$-module spectrum. By [B3, Theorem 4.5], we have an isomorphism $K^{st}(\mathbb{C}) \simeq bu$, where the latter denotes the connective topological $K$-theory spectrum. Choosing a generator $\beta \in \pi_{2}(bu)$, one then defines the topological $K$-theory of a dg-category to be

$$K^{\text{top}}(\mathcal{C}) := K^{st}(\mathcal{C})[\beta^{-1}] = |K(\mathcal{C})| \otimes_{bu} bu[\beta^{-1}]$$

We will also use the construction of a Chern character map $\text{Ch} : K^{\text{top}}(\mathcal{C}) \to C^{\text{per}}_{\bullet}(\mathcal{C})$. First, one obtains a map of presheaves $K(\mathcal{C}) \to C^{\text{per}}_{\bullet}(\mathcal{C})$ from the usual Chern character in algebraic $K$-theory, where $C^{\text{per}}_{\bullet}(\mathcal{C})$ denotes the presheaf $A \mapsto C^{\text{per}}_{\bullet}(A \otimes \mathbb{C} \mathcal{C})$. Using a version of the Kunneth formula for periodic cyclic homology, one obtains an equivalence $|C^{\text{per}}_{\bullet}(\mathcal{C})| \simeq C^{\text{per}}_{\bullet}(\mathcal{C}) \otimes \mathbb{C}[u^{\pm}] |C^{\text{per}}_{\bullet}(\mathcal{C})|$. Then one essentially has an isomorphism of presheaves $C^{\text{per}}_{\bullet}(\mathcal{C}) \simeq H_{B}(\mathbb{C}[u^{\pm}])$, which leads to a map $|C^{\text{per}}_{\bullet}(\mathcal{C})| \to \mathbb{C}[u^{\pm}]$. Combining these provides a map

$$K^{st}(\mathcal{C}) \to C^{\text{per}}_{\bullet}(\mathcal{C}) \otimes \mathbb{C}[u^{\pm}] \frac{|C^{\text{per}}_{\bullet}(\mathcal{C})|}{\to} C^{\text{per}}_{\bullet}(\mathcal{C})$$

which give the Chern character after inverting $\beta$. The main result we use is [B3, Proposition 4.32], which states that for a finite type $\mathbb{C}$-scheme, $X$, the Chern character induces an equivalence $K^{\text{top}}(\text{Perf}(X)) \otimes \mathbb{C} \to C^{\text{per}}_{\bullet}(\text{Perf}(X))$. Furthermore, there is a natural topologization map which is an equivalence $K^{\text{top}}(\text{Perf}(X)) \to K(X^{an})$, and under this equivalence $\text{Ch}$ can be identified with a twisted form of the usual Chern character for $X^{an}$ under a canonical isomorphism $C^{\text{per}}_{\bullet}(X) \to H_{\text{Betti}}(X; \mathbb{Q}) \otimes \mathbb{C}[u^{\pm}]$. More precisely, Blanc’s Chern character provides an equivalence $K^{\text{top}}(\text{Perf}(X)) \otimes \mathbb{Q} \simeq H_{\text{Betti}}(X; \mathbb{Q}) \otimes \mathbb{Q}(\frac{u}{2\pi i}) \subset H_{\text{Betti}}(X^{an}; \mathbb{Q}) \otimes \mathbb{C}(u))$, which we can alternatively express as an isomorphism

$$\pi_{n}(K^{\text{top}}(\text{Perf}(X))) \otimes \mathbb{Q} \simeq \bigoplus_{p} H^{2p-n}(X^{an}; \mathbb{Q}(p)),$$

where $\mathbb{Q}(p) \subset \mathbb{C}$ denotes the subgroup $(2\pi i)^{p}Q$.

3.1. Equivariant $K$-theory: Atiyah-Segal versus Blanc. In this section we consider a reductive group $G$ with maximal compact subgroup $M \subset G$, and a $G$-quasi-projective scheme $X$, which need not be smooth. The goal of this section will be to construct a comparison isomorphism between $K^{\text{top}}(D^{b}\text{Coh}(X/G))$ and topological $M$-equivariant $K$-homology of $X^{an}$ with locally compact supports. We will consider two presheaves on the category of $G$-quasi-projective schemes,

$$E(X) = K^{\text{top}}(\text{Perf}(X/G)), \quad \text{and } E(X) = K_{M}(X^{an}),$$

In order to be consistent with the rest of the paper, we use the notation $C^{\text{per}}_{\bullet}$ for the periodic cyclic homology complex of a dg-category, rather than the notion $HP$ used in [B3]. In addition, we use the notation $\otimes$ rather than $\wedge$ for the smash product of spectra and module spectra. For example $K^{\text{top}}(\mathcal{C}) \otimes \mathbb{C}$ is the $\mathbb{C}$-module spectrum, which we canonically identify with a complex of $\mathbb{C}$-modules, which is denoted $K^{\text{top}}(\mathcal{C}) \wedge HC$ in [B3].
where the latter refers to the topological $K$-cohomology theory for topological $M$-spaces constructed in [AS+].\footnote{Below we use the more systematic description of $K_M(X^{an})$ in terms of equivariant stable homotopy theory as the spectrum obtained by taking level-wise $M$-equivariant mapping spaces from $X$ to the naive $M$-spectrum underlying the $M$-spectrum $bu_M$. For details on the non-equivariant and equivariant stable homotopy category, we refer the reader to [LSM] and [M+].}

We will also consider the Atiyah-Segal equivariant $K$-homology with locally compact supports $K_M^{c,\mathcal{V}}(X^{an})$. This theory was studied in [T5, Section 5] under the notation $G^{AS}(G,X)$, and our discuss follows this reference closely. In particular, we refer the reader there for a nice discussion contextualizing $K_M^{c,\mathcal{V}}(\cdot)$ with respect to several other versions of equivariant $K$-theory. We have chosen to denote the $M$-equivariant $K$-homology with locally compact supports as $K_M^{c,\mathcal{V}}(X^{an})$ because it is the $M$-equivariant Spanier-Whitehead dual of the $M$-spectrum of equivariant $K$-theory with compact supports constructed in [S2], which we denote $K_M^c(X^{an})$.

Consider the category Pairs$_G$ consisting of pairs $(X,U)$ of a smooth quasi-projective $G$-scheme $X$ along with a $G$-equivariant open subscheme $U \subset X$. A map $f : (X_0, U_0) \to (X_1, U_1)$ is a $G$-equivariant map $f : X_0 \to X_1$ such that $f(U_0) \subset f(U_1)$. Given a presheaf of spectra $E$ on the category Sm$_G$ of smooth $G$-schemes, we can define a presheaf on pairs

$$E(X,U) := \text{fib} (E(X) \to E(U)).$$

\textbf{Definition 3.1.} An \textit{equivariant Borel-Moore (BM)-type homology theory} is a presheaf of spectra $E : \text{Sm}_G \to \text{Sp}$ such that for any smooth $G$-scheme $X$:

1. for any $G$-equivariant smooth closed subscheme $X' \hookrightarrow X$ and open $G$-subscheme $U \subset X$ such that $X' \cup U = X$

$$(E(X', U) \to E(X, U \cap X))$$

is an equivalence; and

2. if $V \to X$ is a torsor for a $G$-equivariant locally free sheaf on $X$, then the pullback map $E(X) \to E(V)$ is an equivalence.

Given such an $E$, one defines $E(Z)$ for any $G$-quasi-projective scheme as $E(Z) := E(X, X - Z)$ for some equivariant closed embedding in a smooth quasi-projective $G$-scheme $X$.

\textbf{Lemma 3.2.} In the previous definition, $E(Z) := E(X, X - Z)$ is independent of the equivariant closed embedding $Z \hookrightarrow X$.

This is essentially proved in [T5], which is an extension to the equivariant setting of [T3]. For the benefit of the reader, we explain the conceptual core of argument:

\textit{Proof.} Define a category Emb whose objects are $G$-quasi-projective schemes and whose morphisms $Z \leadsto X$ consist of a $G$-equivariant closed subscheme $V \hookrightarrow X$ along with a $G$-equivariant map $V \to Z$ which can be factored as a composition of maps which are torsors for locally free sheaves. Composition is given by pullback of closed subschemes. Then in the proofs of [T5, T3], Thomason shows that given two maps $Z_1 \leadsto X_1, X_2$, there is a linear action of $G$ on $\mathbb{A}^n$ and maps $X_1, X_2 \leadsto \mathbb{A}^n$ such that the two compositions $Z_i \leadsto \mathbb{A}^n$ agree.\footnote{More precisely, the proof of [T5, Proposition 5.8] shows that for any $G$-quasi-projective $X$ there is a map $X \leadsto \mathbb{A}^n$ for some linear representation of $G$. Thus it suffices to consider the case of two maps $Z \leadsto \mathbb{A}^{n_i}$, $i = 1, 2$. Next if $V \to Z$ is a composition of torsors for locally free sheaves and $V \leadsto \mathbb{A}^{n_i}$, $i = 1, 2$ are two $G$-equivariant closed embeddings, then the proof of [T3, Lemma 4.2] works equivariantly to construct an equivariant embeddings $\mathbb{A}^{n_1} \leadsto \mathbb{A}^{n_1} \times \mathbb{A}^{n_2}$ such that the two induced embeddings $V \leadsto \mathbb{A}^{n_1} \times \mathbb{A}^{n_2}$ agree. Thus it suffices to show that for any two maps $Z \leadsto \mathbb{A}^{n_i}$ corresponding to two fibrations $V_i \to Z$, one can compose with maps $\mathbb{A}^{n_i} \leadsto \mathbb{A}^{n_i}$ such that if $V_i \subseteq \mathbb{A}^{n_i}$ corresponds to the compositions $Z \leadsto \mathbb{A}^{n_i}$, then $V_i \simeq V_j$ over $Z$. This follows from the proof of [T3, Proposition 4.7], which also works equivariantly.} In particular, the under-category Emb$_Z$ is filtered.
Given any map $Z \hookrightarrow X$ for a smooth $G$-scheme $X$, corresponding to $(V \subset X, \pi : V \to Z)$, we can define $E(Z) := E(X, X - V)$. This generalizes the definition in Definition 3.1, which is the case $V = Z$. Given a further map of smooth quasi-projective $G$-schemes $X \hookrightarrow X'$, corresponding to $(V' \subset X', \pi' : V' \to X)$, the composition $Z \hookrightarrow X'$ corresponds to $(\pi')^{-1}(V) \to Z$. We have canonical isomorphisms

$$E(X, X - V) \xrightarrow{(2)} E(V', V' - (\pi')^{-1}(V)) \xleftarrow{(1)} E(X', X' - (\pi')^{-1}(V)).$$

It follows from this and the fact that the under-category of maps $Z \hookrightarrow X$ to a smooth $X$ is filtered that $E(Z)$ defined as $E(X, X - V)$ is canonically independent of the smooth embedding $Z \hookrightarrow X$.

Given a closed immersion $i : Z_0 \hookrightarrow Z_1$, we can choose an embedding in a smooth quasi-projective $G$-scheme $Z_1 \hookrightarrow X$ and regard the restriction map $E(X, X - Z_0) \to E(X, X - Z_1)$ as a pushforward functor $i_* : E(Z_0) \to E(Z_1)$. If $E$ is an equivariant BM-type homology theory, then $i_*$ is independent of the choice of embedding $Z_1 \hookrightarrow X$ in the sense that if $X \hookrightarrow X'$ is a further embedding in a smooth quasi-projective $G$-scheme, the two definitions of $i_*$ are intertwined by the equivalences $E(X', X' - Z_1) \xrightarrow{\sim} E(X, X - Z_1)$.

**Lemma 3.3.** Let $E$ be a BM-type homology theory, and let $i : Z_0 \hookrightarrow Z_1$ be a closed immersion of quasi-projective $G$-schemes. Then there is a fiber sequence

$$E(Z_0) \xrightarrow{i_*} E(Z_1) \xrightarrow{j^*} E(Z_1 - Z_0).$$

**Proof.** This follows formally from the definition of the pushforward functor and the observation that the pair $(X - Z_0, X - Z_1)$ can be used to define $E((Z_1 - Z_0))$. □

**Remark 3.4.** Despite the notation $E(Z)$ we will always make use of explicit smooth embeddings $Z \hookrightarrow X$ when we discuss the functoriality of the construction of $E(Z)$.

**Proposition 3.5.** Both of the presheaves of spectra $E : \text{Sm}_{G}^{op} \to \text{Sp}$ defined in (2) are equivariant BM-type homology theories.

The proof amounts to the following two lemmas. Property (1) of Definition 3.1 follows from the fact that $E(X, X - Z)$ only depends on $Z$ for an equivariant embedding in a smooth $G$-scheme $Z \hookrightarrow X$, which follows from:

**Lemma 3.6.** Let $Z \hookrightarrow X$ be a closed immersion from a $G$-scheme into a smooth $G$-scheme. Then

for $E(-) = K_{M}((-)^{an})$, $E(X, X - Z) \simeq K_{M}^{c,\vee}(Z^{an})$, and

for $E(-) = K^{top}(\text{Perf}(-/G))$, $E(X, X - Z) \simeq K^{top}(D^{b}\text{Coh}(Z/G))$.

