THE DERIVED CATEGORY OF A GIT QUOTIENT
(DRAFT)

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ABSTRACT. Given a quasiprojective algebraic variety with a reductive
group action, we describe a relationship between its equivariant derived
category and the derived category of its geometric invariant theory quo-
tient. This generalizes classical descriptions of the category of coherent
sheaves on projective space and categorifies several results in the theory
of Hamiltonian group actions on projective manifolds.

This perspective generalizes and provides new insight into examples
of derived equivalences between birational varieties. We provide a cri-
teron under which two different GIT quotients are derived equivalent,
and apply it to prove that any two generic GIT quotients of an equiv-
ariantly Calabi-Yau projective-over-affine variety by a torus are derived
equivalent.

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1. INTRODUCTION

We describe a relationship between the derived category of equivariant coherent sheaves on a smooth projective-over-affine variety, $X$, with an action of a reductive group, $G$, and the derived category of coherent sheaves on a GIT quotient of that action. The main theorem connects three classical circles of ideas:

- Serre’s description of quasicoherent sheaves on a projective variety in terms of graded modules over its homogeneous coordinate ring,
- Kirwan’s theorem that the canonical map $H^*_G(X) \to H^*(X//G)$ is surjective,[16] and
- the “quantization commutes with reduction” theorem from geometric quantization theory equating $h^0(X, \mathcal{L})^G$ with $h^0(X//G, \mathcal{L})$ when the linearization $\mathcal{L}$ descends to the GIT quotient[22].

A $G$-linearized ample line bundle $\mathcal{L}$ defines an open semistable locus $X^{ss} \subset X$, the complement of the base locus of invariant global sections of $\mathcal{L}^k$ for $k \gg 0$. We denote the quotient stack $\mathfrak{X} = X/G$ and $\mathfrak{X}^{ss} = X^{ss}/G$. In this paper, the term “GIT quotient” will refer to the quotient stack $\mathfrak{X}^{ss}/G$, as opposed to the coarse moduli space of $X^{ss}/G$.

In order to state the main theorem, we will need to recall the equivariant “Kirwan-Ness (KN) stratification” of $X \backslash X^{ss}[12]$ by connected locally-closed subvarieties. I will formally define a KN stratification and discuss its properties in Section ???. The stratification is determined by a set of distinguished one-parameter subgroups $\lambda_i : \mathbb{C}^* \to G$, and open subvarieties of the fixed locus of $\lambda_i$ denoted $\sigma_i : Z_i \hookrightarrow X$. We will also define integers $\eta_i \geq 0$ in (??). Because $Z_i$ is fixed by $\lambda_i$, the restriction of an equivariant coherent sheaf $\sigma_i^* F$ is graded with respect to the weights of $\lambda_i$.

We denote the bounded derived category of coherent sheaves on $\mathfrak{X}$ by $D^b(\mathfrak{X})$, and likewise for $\mathfrak{X}^{ss}$. Restriction gives an exact dg-functor $i^* : D^b(X/G) \to D^b(\mathfrak{X}^{ss}/G)$, and in fact any bounded complex of equivariant coherent sheaves on $X^{ss}$ can be extended equivariantly to $X$. The main result of this paper is the construction of a functorial splitting of $i^*$.

Theorem 1.1 (derived Kirwan surjectivity, preliminary statement). Let $X$ be a smooth projective-over-affine variety with a linearized action of a reductive group $G$, and let $\mathfrak{X} = X/G$. Specify an integer $w_i$ for each KN

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1On a technical note, all of the categories in this paper will be pre-triangulated dg-categories, so $D^b(\mathfrak{X})$ denotes a dg-enhancement of the triangulated category usually denoted $D^b(\mathfrak{X})$. However, all of the results will be statements that can be verified on the level of homotopy categories, such as semiorthogonal decompositions and equivalences of categories, so I will often write proofs on the level of the underlying triangulated category.
stratum of the unstable locus $X \setminus X^{ss}$. Define the full subcategory of $D^b(X)$

$$
G_w := \left\{ F^* \in D^b(X) \mid \mathcal{H}^*(L\sigma_i^* F^*) \text{ supported in weights } [w_i, w_i + \eta_i] \right\}
$$

Then the restriction functor $i^* : G_w \to D^b(X^{ss})$ is an equivalence of categories.

**Remark 1.2.** The general version, described in Section 2, identifies $G_w$ as piece of a semiorthogonal decomposition of $D^b(X)$, and applies to any (possibly singular) stack $X$ such that $X \setminus X^{ss}$ admits a KN stratification (Definition 2.1) satisfying Properties (L+) and (A).