**Proof.** The key feature of $K$-homology with locally compact supports is a version of Poincaré duality for smooth $G$-schemes $X$: $K_{M}^{c,\vee}(X^{an}) \simeq K_{M}(X^{an})$. More generally for any closed $G$-embedding $i : Z \hookrightarrow X$ where $X$ is smooth, we have a fiber sequence [15, Section 5]

$$K_{M}^{c,\vee}(Z^{an}) \xrightarrow{i_*} K_{M}(X^{an}) \xrightarrow{j^*} K_{M}(X^{an} - Z^{an}),$$

which is a version of Alexander duality, hence the first claim.

For the second claim, we can consider more generally an algebraic stack with closed substack $Z \subset X$. Recall that we have an exact triangle

$$K(D^{b}\text{Coh}(X)_{Z}) \to K(D^{b}\text{Coh}(X)) \to K(D^{b}\text{Coh}(X - Z)).$$

Furthermore, $D^{b}\text{Coh}(X)_{Z} \simeq \text{hocolim}_{\varepsilon} D^{b}\text{Coh}(Z')$, where the colimit is taken with respect to push forward along all infinitesimal thickenings of $Z$ in $X$ [16, Section 7.4]. Because $K$ commutes with
filtered colimits, it follows that pushforward induces an equivalence \( K(D^b\text{Coh}(\mathcal{Z})) \simeq K(D^b\text{Coh}(\mathcal{X})) \).

For any smooth affine scheme, \( T \), we have a canonical equivalence \( D^b\text{Coh}(\mathcal{X}) \otimes_{\mathcal{O}_T} \simeq D^b\text{Coh}(\mathcal{X} \times T) \), because \( D^b\text{Coh}(\mathcal{X}) \otimes D^b\text{Coh}(\mathcal{Y}) \simeq D^b\text{Coh}(\mathcal{X} \times \mathcal{Y}) \) for all QCA stacks, and \( \text{Perf}(T) = D^b\text{Coh}(T) \). It follows that we have a level-wise fiber sequence of presheaves of spectra on the category of smooth affine schemes

\[
K(D^b\text{Coh}(\mathcal{Z}) \otimes \mathcal{O}_T) \rightarrow K(D^b\text{Coh}(\mathcal{X}) \otimes \mathcal{O}_T) \rightarrow K(D^b\text{Coh}(\mathcal{X} - \mathcal{Z}) \otimes \mathcal{O}_T)
\]

and thus we have a fiber sequence on their geometric realizations (geometric realization over the category of smooth affine schemes agrees with geometric realization over the category of all affine schemes by [B3, Proposition 3.22]). After inverting the Bott element this leads to a fiber sequence

\[
K^{\text{top}}(D^b\text{Coh}(\mathcal{Z})) \xrightarrow{i_\ast} K^{\text{top}}(D^b\text{Coh}(\mathcal{X})) \xrightarrow{j_\ast} K^{\text{top}}(D^b\text{Coh}(\mathcal{X} - \mathcal{Z}))
\]

and the second claim follows.

\[\square\]

Property (2) of Definition 3.1 follows from:

Lemma 3.7. Let \((X, U) \in \text{Pairs}_G\), and let \( \mathcal{V} \) be a \( G\)-equivariant locally free sheaf of rank \( n \). Then for the BM-type homology theory associated to either of (2), we have:

1. If \( \pi : \mathcal{P}(\mathcal{V}) \rightarrow X \) is the projection, then the pullback map \( E(X, U) \rightarrow E(\mathcal{P}(\mathcal{V}), \pi^{-1}(U)) \) followed by \((-) \otimes \mathcal{O}_{\mathcal{P}(\mathcal{V})}(k)\) is a split injection.

2. The previous maps, where \( k \) ranges from 0, \ldots, \( n - 1 \), define a canonical equivalence \( E(\mathcal{P}(\mathcal{V}), \pi^{-1}(U)) \simeq E(X, U)^{\oplus n} \).

3. If \( \pi : \mathcal{V} \rightarrow X \) is a \( G\)-equivariant torsor for \( \mathcal{V} \), then \( \pi^* : E(X, U) \rightarrow E(\mathcal{V}, \pi^{-1}(U)) \) is an equivalence.

Proof. (1) and (2) are classical for Atiyah-Segal equivariant \( K\)-theory – See [S2] for these statements for \( K\)-cohomology with compact supports. For the presheaf \( E(-) = K^{\text{top}}(\text{Perf}(-/G)) \), we have a semiorthogonal decomposition

\[
\text{Perf}(\mathcal{P}(\mathcal{V})/G) = \langle \text{Perf}(X/G), \text{Perf}(X/G) \otimes \mathcal{O}(1), \ldots, \text{Perf}(X/G) \otimes \mathcal{O}(n - 1) \rangle,
\]

where each semiorthogonal factor is the essential image of \( \pi^* \) twisted by a power of the Serre bundle. It follows that any additive invariant of dg-categories applied to \( \text{Perf}(\mathcal{P}(\mathcal{V})/G) \) splits as a direct sum. Furthermore, the \( \pi^{-1}(U) = \mathcal{P}(\mathcal{V}|_U) \), and the restriction functor to respects the respective splittings, so we also have the desired splitting for \( E(\mathcal{P}(\mathcal{V}), \pi^{-1}(U)) \).

In order to prove (3), we note that Thomason’s proof when \( E(-) \) is algebraic \( K\)-theory in [T4, Theorem 4.1] works for any additive dg-invariant, as well as for Atiyah-Segal equivariant \( K\)-theory. He constructs\(^{14}\) a surjection of equivariant locally free sheaves \( \mathcal{W} \rightarrow \mathcal{F} \) such that the complement of the embedding \( \mathcal{P}(\mathcal{F}) \hookrightarrow \mathcal{P}(\mathcal{W}) \) is isomorphic to \( V \) over \( X \).

One can use the direct sum decomposition of \( E(\mathcal{P}(\mathcal{W})) \) and \( E(\mathcal{P}(\mathcal{F})) \) from (2) as in [T4] to show that \( E(i_\ast) : E(\mathcal{P}(\mathcal{F})) \rightarrow E(\mathcal{P}(\mathcal{W})) \) is a split injection, and to identify the cofiber with \( E(X) \). It follows from the localization sequence that the cofiber of \( E(i_\ast) \) can be canonically identified with \( E(\mathcal{P}(\mathcal{W}) - \mathcal{P}(\mathcal{F})) \), so we have our equivalence \( E(X) \simeq E(V) \). The claim (3) follows again from the observation that \( \mathcal{V}|_U \) is a torsor for \( \mathcal{V}|_U \).

\[\square\]

Remark 3.8. Using the decomposition for \( E(\mathcal{P}(\mathcal{V})) \) in Lemma 3.7 one can construct as in [T5, Proposition 5.8] a pushforward \( f_* : E(Y) \rightarrow E(Z) \) for a proper \( G\)-equivariant maps \( Y \rightarrow Z \) by factoring such a map an equivariant closed immersion followed by a projection, \( Y \hookrightarrow \mathbb{P}^n \times Z \rightarrow Z \), for some linear action of \( G \) on \( \mathbb{P}^n \). This construction is independent of the choices involved and leads

\(^{14}\)His construction is fppf local and thus works for arbitrary stacks.
to a functor from $G$-quasi-projective schemes to the homotopy category $\text{Ho}(\text{Sp})$ which is covariantly functorial for proper maps and contravariantly functorial for smooth maps. This would lead to a theory satisfying more of the usual axioms for a Borel-Moore homology theory [LM], but with values in $\text{Ho}(\text{Sp})$ rather than graded abelian groups.

Let $H \subset G$ be a reductive subgroup which is equivalent to the complexification of $H_c := M \cap H$. Note that there is a functor $\text{Pairs}_H \to \text{Pairs}_G$ given by $(X,U) \mapsto (G \times_H X, G \times_H U)$. Thus we can define a restriction of groups functor from presheaves of spectra on $\text{Pairs}_G$ to presheaves of spectra on $\text{Pairs}_H$.

**Theorem 3.9.** There is an equivalence of equivariant BM-type homology theories associated to $K^{\text{top}}(\text{Perf}(\neg/G))$ and $K_M((\neg)^{an})$. In particular for any $G$-quasi-projective scheme $X$ we have a canonical equivalence of spectra

$$\rho_{G,X} : K^{\text{top}}(D^b \text{Coh}(X/G)) \xrightarrow{\cong} K_M^{c,v}(X^{an}),$$

whose formation commutes up to homotopy with pushforward along closed immersions, restriction to open $G$-subschemes, and restriction to reductive subgroups $H \subset G$ such that $H$ is the complexification of $H_c := H \cap M$.

We will denote the internal function spectrum in the homotopy category of spectra as $\text{Hom}_{\text{Sp}}(\bullet, \bullet)$.

**Lemma 3.10.** For any space $Y$ and $M$-space $X$, we have a natural isomorphism in $\text{Ho}(\text{Sp})$,

$$\text{Hom}_{\text{Sp}}(\Sigma^\infty Y_+, K_M(X)) \cong K_M(X \times Y),$$

where on the right $Y$ is regarded as an $M$-space with trivial $M$ action.

**Proof.** We fix a universe $U$ for forming the equivariant stable homotopy category $\text{Sp}_M$ as in [LSM]. The “change of universe” functor taking an $E \in \text{Sp}_M$ to its underlying naive $M$-spectrum admits a left adjoint, as does the functor from naive $M$-spectra to spectra which applies the $M$-fixed point functor level-wise. We will denote the composition of these to functors as $(-)^M : \text{Sp}_M \to \text{Sp}$, and it therefore has a left adjoint, which we denote $\iota$. By definition we have that

$$K_M(X) := \left(\text{Hom}_{\text{Sp}_M}(\Sigma^\infty_U X_+, bu_M)\right)^M$$

where $bu_M$ is the $M$-spectrum representing equivariant $K$-theory, $\Sigma^\infty_U$ is the stabilization functor from pointed $M$-spaces to $M$-spectra, and $\text{Hom}_{\text{Sp}_M}$ is the internal function spectrum in the symmetric monoidal category of $M$-spectra [LSM, page 72]. Thus by the (spectrally enhanced) adjunction and the definition of inner Hom in a symmetric monoidal category we have

$$\text{Hom}_{\text{Sp}}(\Sigma^\infty Y_+, K_M(X)) \cong \left(\text{Hom}_{\text{Sp}_M}(\iota(\Sigma^\infty Y_+) \land \Sigma^\infty_U(X_+), bu_M)\right)^M$$

The claim now follows from the natural isomorphism $\iota(\Sigma^\infty Y_+) \cong \Sigma^\infty_U(Y_+)$ [LSM, Remark II.3.14(i)], where $Y$ is regarded as an $M$-space with trivial $M$ action, the fact that $\Sigma^\infty_U$ maps smash products of pointed $M$-spaces to smash products of $M$-spectra [LSM, Remark II.3.14(iii)], and the fact that $Y_+ \land X_+ \cong (Y \times X)_+$ for pointed $M$-spaces.

**Lemma 3.11.** If $M \subset U(n)$ is an embedding of Lie groups, then the canonical restriction map

$$K_{U(n)}(\text{GL}_n \times_G X) \to K_{U(n)}(U(n) \times_M X) \cong K_M(X)$$

is an equivalence of spectra.

**Proof.** We claim that the map $i : U(n) \times_M X \to \text{GL}_n \times_G X$ is a $U(n)$-equivariant homotopy equivalence. To see this, we factor $i$ as
We have the global Cartan decomposition
\[ j : U(n) \times_M X \longrightarrow GL_n \times_M X = GL_n \times_G (G/M \times X) \]
\[ \pi \]
\[ GL_n \times_G X \]
We have the global Cartan decomposition
\[ U(n) \times p_{gl_n} \overset{\sim}{\rightarrow} GL_n \]
\[ (u, p) \rightarrow u \cdot \exp(p) \]
and an analogous decomposition for \( G \) which is compatible with the inclusion of \( G \subset GL_n \). Furthermore, the Lie sub-algebras \( p_{gl_n} \) and \( p_g \) are \( \text{Ad} \)-invariant under the action of \( U(n) \) and \( M \) respectively, which implies that the decomposition is invariant under conjugation. The Cartan decomposition allows us to \( U(n) \)-equivariantly retract \( GL_n \) onto \( U(n) \) by scalar multiplication in \( p_{gl_n} \). This retraction respects the right action by \( M \) because \( (u \cdot \exp(tp)) \cdot m^{-1} = um \cdot \exp(\text{Ad}(m) \cdot tp) \).

Note that this computation also implies that the action of any compact subgroup \( K \) of \( M \) is linear on \( G/M \). We may therefore extend this retraction to obtain an equivariant retraction of \( GL_n \times_M X \) onto \( U(n) \times_M X \).

To conclude that the map \( \pi \) is a \( U(n) \)-equivariant homotopy equivalence, it suffices to check that it induces an ordinary homotopy equivalence on fixed point spaces \( \pi^K : (GL_n \times_G (G/M \times X))^K \rightarrow (GL_n \times_G X)^K \) for any closed subgroup \( K \subset U(n) \). The fibers of the map \( \pi \) are all isomorphic to \( G/M \). A general fixed point in \( (GL_n \times_G X)^K \) takes the form \( (a, x) \) with \( a^{-1}Ka = \): \( K' \subset G \) and \( x \) fixed by \( K' \). As observed above, if \( K' \subset M \), then the action of \( K' \) on \( G/M \) is linear, and so the fixed points \( (G/M)^{K'} \) are linear subspaces and hence contractible. In general, we have that \( g^{-1}(a^{-1}Ka)g \subset M \) because any compact subgroup of \( G \) is conjugate to a closed subgroup of \( M \) by an element \( g \in G \). It follows that the action of this subgroup is linear on the fiber \( G/M \) as well after changing coordinates under the diffeomorphism \( g^{-1} : G/M \rightarrow G/M \). We therefore obtain that the bundles \( \pi^K \) are locally-trivial with contractible fibers so it follows that this is a homotopy equivalence as required.

\[ \square \]

**Proof of Theorem 3.9. Construction of the comparison map:**

In order to construct a comparison natural transformation for the presheaves on \( \text{Pairs}_G \), it suffices to construct a natural transformation of presheaves \( \rho_G : K_{\text{top}}(\text{Perf}(-/G)) \rightarrow K_M((-)^{an}) \).

This comparison map will respect Bott periodicity, i.e. it will respect the \( S((\beta)) \)-linear structure, where \( S \) denotes the sphere spectrum. Thus defining our comparison map is equivalent to giving a \( S[\beta] \)-linear natural transformation \( \rho_G : K^{st}(\text{Perf}(-/G)) \rightarrow K_M((-)^{an}) \).

By the adjunction defining the geometric realization [B3, Definition 3.13] and hence \( K^{st} \), constructing a comparison map is equivalent to defining a map of presheaves
\[ K(\text{Perf}(X/G \times T)) \rightarrow \text{Hom}_{Sp}(\Sigma^\infty T_+^{an}, K_M(X^{an})) , \]
where both sides are regarded simultaneously as presheaves in the smooth \( G \)-scheme \( X \) and the affine scheme \( T \). Here we have used the natural Morita equivalence \( \text{Perf}(X/G) \otimes_C \Omega_T \simeq \text{Perf}(X/G \times T) \).

Observe that we have a natural transformation of presheaves of spectra
\[ K(\text{Perf}(X/G \times T)) \rightarrow K_M(X^{an} \times T^{an}) \]
which is induced by the functor that sends an algebraic \( G \)-vector bundle to its underlying complex topological vector bundle equipped with the induced action of \( M \) (This functor is symmetric monoidal, and hence induces a map of \( K \)-theory spectra [T5, Section 5.4]). By the lemma above, the presheaf \( K_M(X^{an} \times (-)^{an}) \) is equivalent to \( \text{Hom}_{Sp}(\Sigma^\infty(-)^{an}, K_M(X^{an})) \), so we have our map
The resulting natural transformation $\rho_{G,X} : K^{st}(\text{Perf}(X/G)) \to K_M(X^{an})$ will be $bu$-linear by construction and hence $S[[\beta]]$-linear.

The resulting comparison map for the corresponding BM-type homology theories will automatically be compatible the pushforward along closed immersions and restriction to open subsets, as it is a natural transformation of presheaves of spectra on $\text{Pairs}_G$. The fact that the formation of $\rho_{G,X}$ is compatible with restriction to a reductive subgroup $H \subset G$ follows from the the fact that the natural analytification map $K(\text{Perf}(X/G)) \to K_M(X^{an})$ commutes with restriction to subgroups.

**Verification that $\rho_{G,X}$ is an equivalence:**

Again it suffices to show that the comparison map $\rho_{G,X} : K^{\text{top}}(\text{Perf}(X/G)) \to K_M(X^{an})$ is an equivalence for smooth quasi-projective $X$, and the fact that the comparison map for pairs is an equivalence follows formally. We can choose an embedding in a unitary group $M \hookrightarrow U_n$ for some $n$. The inclusion $\{\text{id}\} \times X \hookrightarrow GL_n \times_G X$ is equivariant with respect to the embedding $G \subset GL_n$ and induces an equivalence of quotient stacks, so the canonical restriction functor

$$K^{\text{top}}(\text{Perf}(GL_n \times_G X/ GL_n)) \to K^{\text{top}}(\text{Perf}(X/G))$$

is an equivalence of spectra. Likewise on the topological side, the restriction map is equivariant with respect to the embedding $M \hookrightarrow U_n$, and the corresponding map

$$K_{U(n)}(GL_n \times_G X) \to K_M(X)$$

is an equivalence of spectra by Lemma 3.11. Furthermore, the comparison map induced by the symmetric monoidal functor assigning an algebraic vector bundle to its underlying topological bundle commutes with these restriction functors. Therefore it suffices to prove the claim when $G = GL_n$, which we assume for the remainder of the proof.\(^{15}\)

We fix a maximal torus and Borel subgroup $T \subset B \subset G$ such that $T$ is the complexification of $T_c := T \cap M$. Our first goal is to show that for either of the theories $(2)$, $E_G(X) \to E_T(X)$ is a split injection, and both the restriction map and its splitting commute with $\rho_X$. The morphism $X/B \to X/G$ is an fppf-locally trivial fiber bundle with fiber the complete flag variety $G/B$. It follows that $\text{Perf}(X/B)$ admits a semiorthogonal decomposition and that $K^{\text{top}}(\text{Perf}(X/B))$ admits a decomposition analogous to part $(2)$ of Lemma 3.7, where each factor is equivalent to $K^{\text{top}}(\text{Perf}(X/G))$, and they are the essential images of the fully faithful pullback functor twisted by representations of $B$ corresponding to the Kapranov full exceptional collection $[K2]$.

Furthermore $\text{Perf}(X/T) \to \text{Perf}(X/B)$ is an fppf-locally trivial bundle whose fiber is the affine space $B/T$. Realizing $B$ as a sequence of extensions by $G_a$-torsors, we have that the restriction functor $K^{\text{top}}(\text{Perf}(X/B)) \to K^{\text{top}}(\text{Perf}(X/T))$ is an equivalence by part $(3)$ of Lemma 3.7. The upshot is that $K^{\text{top}}(\text{Perf}(X/T))$ admits a direct sum decomposition where each factor is identified with $K^{\text{top}}(\text{Perf}(X/G))$ by the (fully-faithful) pullback functor followed by tensor product with the character of $T$ inducing the various vector bundles on $G/B$ forming the Kapranov full exceptional collection.

It is a classical fact that the pullback functor $K_M(X^{an}) \to K_{T_c}(X^{an})$ is a split injection, identifying $K_M(X^{an})$ with the $W$-invariant piece of $K_{T_c}(X^{an})$, where $W$ is the Weyl group of $T_c$. Furthermore $K_M(X^{an})$ admits a direct sum decomposition of the same form, whose factors are the essential image of the pullback functor tensored with characters of $T_c$ corresponding to the Kapranov collection. It follows that the comparison map $\rho_T : K^{\text{top}}(\text{Perf}(X/T)) \to K_{T_c}(X^{an})$ respects this direct sum decomposition, and hence it suffices to prove that the comparison map is an equivalence in this case, where $G = T$.

\(^{15}\)Here one encounters a minor difference between our proof and the proof of [T5, Theorem 5.9]. Thomason’s proof gave a different reduction to the case of a torus which required the construction of proper pushforward maps $f_* : E(Y) \to E(X)$ and relied on the fact that for a rational map $f_* f^* \simeq \text{id}_{E(X)}$. We felt that the proof here was simpler in our context.
We can stratify $X/T$ by smooth $T$-schemes of the form $U \times (T/T')$, where $T' \subset T$ is an algebraic subgroup and $T$ acts trivially on $U$.\[\text{Using Lemma 3.3, it thus suffices to prove the claim for schemes of this form.}\]

The fact that $\text{Perf}(U \times (T/T'))/T \simeq \text{Perf}(U \times BT') \simeq \bigoplus_{\chi} \text{Perf}(U)$, where $\chi$ ranges over the group of characters of the diagonalizable group $T'$, implies that $K^{top}(\text{Perf}(U \times (T/T'))/T) \simeq \bigoplus_{\chi} K^{top}(\text{Perf}(U))$. There is an analogous decomposition of $K_{T_c}(U \times T/T')$, and $\rho_{T,U \times T/T'}$ respects this direct sum decomposition because the summands are the essential image of pullback along the map $U \times T/T' \to U$ followed by tensoring with the various characters of $T'$. We note that when the group is trivial, our comparison map agrees with the one constructed in [B3, Proposition 4.32], therefore it is an equivalence, and the claim follows. \[\square\]

**Remark 3.12.** If $G$ is not necessarily reductive, then one can choose a decomposition $G = U \times H$, where $H$ is reductive and $U$ is a connected unipotent group. As in the first step in the proof of [T5, Theorem 5.9], one shows that the map of stacks $X/H \to X/G$ can be factored as a sequence of torsors for vector bundles, so the canonical restriction map $K^{top}(\text{Perf}(X/G)) \to K^{top}(\text{Perf}(X/H))$ is an equivalence by Lemma 3.7. Combining this with the previous theorem shows that for a maximal compact subgroup $M \subset H \subset G$, the topologization functor is an equivalence $K^{top}(\text{Perf}(X/G)) \to K_{M}(X^{an})$ as presheaves of spectra on $\text{Sm}_{G}$, and we have a comparison isomorphism $\rho_{G,X} : K^{top}(\text{D}^{b}\text{Coh}(X/G)) \to K_{M}^{c,v}(X^{an})$.

**Remark 3.13.** A version of Theorem 3.9 holds for algebraic spaces with $G$-action, with the same proof, under the caveat that the proof of Lemma 3.2 does not apply. As a result, we have that for any smooth algebraic space over $\mathbb{C}$ with $G$-action, the canonical topologization map $\rho_{G,K} : K^{top}(\text{Perf}(X/G)) \to K_{M}(X^{an})$ is an equivalence of spectra. It follows from Lemma 3.6 (whose proof does not use quasi-projectivity) that for any algebraic $G$-space which admits an embedding into a smooth $G$-space $X \hookrightarrow Z$, one obtains an equivalence $\rho_{G,X/Z} : K^{top}(\text{D}^{b}\text{Coh}(X/G)) \to K_{M}^{c,v}(X^{an}),$ which could depend on the embedding a priori.

**Remark 3.14.** The comparison isomorphism $\rho_{G,X}$ is compatible with the canonical direct sum decomposition $E(Z) \simeq E(X,U)^{G}_{an}$ of Lemma 3.7 by construction. It follows that for a proper equivariant map of $G$-quasi-projective schemes $f : Y \to Z$, the pushforward $f_{*} : E(Y) \to E(Z)$ described in Remark 3.8 commutes up to homotopy with the equivalence $\rho_{G,X}$ as well.

### 3.2. The case of smooth Deligne-Mumford stacks.

Here we provide an explicit computation of the periodic cyclic homology of $\text{Perf}(\mathcal{X})$ for a smooth Deligne-Mumford stack of finite type over $\mathbb{C}$ and study its noncommutative Hodge theory when it is proper. The results of this section are likely known to experts.

Given a smooth scheme $U$, we can consider its de Rham complex, $0 \to \mathcal{O}_{U} \to \Omega_{U}^{1} \to \cdots$, a complex of vector spaces. We can regard this as a $\Lambda$-module $\Omega_{U}$ by defining $\Omega_{p}(U) := \Omega_{U}^{-p}$ and letting $B$ act via the de Rham differential. Even though the $\Lambda$-module structure is not $\mathcal{O}_{U}$-linear, it still defines a sheaf of $\Lambda$-modules on the small site $\mathcal{X}_{et}$ for any smooth DM stack $\mathcal{X}$. We define the de Rham cohomology of a smooth Deligne-Mumford stack $\mathcal{X}$ to be the $\Lambda$-module $H_{dR}(\mathcal{X}) := R\Gamma(\mathcal{X}_{et}, \Omega_{*}).$

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\[16\]When $X$ is an algebraic space as in Remark 3.13, one can stratify by normal $G$-schemes, then refine this to a stratification by quasi-projective $G$-schemes by Sumihiro’s theorem, then reduce to the claim for projective space.

\[17\]For a smooth $G$-scheme $X$, the comparison map $\rho_{G,X}$ agrees with the comparison map of pairs under the identification $E(X) = E(X, \emptyset) \simeq E(X', X' - X)$ for any closed $G$-embedding in a smooth $G$-scheme $X \hookrightarrow X'$.
There are several other ways to present $H_{dR}(\mathcal{X})$. First note that we can equivalently restrict to
the sub-site of étale maps $U \to \mathcal{X}$ for which $U$ is affine, which we denote $\mathcal{X}^{\text{aff}}$, because it has an
equivalent topos of sheaves, i.e. the canonical map is an equivalence

$$H_{dR}(\mathcal{X}) \xrightarrow{\sim} R\Gamma(\mathcal{X}^{\text{aff}}, \Omega_\bullet).$$

We can consider the sheaf of $\Lambda$-modules on $\mathcal{X}$. Vertical arrows are closed immersions:

Here $\Gamma$ denotes the graph of the morphism $U \to X$. Then by definition $I_X$ is the derived self
intersection of the closed subspace $X \to X \times X$, so in order to prove the claim it will suffice to show
that $\Gamma \times_{U \times X} \Gamma$ is isomorphic to $I_U$ as a derived scheme over $U$.

**Lemma 3.15.** The canonical maps

$$R\Gamma(\mathcal{X}^{\text{aff}}, \Omega_\bullet) \leftarrow R\Gamma(\mathcal{X}^{\text{aff}}, C_\bullet(\mathcal{O}_\bullet)) \to R\Gamma(\mathcal{X}^{\text{aff}}, C_\bullet(\text{Perf}(\mathcal{O}_\bullet)))$$

are all equivalences of $\Lambda$-modules.

**Proof.** These maps are all equivalences for affine $U$ at the level of underlying complexes. The result
follows formally from the fact that a map of $\Lambda$-modules is an equivalence if and only if the underlying
map of complexes is an equivalence, and the forgetful functor taking a $\Lambda$-module to its underlying
complex commutes with limits, hence commutes with $R\Gamma$. □

The following is due to Toen, and essentially follows the argument of [T6] in the case of algebraic
$K$-theory. We will need to use both the derived inertia stack $I_X$ and its underlying classical stack
$I_X^{\text{cl}} \subset I_X$.

**Proposition 3.16** (Toen, unpublished). Let $\mathcal{X}$ be a smooth Deligne-Mumford stack, and let $I_X^{\text{cl}}$
denote its classical inertia stack. There is a natural isomorphism of $\Lambda$-modules $C_\bullet(\text{Perf}(\mathcal{X})) \to
H_{dR}(I_X^{\text{cl}})$.

The idea of the proof is to show that the formation of both complexes is local in the étale topology
over the coarse moduli space of $\mathcal{X}$, so one can reduce to the case of a global quotient. Thus a key
observation is that the formation of the derived inertia stack $I_X$ is étale local.

**Lemma 3.17.** Let $\mathcal{X} \to X$ be a map from a stack to a separated algebraic space, and let $\mathcal{U} \to U$ be
the base change along an étale map $U \to X$. Then $I_U \simeq I_X \times_X U$.

**Proof.** This can be seen for the derived inertia stack from a functor-of-points definition of $I_U$. We
let $\mathcal{U}(T)$ denote the $\infty$-groupoid of maps from $T$ to $\mathcal{U}$ for a derived affine scheme $T$.

$$I_U(T) = \mathcal{U}(T) \times_{\mathcal{U}(T) \times \mathcal{U}(T)} \mathcal{U}(T)$$

$$\sim \text{Map}(S^1, \mathcal{U}(T))$$

$$\sim \text{Map}(S^1, \mathcal{X}(T)) \times_{\text{Map}(S^1, \mathcal{X}(T))} \text{Map}(S^1, U(T))$$

So in order to show that $I_U(T) \simeq I_X(T) \times_X U(T)$, it will suffice to show that $I_U \simeq I_X \times_X U$
in the derived sense. Consider the following diagram, in which each square is Cartesian and the
vertical arrows are closed immersions:

$$\begin{array}{ccc}
U & \xrightarrow{\sim} & U \times_X U \\
\downarrow & & \downarrow \\
U \times U & \xrightarrow{\sim} & U \times X \\
\downarrow & & \downarrow \\
U \times U & \xrightarrow{\sim} & X \times X
\end{array}$$

$$\Gamma \sim \text{Graph}(U \to X)$$

Here $\Gamma$ denotes the graph of the morphism $U \to X$. Then by definition $I_X$ is the derived self
intersection of the closed subspace $X \to X \times X$, so in order to prove the claim it will suffice to show
that $\Gamma \times_{U \times X} \Gamma$ is isomorphic to $I_U$ as a derived scheme over $U$. 29
The map $U \to \Gamma$ is an isomorphism on underlying classical algebraic spaces, and it follows from the fact that $U \to U \times_X U$ is an étale closed immersion of closed substacks of $U \times U$ that the induced map $I_U \to \Gamma \times_{U \times X} \Gamma$ induces an isomorphism on cotangent complexes as well, hence it is an isomorphism.

Proof of Proposition 3.16. The pullback functor along the projection $I_X^d \to \mathcal{X}$ induces a map $C_\bullet(\text{Perf}(\mathcal{X})) \to C_\bullet(\text{Perf}(I_X^d))$. For any étale $U/I_X^d$, the pullback functor induces a natural map $C_\bullet(\text{Perf}(I_X^d)) \to C_\bullet(\text{Perf}(U))$. Thus we get a map of presheaves of $\Lambda$-modules

$$C_\bullet(\text{Perf}(\mathcal{X})) \to R\Gamma((I_X^d)^{\text{aff}}, C_\bullet(\text{Perf}(-))) \simeq H_{\text{dR}}(I_X^d).$$

Note that if $p : \mathcal{X} \to X$ is the coarse moduli space of $\mathcal{X}$, then the map constructed above is functorial with respect to pullback along maps $U \to X$.

We claim that $C_\bullet(\text{Perf}(\mathcal{X}))$, regarded as a presheaf over $X$, has étale descent. Indeed, consider any étale map $U \to X$, and let $\mathcal{U} = \mathcal{X} \times_X U$. Because the derived category of $\mathcal{U}$ is compactly generated [HR], we can identify

$$C_\bullet(\text{Perf}(\mathcal{U})) \simeq R\Gamma(\mathcal{U}, \Delta^* \Delta_* (\mathcal{O}_{\mathcal{U}})) \simeq R\Gamma(U, (p_U)_* \mathcal{O}_{I_U}),$$

where $\Delta : U \to U \times U$ is the diagonal, $p_U : U \to U$ is the base change of $p$, and $\mathcal{O}_{I_U}$ is the structure sheaf of the derived inertia stack, regarded as a finite algebra over $\mathcal{O}_U$. In the previous lemma, we saw that the formation of $I_U$ commutes with étale base change, so this combined with the projection formula implies that $R\Gamma(U, (p_U)_* \mathcal{O}_{I_U}) \simeq R\Gamma(U, (p_U)_*(\mathcal{O}_I)_{|U})$, functorially in $U$. The presheaf $U/X \mapsto R\Gamma(U, (p_U)_*(\mathcal{O}_{I_X})_{|U})$ has étale descent, so $U \mapsto C_\bullet(\text{Perf}(\mathcal{U}))$ does as well.

Thus in order to show that $C_\bullet(\text{Perf}(\mathcal{X})) \to H_{\text{dR}}(I_X^d)$ is an equivalence, it suffices to verify this after base change to an étale cover of $X$. We can find such a $U \to X$ such that $\mathcal{U} = \mathcal{X} \times_X U$ is a global quotient of a scheme by a finite group action. In that case, the result is shown in [B1, Proposition 4].

Finally after applying the Tate construction, i.e. passing to periodic cyclic homology, we can compare this to the cohomology of $|\mathcal{X}^\text{an}|$, the geometric realization of the underlying topological stack (in the analytic topology) associated to $\mathcal{X}$ [N], as well as the cohomology of a coarse moduli space $X \to \mathcal{X}$.

Lemma 3.18. Let $\mathcal{X}$ be a Noetherian separated DM stack of finite type over a Noetherian base scheme. Assume that $X$ has finite dimension. Then $\mathcal{X}$ has finite étale cohomological dimension with $\mathbb{Q}$-linear coefficients, and the functor $R\Gamma(\mathcal{X}_{\text{et}}, -)$ commutes with filtered colimits.

Proof. We first claim that the pushforward along the projection to the coarse moduli space $p : \mathcal{X} \to X$ is exact. Indeed this can be checked étale locally on $X$, and so we may assume that $\mathcal{X}$ is a global quotient $U/G$, where $G$ is a finite group. One can factor $p$ as $U/G \to X \times BG \to X$ – pushforward along the first is exact by [S3, Tag 03QP], and the second is exact because we are using characteristic 0 coefficients.

It now suffices to prove the claim when $\mathcal{X} = X$ is a Noetherian separated algebraic space algebraic space of finite type over a Noetherian base scheme. In this case, we can apply the induction principle of [S3, Tag 08GP] and the fact that étale cohomology takes elementary excision squares to homotopy cartesian squares to reduce to the case of affine schemes. In this case, the result follows from the fact that derived global sections of characteristic 0 sheaves on a Noetherian scheme in the étale topology agrees with that in the Nisnevich topology, and the Nisnevich topology has cohomological dimension $\leq d$.

Finally, the implication that finite cohomological dimension implies commutation with filtered colimits in the unbounded derived category is [CD, Lemma 1.1.7].
Lemma 3.19. There are natural isomorphisms
\[ H_{dR}(X)^\text{Tate} \simeq C^*_\text{sing}(|\mathcal{X}^{an}|; \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}((u)) \simeq C^*_\text{sing}(X; \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}((u)) \]

Proof. The de Rham isomorphism gives a canonical isomorphism of pre-sheaves of \( \mathbb{C}((u)) \)-modules on \( \mathcal{X}^{\text{aff}}_\text{et} \) between \( U \mapsto (\Omega_*(U))^\text{Tate} \) and \( U \mapsto C^*_\text{sing}(U^{an}; \mathbb{C})((u)) \), so we have a canonical isomorphism\(^{18}\)
\[ C^*_\text{sing}(|\mathcal{Y}^{an}|; \mathbb{C}) \otimes \mathbb{C}((u)) \simeq R\Gamma(\mathcal{X}^{\text{aff}}_\text{et}, \Omega^*_\text{Tate}) \]

It therefore suffices to show that the Tate construction commutes with taking derived global sections for the sheaf of \( \Lambda \)-modules \( \Omega_* \). For this we observe that the functor \( M \mapsto M^{S^1} \) commutes with homotopy limits, and hence with derived global sections, and \( M^{\text{Tate}} \) is the filtered colimit of \( M^{S^1} \to M^{S^1}[2] \to M^{S^1}[4] \to \cdots \), so its formation commutes with \( R\Gamma \) by the previous lemma.

Finally, one can check that the pullback map \( C^*_\text{sing}(Y^{an}; \mathbb{Q}) \to C^*_\text{sing}(|\mathcal{Y}^{an}|; \mathbb{Q}) \) is an equivalence locally in the analytic topology on \( Y^{an} \). Locally \( \mathcal{Y}^{an} \) is isomorphic to a global quotient of a scheme by a finite group, for which the fact is well-known. \( \square \)

3.3. Equivariant \( K \)-theory and periodic cyclic homology. For a dg-category, \( \mathcal{C} \), it is natural to ask if the Chern character induces an equivalence \( K^\text{top}(\mathcal{C}) \otimes \mathbb{C} \to \mathcal{C}^\text{per}(-) \). This is referred to as the lattice conjecture in [B3], where it is conjectured to hold for all smooth and proper dg-categories. Here we observe some situations in which the lattice conjecture holds, even for categories which are not smooth and proper.

Theorem 3.20 (Lattice conjecture for smooth quotient stacks). Let \( G \) be an algebraic group acting on a smooth quasi-projective scheme \( X \). If \( X/G \) admits a semi-complete KN stratification (Definition 2.1), then the Chern character induces an equivalence \( K^\text{top}(\text{Perf}(X/G)) \otimes \mathbb{C} \to \mathcal{C}^\text{per}(\text{Perf}(X/G)) \).

Lemma 3.21. Let \( \mathcal{X} \) be a smooth Deligne-Mumford stack, and let \( i : \mathcal{Z} \to \mathcal{X} \) be a smooth closed substack. Then the pushforward functor fits into a fiber sequence
\[ C^\text{per}(\text{Perf}(\mathcal{Z})) \overset{i_*}{\to} C^\text{per}(\text{Perf}(\mathcal{X})) \overset{j^*}{\to} C^\text{per}(\text{Perf}(\mathcal{X} - \mathcal{Z})). \]

Proof. This follows from Proposition 3.16, combined with Lemma 3.19 and the usual Gysin sequence for the regular embedding of inertia stacks \( I_Z \to I_X \). \( \square \)

Proof of Theorem 3.20. Because \( K^\text{top}(-) \otimes \mathbb{C} \) and \( \mathcal{C}^\text{per}(-) \) are both additive invariants, proving that the natural transformation
\[ K^\text{top}(\text{Perf}(X/G)) \otimes \mathbb{C} \to \mathcal{C}^\text{per}(\text{Perf}(X/G)) \]
is an equivalence for smooth projective-over-affine \( X \) and reductive \( G \) reduces to the case where \( X/G \) is Deligne-Mumford by Theorem 2.7.

Note that the only point in the proof of Lemma 3.7 which does not immediately apply to an arbitrary additive invariant is the localization sequence for a closed immersion. Therefore Lemma 3.21 implies that Lemma 3.7 applies to the presheaf \( \mathcal{C}^\text{per}(\text{Perf}(-)) \), because the only stacks that appear in the proof are DM.

We can now imitate the proof of Theorem 3.9: \( \text{Perf}(X/G) \) is a retract of \( \text{Perf}(X/B) \), and \( \text{Perf}(X/B) \to \text{Perf}(X/T) \) induces an equivalence for both invariants \( K^\text{top}(\bullet) \) and \( \mathcal{C}^\text{per}(\bullet) \), by Lemma 3.7. Thus it suffices to consider smooth DM stacks of the form \( X/T \). Any such stack admits a stratification by smooth stacks of the form \( U \times B \Gamma \) for some finite group \( \Gamma \), and by Lemma 3.21 it suffices to prove the theorem for such stacks. Thus \( \text{Ch} \otimes \mathbb{C} \) is an equivalence because it is an equivalence for smooth schemes and \( \text{Perf}(U \times B \Gamma) \simeq \bigoplus_{\chi} \text{Perf}(U) \), the sum ranging over characters of \( \Gamma \). \( \square \)

\(^{18}\)All of the singular complexes we will encounter have finite dimensional total cohomology, so \( M((u)) \simeq M \otimes_{\mathbb{C}} \mathbb{C}((u)). \)
3.4. Hodge structure on equivariant $K$-theory. We can now prove the final result of this paper, the construction of a pure Hodge structure on the equivariant $K$-theory. What we mean by a pure Hodge structure on a spectrum $E$ in this case is simply a Hodge structure on the homotopy groups of that spectrum $\pi_n(E)$, i.e. for each $n$ a weight $n$ Hodge structure on $\pi_n(E)$ is a descending filtration of $\pi_n(E) \otimes \mathbb{C}$ such that

$$\pi_n(E) \otimes \mathbb{C} = F^p\pi_n(E) \otimes \mathbb{C} \oplus \overline{F^{n+1-p}\pi_n(E)} \otimes \overline{\mathbb{C}}, \forall p$$

Theorem 3.22. Let $X$ be a smooth quasi-projective $\mathbb{C}$-scheme, let $M$ be a compact Lie group whose complexification $G$ acts on $X$. Then if $X/G$ admits a complete KN stratification, the Chern character isomorphism

$$K_M(X^{an}) \otimes \mathbb{C} \to C^\per_{\bullet}(\text{Perf}(X/G))$$

combined with the noncommutative Hodge-de Rham sequence induces a pure Hodge structure of weight $n$ on $K_M^n(X^{an})$ with a canonical isomorphism

$$\text{gr}_{\text{Hodge}}^p(K_M^n(X^{an}) \otimes \mathbb{C}) \simeq H^{n-2p} \Gamma(I_X, \mathcal{O}_{I_\chi}),$$

where $I_\chi$ denotes the derived inertia stack of $\chi := X/G$. The Hodge filtration on $K_M^n(X^{an})$ is compatible with pullback maps, and in particular it is a filtration of $\text{Rep}(M)$-modules.

Remark 3.23. As we will see in the proof, this claim also holds for arbitrary smooth and proper DM stacks over $\mathbb{C}$, without requiring that $X$ is a global quotient.

Proof. The degeneration property follows from Corollary 2.20, and we have a Chern character isomorphism from Theorem 3.9 combined with Theorem 3.20, so all we have to do is check that the filtration on $C^\per_{\bullet}(\text{Perf}(X/G))$ coming from the HdR spectral sequence combined with the rational structure coming from the Chern character defines a weight $n$ pure Hodge structure on $\pi_{-n}(K^{top}(\text{Perf}(X/G)) \otimes \mathbb{Q})$. This claim is closed under arbitrary direct sums and summands in $\mathcal{M}_k$, so by Theorem 2.7 it suffices to prove this claim for smooth and proper DM stacks.

For a smooth and proper DM stack, Proposition 3.16 gives an isomorphism of $\Lambda$-modules

$$C_{\bullet}(\text{Perf}(X)) \simeq H_{dR}(I_{X})$$

Note that $I_{X}$ is itself a smooth and proper DM stack, and for any smooth DM stack $\mathcal{Y}$ the complex $H_{dR}(\mathcal{Y})_{\text{Tate}}$ is canonically equivalent to the usual de Rham complex of $[S1]$ tensored with $\mathbb{C}((u))$,

$$\Gamma\bigl(\mathcal{Y}, [0 \to \mathcal{O}_{\mathcal{Y}} \to \Omega^1_{\mathcal{Y}} \to \cdots]\bigr) \otimes_{\mathbb{C}} \mathbb{C}((u)).$$

However, the usual Hodge filtration differs slightly from the noncommutative one. We have a canonical isomorphism

$$H^n(H_{dR}(\mathcal{Y})_{\text{Tate}}) \simeq \bigoplus_{l \equiv n \mod 2} H^l(\mathcal{Y}; \mathbb{C})$$

$$\text{gr}^p F_{\text{nc}}^i H^n(H_{dR}(\mathcal{Y})_{\text{Tate}}) \simeq \bigoplus_{i} \Gamma(\mathcal{Y}, \Omega^{i}_{\mathcal{Y}}[i-2p])$$

Because the cyclic complex $H_{dR}(\mathcal{Y})$ has the degeneration property, we may commute taking cohomology $H^n$ and taking associated graded $\text{gr}^p$, so we have

$$\text{gr}^p F_{\text{nc}}^i H^n(H_{dR}(\mathcal{Y})_{\text{Tate}}) \simeq \bigoplus_{i} H^{n+i-2p}(\mathcal{Y}, \Omega^{i}_{\mathcal{Y}})$$

Thus on each direct summand $H^l(\mathcal{Y}; \mathbb{C})$ of $H^n(H_{dR}(\mathcal{Y})_{\text{Tate}})$, the subquotient $H^{l-p'}(\mathcal{Y}, \Omega^{p''}_{\mathcal{Y}})$ shows up in $F_{\text{nc}}^p$ if and only if $l-p' = n + p - 2p''$ for some $p'' \geq p$. In other words the subquotients appearing are those for which $p' \geq p + \frac{t-n}{2}$. It follows that under the direct sum decomposition above we have

$$F_{\text{nc}}^p H^n(H_{dR}(\mathcal{Y})_{\text{Tate}}) \simeq \bigoplus_{l \equiv n \mod 2} F_{\text{classical}}^{p + \frac{t-n}{2}} H^l(\mathcal{Y}; \mathbb{C})$$
Thus under the isomorphism $H^n(H_{dR}(Y)_{\text{Tate}}) \simeq \bigoplus_{l \equiv n \mod 2} H^l(Y; \mathbb{Q}) \otimes \mathbb{C}$ of Lemma 3.19, the noncommutative Hodge filtration corresponds to the Hodge filtration on $\bigoplus_{l \equiv n \mod 2} H^l(Y; \mathbb{Q})(\frac{l-n}{2})$.

We claim that this rational structure on $H^n(H_{dR}(I^d_{cl})_{\text{Tate}})$ agrees with the one induced by the equivalence $K^\text{top}(\text{Perf}(X)) \otimes \mathbb{C} \simeq H_{dR}(I^d_{cl})_{\text{Tate}}$ of Theorem 3.20 and Lemma 3.19, so that we have an isomorphism of Hodge structures

$$\pi_{-n}(K^\text{top}(\text{Perf}(X))) \otimes \mathbb{Q} \simeq \bigoplus_{l \equiv n \mod 2} H^l(I^d_{cl}; \mathbb{Q})(\frac{l-n}{2}). \quad (4)$$

The Hodge structure on the $l^{th}$ rational cohomology of the de Rham complex of a smooth DM stack has weight $l$ (see [S4]), so it would follow that $\pi_{-n}(K^\text{top}(\text{Perf}(X))) \otimes \mathbb{Q}$ has a Hodge structure of weight $n$.

To prove the claim about the rational structure of $H_{dR}(I^d_{cl})$, note that the isomorphism $K^\text{top}(\text{Perf}(X)) \otimes \mathbb{C} \simeq H_{dR}(I^d_{cl})_{\text{Tate}}$ results from applying the derived global sections functor to isomorphic sheaves on the étale site of $I^d_{cl}$:

$$\text{RG}((I^d_{cl})_{\text{aff}}, K^\text{top}(\text{Perf}(\text{Sing}(\mathcal{M})) \otimes \mathbb{C}) \simeq \text{RG}((I^d_{cl})_{\text{et}}, C^\text{per}^\bullet(\text{Perf}(\text{Sing}(\mathcal{M}))))$$

$$\simeq \text{RG}((I^d_{cl})_{\text{et}}, C^\text{sing}^\bullet((-)^{\text{an}}; \mathbb{C})(\mathcal{M})).$$

But according to [B3, Proposition 4.32], the noncommutative Chern character for smooth $\mathbb{C}$-schemes factors through the twisted Chern character under the natural equivalence $C^\text{per}^\bullet(\text{Perf}(X)) \simeq H_{dR}(X)_{\text{Tate}} \simeq C^\text{sing}^\bullet(X; \mathbb{C})(\mathcal{M})$. It follows that the isomorphism above is the complexification of a map of presheaves of $\mathbb{C}$-complexes on $(I^d_{cl})_{\text{et}}$

$$K^\text{top}(\text{Perf}(\text{Sing}(\mathcal{M}))) \otimes \mathbb{Q} \to C^\text{sing}^\bullet((-)^{\text{an}}; \mathbb{Q}) \otimes \mathbb{Q}(\frac{u}{2\pi i}),$$

which is also a level-wise weak equivalence. Thus the rational structure on $H^n(H_{dR}(I^d_{cl})_{\text{Tate}})$ agrees with that of the Hodge structure of Equation 4.

\[\square\]

**Remark 3.24.** For any of the quotient stacks appearing in Amplification 2.22, the above theorem still holds for $\text{DbCoh}(\mathcal{X})$ with the same proof, with the exception of the explicit computation of $\text{gr}^p H^n(C^\text{per}_{\text{DbCoh}}(\text{DbCoh}(\mathcal{X})))$ when $\mathcal{X}$ is not smooth. In particular we have:

- A canonical isomorphism $K_M(X^{\text{an}}) \otimes \mathbb{C} \to C^\text{per}_{\text{DbCoh}}(\text{DbCoh}(X/G))$ factoring through $K^\text{top}(\text{DbCoh}(X/G))$;
- The degeneration property for $\text{DbCoh}(X/G)$; and
- A pure Hodge structure of weight $n$ on $\pi_{-n} K^\text{top}(\text{DbCoh}(\mathcal{X}))$ coming from the degeneration of the noncommutative Hodge-de Rham sequence which is $\text{Rep}(G)$-linear.

## 4. Computations of Hochschild invariants

### 4.1. Generalities on Hochschild invariants.

We begin with some abstract considerations and for the moment we let $\mathcal{X}$ be a smooth $k$-stack with affine stabilizers at geometric points. We will later specialize to the cases of interest, where we will obtain more explicit results. We denote the derived inertia stack (or loop space) $\mathcal{X} \times_k \mathcal{X} \times_k \mathcal{X}$ by $\mathcal{I}_\mathcal{X}$. Recall that the Hochschild cohomology $\text{HH}^\bullet(\mathcal{C})$ of a $k$-linear dg-category is the complex of endomorphisms of the identity functor $\mathcal{C} \to \mathcal{C}$.

The following proposition is a direct analogue of a well-known result for smooth schemes.

**Proposition 4.1.** Let $\mathcal{X}$ be as above, and let $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ be the diagonal. Then we have an identification

$$\text{HH}^\bullet(\text{Perf}(\mathcal{X})) \cong \text{Hom}_{\mathcal{X} \times \mathcal{X}}(\Delta_\ast \omega_\mathcal{X}, \Delta_\ast \omega_\mathcal{X})$$

$$\cong \text{Hom}_{\mathcal{X} \times \mathcal{X}}(\Delta_\ast \mathcal{O}_\mathcal{X}, \Delta_\ast \mathcal{O}_\mathcal{X})$$
We also have an identification:
\[ C_\bullet(\text{Perf}(\mathcal{X})) \cong R\Gamma(\emptyset_{\mathcal{X}}) \]

**Proof.** The computation of Hochschild cohomology is a standard consequence of Morita theory for perfect stacks [BZFN]. To compute \( C_\bullet \), we use the Morita invariant definition of the Hochschild homology of a compactly generated dg-category as the trace of the the identity functor. Thus we must compute the trace
\[ \text{tr} : \text{QCoh}(\mathcal{X} \times \mathcal{X}) \rightarrow \text{QCoh}(\text{Spec } k). \]

On sheaves of the form \( \pi_1^*(P_1) \otimes \pi_2^*(P_2) \), with \( P_1, P_2 \in \text{Perf}(\mathcal{X}) \), we have that the trace is given by
\[
\text{tr}(\pi_1^*(P_1) \otimes \pi_2^*(P_2)) := \text{RHom}(\mathcal{H}om(P_1, \omega_{\mathcal{X}}), P_2) \\
\cong R\Gamma((\mathcal{X}, \Delta^* (\pi_1^*(P_1) \otimes \pi_2^*(P_2)) \otimes \omega_{\mathcal{X}}^{-1})).
\]

Since the category \( \text{QC}(\mathcal{X} \times \mathcal{X}) \) is the colimit completion of sheaves of this form, we have that for an arbitrary object \( F \in \text{QC}(\mathcal{X} \times \mathcal{X}) \), the trace can be computed by
\[ F \rightarrow R\Gamma(\Delta^*(F) \otimes \omega_{\mathcal{X}}^{-1}). \tag{5} \]

It follows that we have an isomorphism
\[ C_\bullet(\text{Perf}(\mathcal{X})) \cong R\Gamma(\Delta^* \Delta_\ast \omega_{\mathcal{X}} \otimes \omega_{\mathcal{X}}^{-1}). \]

We furthermore have that \( \Delta^* \Delta_\ast \omega_{\mathcal{X}} \cong \Delta^* \Delta_\ast \mathcal{O}_{\mathcal{X}} \otimes \omega_{\mathcal{X}} \). Substituting this expression into our formula for the trace gives the desired result. \( \square \)

Now suppose that we have in addition the data of a function \( W : \mathcal{X} \rightarrow \mathbb{A}^1 \). It follows from Theorem 1.16 and Proposition 4.1 that the \( k((\beta)) \)-linear Hochschild cohomology \( \text{HH}^\bullet_{k((\beta))}(\text{MF}(\mathcal{X}, W)) \) is determined by \( \text{HH}^\bullet(\text{Perf}(\mathcal{X})) \) equipped with a natural \( \Lambda \)-module structure.

**Lemma 4.2.** The Hochschild cohomology of \( \text{MF}(\mathcal{X}, W) \) and \( \text{Perf}(\mathcal{X}) \) are related by the following chain of isomorphisms:
\[
\text{HH}^\bullet_{k((\beta))}(\text{MF}(\mathcal{X}, W)) \cong \text{Hom}_{\text{MF}^\infty(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W)}(\Delta_\ast \omega_{\mathcal{X}}, \Delta_\ast \omega_{\mathcal{X}}) \\
\cong \text{HH}^\bullet(\text{Perf}(\mathcal{X}))^{\text{Tate}}.
\]

Here the Tate construction is performed with respect to the \( S^1 \)-action of Lemma 1.9.

**Proof.** This is a direct application of Theorem 1.16, Proposition 4.1, and Lemma 1.9. Let \( \text{QC}(\mathcal{X})_{<\infty} \) and \( \text{IndCoh}(\mathcal{X})_{<\infty} \) denote the categories of homologically bounded-above complexes. We have that the natural functor
\[ \text{QC}((\mathcal{X} \times \mathcal{X})_0)_{<\infty} \rightarrow \text{IndCoh}((\mathcal{X} \times \mathcal{X})_0)_{<\infty} \]
is an equivalence and in particular fully-faithful. Because \( \Delta_\ast \omega_{\mathcal{X}} \) is bounded, and therefore in \( \text{IndCoh}_{<\infty} \), we have an equality as \( k \)-vector spaces,
\[
\text{Hom}_{\text{PreMF}^\infty(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W)}(\Delta_\ast \omega_{\mathcal{X}}, \Delta_\ast \omega_{\mathcal{X}}) \cong \text{Hom}_{(\mathcal{X} \times \mathcal{X})_0}(\Delta_\ast \omega_{\mathcal{X}}, \Delta_\ast \omega_{\mathcal{X}})
\]
The proof of Lemma 1.9 therefore applies unchanged and we may therefore prove the lemma by base change to \( k((\beta)) \). \( \square \)

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19In order to be consistent with Theorem 1.16, we use the version of Morita theory which results from Grothendieck duality \( \mathcal{H}om(-, \omega_{\mathcal{X}}) : \text{Perf}(\mathcal{X})^{\text{op}} \rightarrow \text{Perf}(\mathcal{X}) \) rather than the more standard linear duality \( \mathcal{H}om(-, \mathcal{O}_{\mathcal{X}}) \). As a result the identity functor corresponds to \( \Delta_\ast \omega_{\mathcal{X}} \).
To obtain a similar description of $C^b_{\bullet}(\beta)\left(MF(\mathcal{X}, W)\right)$, we must begin by describing the $k[[\beta]]$-linear trace map

$$\hat{\text{tr}}_{k[[\beta]]} : \text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0}^{\infty}(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W) \to k[[\beta]] - \text{Mod}.$$ 

Note that by Lemma 1.9 and the discussion preceding it, the composition

$$\begin{array}{ccc}
\text{PreMF}(\mathcal{X}, W)^{\text{op}} \otimes_{k[[\beta]]} \text{PreMF}(\mathcal{X}, W) & \to & \text{Perf}(\mathcal{X} \times \mathcal{X}) \\
\text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0}(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W) & \to & k - \text{Mod},
\end{array}$$

which maps

$$F \otimes G \mapsto \text{Hom}_\mathcal{X}(\mathcal{H}\text{om}_\mathcal{X}(i_*F, \omega_\mathcal{X}), i_*G) \simeq \text{Hom}_\mathcal{X}(i_*\mathcal{H}\text{om}_{\mathcal{X}_0}(F, \omega_{\mathcal{X}_0}), i_*G),$$

can be canonically enhanced to a functor with values in $\Lambda$-modules. Thus we can define an enhanced trace

$$\text{tr} \circ i_* : \text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0}(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W) \to \Lambda - \text{Mod}.$$ 

We let $\text{tr} \circ i_*$ denote the unique extension of this to a continuous functor on $\text{PreMF}^{\infty} := \text{Ind}(\text{PreMF})$ with values in $\Lambda - \text{Mod}$. 

**Lemma 4.3.** For any object $M \in \text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0}(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W)$ whose underlying object of $\text{IndCoh}(\mathcal{X} \times \mathcal{X})$ lies in $\text{IndCoh}(\mathcal{X} \times \mathcal{X})_{<\infty}$,

$$\hat{\text{tr}}_{k[[\beta]]}(M) \simeq \hat{\text{tr}}(i_*M)^{S^1}.$$ 

**Proof.** By definition the $k[[\beta]]$-linear trace corresponds to the functor

$$\text{PreMF}(\mathcal{X}, W)^{\text{op}} \otimes_{k[[\beta]]} \text{PreMF}(\mathcal{X}, W) \to k[[\beta]] - \text{Mod},$$

under the equivalence of Theorem 1.16. Hence Lemma 1.9 implies the proposition for compact objects.

To complete the proof, observe the underlying $k$-linear $\infty$-category of $\text{PreMF}^{\infty}(\mathcal{X}, W)$ is just $\text{IndCoh}(\mathcal{X}_0)$, so by [G2, Proposition 1.2.4] any $M \in \text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0}^{\infty}(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W)_{\leq n}$, the category of objects whose underlying ind-coherent sheaf lies in $\text{IndCoh}(\mathcal{X} \times \mathcal{X})_{\leq n}$, can be written as a filtered colimit of $M_{\alpha} \in \text{PreMF}_{\leq n}$. By the alternative formula for the trace in Equation 5, one deduces that

$$\text{tr} \circ i_*(\text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0}(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W)_{\leq n}) \subset \text{PreMF}_{\mathcal{X}_0 \times \mathcal{X}_0}(\mathcal{X} \times \mathcal{X}, -\pi_1^*W + \pi_2^*W)_{\leq n + K}$$

for some fixed $K > 0$. Thus $\text{tr}(i_*M) \simeq \text{hocolim}_{\alpha} \hat{\text{tr}}(i_\alpha M_{\alpha})$ is a filtered colimit of $\Lambda$-modules which are uniformly homologically bounded below, and $(-)^{S^1}$ commutes with such filtered colimits.

**Remark 4.4.** As before we have a presentation

$$\mathcal{O}(\mathcal{X} \times \mathcal{X})_0 \simeq \mathcal{A} := \mathcal{O}_{\mathcal{X} \times \mathcal{X}}[\epsilon; d\epsilon = -\pi_1^*W + \pi_2^*W].$$

If $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal, then $\Delta^*\mathcal{A} \simeq \mathcal{O}_{\mathcal{X}}[\epsilon; d\epsilon = 0]$, which expresses the fact that we have a cartesian diagram

$$\begin{array}{ccc}
\mathcal{X}_0 \times \text{Spec}(\Lambda) & \longrightarrow & (\mathcal{X} \times \mathcal{X})_0 \\
\downarrow & & \downarrow i \\
\mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}
\end{array}$$

with

$$\begin{array}{c}
\Delta + \epsilon \\
\Delta \end{array}$$

commutes with such filtered colimits.
It follows from the base change formula that for any $M \in \text{PreMF}(\mathcal{X} \times \mathcal{X}, -\pi_* W + \pi_2^* W)$, $\Delta^* i_*(M)$ is a quasi-coherent complex of $\Lambda$-modules and hence by Equation 5
\[ \text{tr}(i_*(M)) = R\Gamma(\mathcal{X}, \Delta^* i_*(M)) \otimes \omega^{-1}_{\mathcal{X}} \]
has a natural $\Lambda$-module structure.

On the full subcategory $\text{PreMF}(\mathcal{X}, W)^{op} \otimes_{k[\beta]} \text{PreMF}(\mathcal{X}, W)$, this corresponds to the bilinear functor
\[ F \otimes G \mapsto R\Gamma(\mathcal{X}, \Delta^* (i_*(M)) \otimes \omega^{-1}_{\mathcal{X}}) \]
\[ \simeq R\Gamma(\mathcal{X}, i_*(\text{Hom}(F, \omega_{\mathcal{X}})) \otimes \omega^{-1}_{\mathcal{X}}) \]
\[ \simeq R\Gamma(\mathcal{X}, \text{Hom}(i_* F, \omega_{\mathcal{X}})) \otimes \omega^{-1}_{\mathcal{X}} \]
\[ = \text{Hom}(i_*(F), i_*(G)) \]
And under these equivalences, the $\Lambda$-module structure is equivalent to the $\Lambda$-module structure described in the discussion leading up to Lemma 1.9. We deduce the alternative formula,
\[ \text{tr}_{k[\beta]}(M) \simeq R\Gamma(\mathcal{X}, \Delta^* (i_*(M)) \otimes \omega^{-1}_{\mathcal{X}})^{S^1}, \]
for any $M \in \text{PreMF}_{\mathcal{X}}(\mathcal{X} \times \mathcal{X}, -\pi_* W + \pi_2^* W)$ which is homologically bounded above.

After base change Lemma 4.3 provides a formula for
\[ \text{tr}_{k[\beta]}(\cdot): \text{MF}_{\mathcal{X}}(\mathcal{X} \times \mathcal{X}, -\pi_* W + \pi_2^* W) \to k((\beta)) - \text{Mod}, \]
and we deduce the following corollary.

**Corollary 4.5.** The following sequence of isomorphisms hold
\[ C^{k((\beta))}_*(\text{MF}(\mathcal{X}, W)) \cong \text{tr}_{k((\beta))}(\Delta_*(\omega_{\mathcal{X}})) \]
\[ \cong R\Gamma(\mathcal{X}, \Delta^* \Delta_*(\omega_{\mathcal{X}})) \otimes \omega^{-1}_{\mathcal{X}}^{\text{Tate}} \]
where the Tate construction is with respect the $S^1$-action described in Remark 4.4.

This will allow us to explicitly describe $C^{k((\beta))}_*(\text{MF}(\mathcal{X}, W))$ when $\mathcal{X}$ is a Deligne-Mumford stack, as in Proposition 3.16. For any affine $U$ with an étale map $U \to \mathcal{X}$, let $\Omega_*(U, W|_U)$ denote the Tate construction on $\bigoplus \Omega^i(U)[i]$ with respect to the $S^1$-action given by $-dW\wedge$, i.e. $\Omega_*(U, W|_U)$ is the $\Lambda((\beta))$-module $\bigoplus \Omega^i_*(U)(\beta)[i]$ with differential $-\beta \cdot dW\wedge$. Letting $B$ act on $\Omega_*(U, W|_U)$ via the de Rham differential as usual, $\Omega_*(-, W)$ defines a sheaf of $\Lambda((\beta))$-modules on $\mathcal{X}^{\text{aff}}_{et}$. We define the global de Rham complex to be the $\Lambda((\beta))$-module
\[ \Omega_*(\mathcal{X}, W) := R\Gamma(\mathcal{X}^{\text{aff}}, \Omega_*(-, W)) \]

**Corollary 4.6.** Let $W: \mathcal{X} \to \mathbb{A}^1$ be an LG-model, where $\mathcal{X}$ is a smooth separated Deligne-Mumford stack. Then there is a natural isomorphism of $\Lambda((\beta))$-modules
\[ C^{k((\beta))}_*(\text{MF}(\mathcal{X}, W)) \cong \Omega_*(I^d_{\mathcal{X}}, W), \]
which induces an equivalence $C^{k((\beta))}_*(\text{MF}(\mathcal{X}, W)) \cong \Omega_*(I^d_{\mathcal{X}}, W)^{\text{Tate}}$.

**Proof.** The proof follows that of Proposition 3.16 exactly. We let $I^{\mathcal{X}}$ denote the derived inertia stack, and let $\mathcal{X} \to X$ be the coarse moduli space. The morphism $I^{\mathcal{X}} \to \mathcal{X}$ has finite Tor amplitude, hence we have pullback functors between $D^b\text{Coh}(W^{-1}(0))$ and $\text{Perf}(W^{-1}(0))$ and thus we have a $k((\beta))$-linear pullback functor $C^{k((\beta))}_*(\text{MF}(\mathcal{X}, W)) \to C^{k((\beta))}_*(\text{MF}(I^{\mathcal{X}}, W))$, and we can further pull back along any étale map to $I^{\mathcal{X}}$. This defines a map of $\Lambda((\beta))$-modules
\[ C^{k((\beta))}_*(\text{MF}(\mathcal{X}, W)) \to R\Gamma \left( (I^{\mathcal{X}})^{\text{aff}}, C^{k((\beta))}_*(\text{MF}(U, W)) \right) \]
The HKR map for affine LG-models \([P, CT]\) induces an equivalence of presheaves on \(Y_{et}\) for any smooth orbifold LG-model

\[\Omega_{\bullet}(-, W) \simeq C^{k(\beta)}_\bullet(MF(-, W))\]

just as in Lemma 3.15. Applying this to the classical inertia stack allows us to define our canonical map \(C^{k(\beta)}_\bullet(MF(X, W)) \to \Omega_{\bullet}(I^d_X, W)\).

If \(X\) is a global quotient of a smooth affine scheme by a finite group, then this comparison map is an equivalence by [CT, Theorem 1.25]. Thus it suffices to show that the presheaf on \(X_{et}\),

\[(U \to X) \mapsto C^{k(\beta)}_\bullet(MF(X \times U, W)),\]

satisfies étale descent, because \(X\) is a quotient of a smooth affine scheme by a finite group étale locally over \(X\). The fact that the formation of \(C^{k(\beta)}_\bullet(MF(X, W))\) satisfies étale descent over \(X\) follows from the formula of Corollary 4.5, because the formation of \(\Delta^*\Delta_*(\omega_X) \otimes \omega_X^{-1}\) is étale local over \(X\) as in the proof of Proposition 3.16, and the Tate construction commutes with \(R\Gamma\) again by Lemma 3.18.

Finally we wish to perform a further Tate construction, this time with respect to the de Rham differential and using the formal variable \(u\). As in Lemma 3.19, this commutes with \(R\Gamma\) because of Lemma 3.18, so we have an equivalence \(C^{k(\beta),per}_\bullet(MF(X, W)) \simeq \Omega_{\bullet}(I^d_X, W)^{per}\). \(\square\)

4.2. Global quotients and gauged linear sigma models. Now let \(X/G\) be a global quotient stack. We consider the scheme \(\mathcal{P} := G \times X \times X\). Denote by \(\Delta : G \times X \to \mathcal{P}\) the map \((g, x) \mapsto (g, x, x)\), and by \(\Gamma : G \times X \to \mathcal{P}\) the map \((g, x) \mapsto (g, x, g \cdot x)\). Both are closed immersions, and we will also use the notation \(\Delta\) and \(\Gamma\) to denote the corresponding subschemes of \(\mathcal{P}\).

**Lemma 4.7.** Let \(X/G\) be a smooth quotient stack. Both \(\Gamma\) and \(\Delta\) are equivariant with respect to the \(G\) action on \(\mathcal{P}\) which sends \(h \cdot (g, x_1, x_2) \mapsto (hgh^{-1}, hx_1, hx_2)\). We have

\[C_\bullet(\text{Perf}(X/G)) \cong R\Gamma(\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Gamma)^G\]

**Proof.** By Proposition 4.1 we must compute the derived global sections of the structure sheaf of the derived inertia stack. First note the alternate presentation for the stack \(X/G \simeq G \times X/G^2\), where the \(G^2\) action in the second presentation is given by

\[(h_1, h_2) \cdot (g, x) = (h_2h_1^{-1}, h_1x).\]

In this presentation the diagonal \(X/G \to X/G \times X/G\) corresponds to the \(G^2\)-equivariant map \(G \times X \to X \times X\) given by \((g, x) \mapsto (x, gx)\).

Let \(G^2\) act on \(\mathcal{P}\) by

\[(h_1, h_2) \cdot (g, x_1, x_2) = (h_2h_1^{-1}, h_1x_1, h_2x_2).\]

Then \(\Gamma\) is \(G^2\)-equivariant, and using the presentation above we see that the diagonal factors as the closed immersion \(\Gamma : G \times X/G^2 \to \mathcal{P}/G^2\) followed by the projection \(\mathcal{P}/G^2 \to X \times X/G^2\), which is smooth and affine. It follows that the derived inertia stack is the derived intersection of \(p_1^{-1}\Gamma\) and \(p_2^{-1}\Gamma\) in \(\mathcal{P} \times_{X^2} \mathcal{P}/G^2\).

Now \(\mathcal{P} \times_{X^2} \mathcal{P} \simeq G \times X \times X\) with \(G^2\)-action given by

\[(h_1, h_2) \cdot (g_1, g_2, x_1, x_2) = (h_2g_1h_1^{-1}, h_2g_2h_1^{-1}, h_1x_1, h_2x_2)\]

The projections \(p_1, p_2 : \mathcal{P} \times_{X^2} \mathcal{P} \to \mathcal{P}\) are given by forgetting \(g_2\) and \(g_1\) respectively. We claim that \(\mathcal{P} \times_{X^2} \mathcal{P}/G^2 \simeq \mathcal{P}/G\), where \(G\) acts on \(\mathcal{P}\) as in the statement of the lemma. Indeed we can present \(\mathcal{P}/G\) as the quotient of \(G \times \mathcal{P}\) by the \(G^2\)-action

\[(h_1, h_2) \cdot (g_1, g_2, x_1, x_2) = (h_2g_1h_1^{-1}, h_1g_2h_1^{-1}, h_1x_1, h_1x_2),\]

and we have a \(G^2\)-equivariant isomorphism \(G \times \mathcal{P} \to \mathcal{P} \times_{X^2} \mathcal{P}\) given by

\[(g_1, g_2, x_1, x_2) \mapsto (g_1, g_1g_2, x_1, g_1x_2)\]
The resulting isomorphism \( \mathcal{P}/G \to \mathcal{P} \times_{X^2} \mathcal{P}/G^2 \) is given by the map \((g, x_1, x_2) \mapsto (1, g, x_1, x_2)\) which is equivariant with respect to the diagonal homomorphism \( G \to G^2 \).

In order to complete the proof, we must identify the closed substacks \( p_1^{-1}(\Gamma/G^2) \) and \( p_2^{-1}(\Gamma/G^2) \) in \( \mathcal{P} \times_{X^2} \mathcal{P}/G^2 \) under the isomorphism with \( \mathcal{P}/G \). The first is the closed subscheme \( p_1^{-1}(\Gamma) \cap (\{1\} \times \mathcal{P}) = \Delta \), regarded as a \( G \)-equivariant closed subscheme of \( \mathcal{P} \), and the second is \( p_2^{-1}(\Gamma) \cap (\{1\} \times \mathcal{P}) = \Gamma \) as a \( G \)-equivariant closed subscheme of \( \mathcal{P} \).

The case when \( X \) is a vector space, \( \mathbb{V} \), and \( G \) acts on \( \mathbb{V} \) via a linear action, is of interest in two-dimensional gauge theory. In this case we make the above derived intersection explicit using a Koszul resolution. Denote by \( \alpha : G \times \mathbb{V} \to \mathbb{V} \) the action morphism \((g, v) \mapsto g \cdot v\). We choose linear coordinates on \( \mathbb{V} \) and identify \( \mathbb{V} \times \mathbb{V} \) with \( \text{Spec}(k[x_1, y_1]) \).

The Koszul complex for the regular sequence \( K_{\mathbb{V} \times \mathbb{V}}(x_i - y_i) \) gives a resolution of the diagonal on \( \mathbb{V} \times \mathbb{V} \). An important point is that, in this case, this resolution is \( G \)-equivariant with respect to diagonal \( G \)-action because the \( G \)-action on \( \mathbb{V} \) is linear. Then

\[
K_{G \times \mathbb{V} \times \mathbb{V}}(x_i - y_i) \to \mathcal{O}_\Delta
\]

is a resolution of \( \mathcal{O}_\Delta \) over \( \mathcal{P} \).

**Corollary 4.8.** \( C_\bullet(\text{Perf}(\mathcal{V}/G)) \cong (K_{G \times \mathbb{V}}(x_i - \alpha^*(x_i)))^G \)

**Proof.** By the above lemma, \( C_\bullet(\text{Perf}(\mathcal{V}/G)) \) is isomorphic to

\[
(K_{G \times \mathbb{V} \times \mathbb{V}}(x_i - y_i) \otimes_{\mathcal{O}_G} \mathcal{O}_\Gamma)^G \cong (\Gamma^*K_{G \times \mathbb{V} \times \mathbb{V}}(x_i - y_i))^G.
\]

Pulled back to \( G \times \mathbb{V} \), the function \( \Gamma^*(y_i) = \alpha^*(x_i) \), hence \( \Gamma^*K_{G \times \mathbb{V} \times \mathbb{V}}(x_i - y_i) = K_{G \times \mathbb{V}}(x_i - \alpha^*(x_i)) \).

We next generalize the above discussion to LG-models \((\mathcal{V}/G, W)\). Begin by assuming that \( G = T \) is a torus and let \( W \in k[\mathbb{V}]^T \). Choose coordinates which diagonalize our action and let \( n_i \) denote the corresponding basis of \( K_{\mathbb{V} \times \mathbb{V}}^1(x_i - y_i) \). Next, consider the difference functions

\[
A_j(W) = \frac{W(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n) - W(x_1, \ldots, x_{j-1}, y_j, y_{j+1}, \ldots, y_n)}{y_j - x_j}
\]

and the special element

\[
A := \sum_j A_j(W) \otimes n_j \in K_{\mathbb{V} \times \mathbb{V}}^1(x_i - y_i).
\]

This element has the key feature that

\[
d_{Kos}(A) = \sum_j \Delta_j(W)(y_j - x_j) = W(y) - W(x).
\]

Because \( A \) is an odd element of the Koszul algebra, \( A \wedge A = 0 \), and we can introduce an \( S^1 \)-action on \( K_{T \times \mathbb{V} \times \mathbb{V}}(x_i - y_i) \) given by the operator

\[
\Delta_W := A \wedge - : K_{T \times \mathbb{V} \times \mathbb{V}}^*(x_i - y_i) \to K_{T \times \mathbb{V} \times \mathbb{V}}^{*+1}(x_i - y_i)
\]

This operator has weight zero with respect to the torus action and satisfies the relations \( \Delta_W^2 = 0 \) and \( d_{Kos} \circ \Delta_W + \Delta_W \circ d_{Kos} = (W(y) - W(x)) \cdot \text{id} \).

For general reductive \( G \), the element \( A \notin (\text{Sym}^*(\mathbb{V}) \otimes \mathbb{V})^G \). However, we have a canonical Reynolds projection operator

\[
\rho_G : \text{Sym}^*(\mathbb{V}) \otimes \mathbb{V} \to (\text{Sym}^*(\mathbb{V}) \otimes \mathbb{V})^G
\]

We then replace \( A \) with \( \rho_G(A) \), and \( \Delta_W = \rho_G(A) \wedge - \). Applying \((-)^G \) to the relations above shows that we still have \( d_{Kos} \circ \Delta_W + \Delta_W \circ d_{Kos} = (W(y) - W(x)) \cdot \text{id} \), and \( \Delta_W^2 = 0 \) again for degree reasons.

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Corollary 4.9. \( C_b((\beta)) (\text{MF}(X, W)) \cong (K_{G \times Y}(x_i - \alpha(x_i))^G((\beta)), d_{K\text{os}} + \beta \Delta_W) \)

Proof. The operator \( \Delta_W \) determines the structure of a module over \( O_{X \times Y}[\epsilon; d\epsilon = -\pi_1^*W + \pi_2^*W] \) via the action \( \epsilon(x) = \Delta_W(x) \). This determines the circle action on the Hochschild complex by Lemma 4.3 and Remark 4.4. The corollary is then a consequence of Corollary 4.5. \( \square \)

4.3. An HKR theorem and quotients of affine varieties. In [BG], Block and Getzler construct for any compact smooth \( M \)-manifold \( X \) an explicit model for the \( M \)-equivariant cyclic homology of the algebra \( C^\infty(X) \) using differential forms on \( X \). Our goal is to translate their construction into algebraic geometry and establish their version of the equivariant Hochschild-Kostant-Rosenberg theorem when \( X = \text{Spec}(A) \) is smooth and affine and \( X/G \) is cohomologically proper. Our proof is an application of Theorem 2.7 and Theorem 3.20. For simplicity, we let \( k = \mathbb{C} \) throughout this section.

To compute the derived intersection appearing in Lemma 4.7, we may use the bar resolution \( B(A) \) of \( A \) as an \( A - A \) bimodule. Namely,

\[
B_n(A) := A \otimes A^{\otimes n} \otimes A
\]

where the differential can be described as the sum \( b = \sum_i (-1)^i \partial_i \), where

\[
\partial_i(a_0' \otimes a_1 \otimes \cdots \otimes a_n \otimes a''_n) := \begin{cases} a_0' a_1 \otimes \cdots \otimes a_n \otimes a''_n, & i = 0 \\ a_0' \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a''_n, & i \neq 0, \neq n \\ a_0' \otimes \cdots \otimes a_n a''_n, & i = n \end{cases}
\]

Our notation is meant to highlight the fact that the first and last variables in the bar complex play a distinguished role from the other \( a_i \). We then have that \( O_G \otimes B(A) \) is a resolution of \( O_X \) which we may restrict to \( \Gamma \). The result is a complex where the \( n \)-th graded piece is

\[
C_n(A, G) := O_G \otimes A^{\otimes n+1} = \Gamma(G \times X^{n+1}, O_{G \times X^{n+1}}).
\]

For any \( c \in \Gamma(O_G \times X^{n+1}) \), the differentials \( \partial_i \) above now take the form

\[
\partial_i c(g, x_0, x_1, \cdots, x_{n-1}) := \begin{cases} c(g, x_0, x_0, x_1, \cdots, x_{n-1}) & i = 0 \\ c(g, x_0, \cdots, x_i, x_i, \cdots, x_{n-1}), & i \neq 0, \neq n \\ c(g, x_0, \cdots, x_{n-1}, g \cdot x_0), & i = n \end{cases}
\]

We define the \( \Lambda \)-module \( C_{\bullet, G}(A) := C_{\bullet}(A, G)^G \), and note the following corollary of Lemma 4.7.

Corollary 4.10. \( C_{\bullet}(\text{Perf}(X/G)) \cong C_{\bullet, G}(A) \)

Let \( C_{G}(A)^\wedge_g \) denote the completion of \( C_{\bullet, G}(A) \) as a complex of modules over the representation ring \( \text{Rep}(G) \otimes \mathbb{C} = \Gamma(O_G)^G \) at the conjugacy class \([g]\). Let us work for the moment with a fixed normal element \( g \in G \). Let \( Y = \text{Spec}(A)^g \) denote the fixed point locus of \( g \) and \( B = \Gamma(O_Y) \). The letter \( Z \) will designate the centralizer of \( g \) and \( \mathfrak{z} \) denotes its Lie algebra, and normality of \( g \) ensures that \( Z \) is the complexification of \( Z_g := Z \cap M \) for a maximal compact subgroup \( M \subset G \). We have embeddings \( j : Z \rightarrow G \) and \( k : Y \rightarrow X \).

Lemma 4.11. When \( \text{Spec}(A)/G \) is cohomologically proper, the natural restriction map gives rise to an isomorphism \( k^* : C_{\bullet, G}(A)^\wedge_g \rightarrow C_{\bullet, Z}(B)^\wedge_g \).

Proof. Note that because \( \text{Spec}(A)/G \) is cohomologically proper, each \( C_{n, G}(A) \) is a coherent \( \text{Rep}(G) \otimes \mathbb{C} \)-module, so completion commutes with taking homology in this case, and it suffices to prove the result on the level of homology. It is known [FHT, Proposition 3.10] that the map \( k^* :
We thus have a chain complex, in fact a CDGA,\[ \text{Spec}(\mathbb{Z}) \]

\[ \text{Rep}(\mathbb{Z}) \]

Note that \( \omega \) is the unique extension of the contraction map \( \Omega^\bullet(Y \otimes \mathbb{C}) \) projective of these decompositions. We observe that the Hodge decompositions

\[ H_* \text{H} \]

\[ H_* \text{Z} \]

are decompositions of \( \text{Rep}(\mathbb{G}) \) and \( \text{Rep}(\mathbb{Z}) \) modules respectively. This follows because both the Hodge filtration and the conjugate filtration are filtrations of \( \text{Rep}(\mathbb{G}) \) modules as can be seen for example by examining the explicit model for \( \text{Rep}^*_{\mathbb{Z}}(\mathbb{G}) \). The lemma now follows by taking completions of these decompositions. \( \square \)

Next we construct a model for \( C^\bullet_{\mathbb{Z}}(B)^\wedge \) based on algebraic differential forms \( \Omega^n_Y \), regarded as a projective \( B \)-module. Recall that the Cartan differential

\[ i : \text{Sym}(\mathfrak{j}^*) \otimes \Omega_Y^0 \rightarrow \text{Sym}(\mathfrak{j}^*) \otimes \Omega_Y^{n-1} \]

is the unique extension of the contraction map \( \Omega_Y^n \rightarrow \mathfrak{j}^* \otimes \Omega_Y^{n-1} \) to a differential satisfying the Liebniz rule. Alternatively, regarding \( \omega \in \text{Sym}(\mathfrak{j}^*) \otimes \Omega_Y^n \) as a section of a quasi-coherent sheaf over \( \mathfrak{j} \), we have

\[ (i\omega)(z) = i_z \omega(z). \]

We thus have a chain complex, in fact a CDGA,

\[ \Omega^\bullet_Y[\mathfrak{j}^*] = \left( \bigoplus_n \text{Sym}(\mathfrak{j}^*) \otimes \Omega_Y^0[n], i \right). \]

Note that \( i \) is \( \mathbb{Z} \)-equivariant, and that it descends to the quotient \( \text{Sym}(\mathfrak{j}^*)/m^k \). Thus we can define

\[ \Omega^\bullet_Y[\mathfrak{j}^*]_k := \left( \bigoplus_n \text{Sym}(\mathfrak{j}^*)/m^k \otimes \Omega_Y^0[n], i \right) \]

\[ \Omega^\bullet_Y[\mathfrak{j}^*]^\mathbb{Z}_k := \left( \bigoplus_n (\text{Sym}(\mathfrak{j}^*)/m^k \otimes \Omega_Y^0[n]), i \right) \]

**Proposition 4.12.** The comparison map of Construction 4.13 below is a quasi-isomorphism of \( \Lambda \)-modules

\[ \text{HKR}^\wedge_g : C^\bullet_{\mathbb{Z}}(B)^\wedge_g \rightarrow \lim_k (\Omega^\bullet_Y[\mathfrak{j}^*]^\mathbb{Z}_k). \]

Hence when \( \text{Spec}(A)/G \) is cohomologically proper, we have a quasi-isomorphism of \( \Lambda \)-modules

\[ \text{HKR}^\wedge_g \circ k^* : C^\bullet_{\mathbb{Z}}(A)^\wedge_g \rightarrow \lim_k (\Omega^\bullet_Y[\mathfrak{j}^*]^\mathbb{Z}_k). \]
Let \( Z_{(k)} \) denote the \( k \)-th infinitesimal neighborhood of the identity in \( Z \). The exponential map provides a compatible system of isomorphisms \( \exp_k : \text{Spec}(\text{Sym}(\mathfrak{g}^*)/m^k)) \to Z_{(k)} \). Note that under this equivalence, \( \mathbb{G}_m \) acts algebraically on \( Z_{(k)} \) by scaling, and this action actually extends to an action of the monoid \( \mathbb{A}^1 \). This is encoded algebraically via a coaction map \( \text{Sym}(\mathfrak{g}^*)/m^k \to \text{Sym}(\mathfrak{g}^*)/m^k \otimes \mathbb{C}[t] \).

**Construction 4.13.** For any \( b \in B \), the coaction of \( \mathcal{O}_Z \) on \( B \), the exponential map \( \exp_k \), and the \( \mathbb{G}_m \)-action on \( \mathfrak{g} \) define an element

\[
\exp_k(-t \cdot z) \cdot b \in B \otimes \text{Sym}(\mathfrak{g}^*)/m^k \otimes \mathbb{C}[t].
\]

We define \( C_{n,k}(B, Z) := \mathcal{O}_{Z_{(k)}} \otimes B^{n+1} \), i.e. the reduction of \( C_\bullet(B, Z) \) modulo \( m^k \), and introduce the map \( \text{HKR}_{g,k} : C_{n,k}(B, Z) \to \Omega^\bullet_Y[[\mathfrak{g}^*]]_k \) given by

\[
\psi \otimes b'_0 \otimes b_1 \cdots b_n \mapsto \psi(g \cdot \exp_k(z)) \int_{\Delta_n} b'_0 d(\exp_k(-t_1 z) \cdot b_1) \wedge \cdots \wedge d(\exp_k(-t_n z) \cdot b_n) dt_1 dt_2 \cdots dt_n. \quad (7)
\]

Here \( d(-) \) denotes the \( \text{Sym}(\mathfrak{g}^*)/m^k \otimes \mathbb{C}[t] \)-linear extension of the exterior derivative

\[
d : B \otimes \text{Sym}(\mathfrak{g}^*)/m^k \otimes \mathbb{C}[t] \to \Omega^1_Y \otimes \text{Sym}(\mathfrak{g}^*)/m^k \otimes \mathbb{C}[t].
\]

The integrand is regarded as an element of \( \text{Sym}(\mathfrak{g}^*)/m^k \otimes \Omega_Y^\bullet \otimes \mathbb{C}[t_1, \ldots, t_n] \), and the integral over the standard \( n \)-simplex \( \Delta_n \) is regarded formally as a linear map \( \mathbb{C}[t_1, \ldots, t_n] \to \mathbb{C} \). This formula is identical to the one used in [BG], so it follows formally from the computations there that \( \text{HKR}_{g,k} \) is a chain map (See for instance [BG, Theorem 3.2]). This map is \( \text{Z} \)-equivariant, so it restricts to a chain maps

\[
\text{HKR}_{g,k} : C_{\bullet,k}(B, Z) \to \Omega^\bullet_Y[[\mathfrak{g}^*]]_k^Z, \quad \text{and}
\]

\[
\text{HKR}_g^\wedge := \varprojlim_k \text{HKR}_{g,k} : C_{\bullet,Z}(B) \to \Omega^\wedge_Y[[\mathfrak{g}^*]]_k^Z.
\]

**Proof of Proposition 4.12.** By the compatibility of the HKR maps with translation by the central element \( g \), it suffices to consider the case \( g = \text{id} \). The maps \( \text{HKR}_{id,k} : C_{\bullet,k}(B, Z) \to \Omega^\bullet_Y[[\mathfrak{g}^*]]_k \) are a compatible family of maps of bounded complexes with coherent homology over \( B \otimes \text{Sym}(\mathfrak{g}^*)/m^k \). \( \text{HKR}_{id,1} \) is the classical HKR map

\[
b'_0 \otimes \cdots \otimes b_n \mapsto \frac{1}{n!} b'_0 db_1 \cdots db_n,
\]

which is an equivalence of \( \Lambda \)-modules. Hence by Nakayama’s lemma each \( \text{HKR}_{id,k} \) is a quasi-isomorphism, and the same is true after taking \( \mathbb{Z} \)-invariants. Hence \( \text{HKR}_g^\wedge \) is a quasi-isomorphism. The final statement of the proposition combines this with the previous lemma. \( \square \)

We finish with a few observations concerning LG-models of the form \( \text{Spec}(A)/G,W \). Then we can equip \( B(A) \) with a \( \mathcal{O}_{X \times X}[\epsilon; d\epsilon = -\pi^*_1 W + \pi^*_2 W] \) module structure quasi-isomorphic to \( \Delta_\bullet(\mathcal{O}_X) \) by defining the operator

\[
\Delta_W(\psi \otimes a_0' \otimes \cdots \otimes a_n') := \sum_{i=1}^n (-1)^i (\psi \otimes a_0' \otimes \cdots \otimes W \otimes a_i \otimes \cdots \otimes a_n')
\]

\[
+ (-1)^{n+1} (\psi \otimes a_0' \otimes \cdots \otimes W \otimes a_0').
\]
This operator is equivariant because the function $W$ is invariant. We again define $\epsilon \cdot x = \Delta_W(x)$. By restricting to $\Gamma$, we obtain an $S^1$-action on $C_\bullet(A,G)$ the explicit formula is given by

$$
\Delta_W(\psi \otimes a'_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{i=n} (-1)^i(\psi \otimes a'_0 \otimes \cdots \otimes W \otimes a_i \cdots \otimes a_n) + (-1)^{n+1}(\psi \otimes a'_0 \otimes \cdots \otimes W)
$$

Let $C^{k(\beta)}_\bullet(A,G,W)$ denote the $\Lambda((\beta))$-module $(C_\bullet(A,G)((\beta)), B + \beta \Delta_W)$

Parallel to Corollary 4.9, we have

**Corollary 4.14.** There is a canonical quasi-isomorphism

$$
C^{k(\beta)}_\bullet(M\!F(X,W)) \cong C^{k(\beta)}_\bullet(A,G,W)^G.
$$

**REFERENCES**


The Stacks Project Authors, *stacks project*, 2015.