The simplest example of Theorem 1.1 is familiar to many mathematicians: projective space $P(V)$ can be thought of as a GIT quotient of $V/C^*$. Theorem 1.1 identifies $D^b(P(V))$ with the full triangulated subcategory of the derived category of equivariant sheaves on $V$ generated by $O_V(q), \ldots, O_V(q + \dim V - 1)$. In particular the semiorthogonal decompositions described in Section 3 refine and provide an alternative proof of Beilinson’s theorem that the line bundles $O_{P(V)}(1), \ldots, O_{P(V)}(\dim V)$ generate $D^b(P(V))$.

Serre’s theorem deals with the situation in which $G = C^*$, $X$ is an affine cone, and the unstable locus consists only of the cone point – in other words one is studying a connected, positively graded $k$-algebra $A$. The category of quasicoherent sheaves on $\text{Proj}(A)$ can be identified with the full subcategory of the category of graded $A$-modules graded in degree $\geq q$ for any fixed $q$. This classical result has been generalized to noncommutative $A$ by M. Artin. D. Orlov studied the derived categories and the category of singularities of such algebras in great detail in [19], and much of the technique of the proof of Theorem 1.1 derives from that paper.

In the context of equivariant Kähler geometry, Theorem 1.1 is a categorification of Kirwan surjectivity. To be precise, one can recover the De Rham cohomology of a smooth stack as the periodic-cyclic homology its derived category [15, 24], so the classical Kirwan surjectivity theorem follows from the existence of a splitting of $i^*$. Kirwan surjectivity applies to topological $K$-theory as well[13], and one immediate corollary of Theorem 1.1 is an analogous statement for algebraic $K$-theory

**Corollary 1.3.** The restriction map on algebraic $K$-theory $K_i(X) \to K_i(X^{ss})$ is surjective.

The fully faithful embedding $D^b(X^{ss}) \subset D^b(X)$ of Theorem 1.1 and the more precise semiorthogonal decomposition of Theorem 2.5 correspond, via Orlov’s analogy between derived categories and motives [18], to the claim that the motive $X^{ss}$ is a summand of $X$. Via this analogy, the results of this paper bear a strong formal resemblance to the motivic direct sum decompositions of homogeneous spaces arising from Białyńcki-Birula decompositions [8]. However, the precise analogue of Theorem 1.1 would pertain to the equivariant motive $X/G$, whereas the results of [8] pertain to the nonequivariant motive $X$. 

The “quantization commutes with reduction” theorem from geometric quantization theory relates to the fully-faithfulness of the functor $i^\ast$. The original conjecture of Guillemin and Sternberg, that $\dim H^0(X/G, \mathcal{L}^k) = \dim H^0(X^{ss}/G, \mathcal{L}^k)$, has been proven by several authors, but the most general version was proven by Teleman in [22]. He shows that the canonical restriction map induces an isomorphism $R\Gamma(X/G, \mathcal{V}) \rightarrow R\Gamma(X^{ss}/G, \mathcal{V})$ for any equivariant vector bundle such that $\mathcal{V}|_{Z_\alpha}$ is supported in weight $>-\eta_\alpha$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two vector bundles in the grade restriction windows of Theorem 1.1, then the fact that $R\text{Hom}_X(\mathcal{V}_1, \mathcal{V}_2) \rightarrow R\text{Hom}_{X^{ss}}(\mathcal{V}_1|_{X^{ss}}, \mathcal{V}_2|_{X^{ss}})$ is an isomorphism is precisely Teleman’s quantization theorem applied to $\mathcal{V}_2 \otimes \mathcal{V}_1^\vee \simeq R\text{Hom}(\mathcal{V}_1, \mathcal{V}_2)$.

In Section 4, we apply Theorem 1.1 to construct new examples of derived equivalences and embeddings resulting from birational transformations, as conjectured by Bondal & Orlov[7]. The $G$-ample cone in $NS^0_G(X)$ has a decomposition into convex conical chambers[12] within which the GIT quotient $X^{ss}(\mathcal{L})$ does not change, and $X^{ss}(\mathcal{L})$ undergoes a birational transformation as $[\mathcal{L}]$ crosses a wall between chambers. Derived Kirwan surjectivity provides a general approach to constructing derived equivalences between the quotients on either side of the wall: in some cases both quotients can be identified by Theorem 1.1 with the same subcategory of $D^b(X/G)$. This principle is summarized in Ansatz 4.11.

For a certain class of wall crossings, balanced wall crossings, there is a simple criterion for when one gets an equivalence or an embedding in terms of the weights of $\omega_X|_{Z_i}$. When $G = T$ is abelian, all codimension-1 wall crossings are balanced, in particular we are able to prove that any two generic torus quotients of an equivariantly Calabi-Yau variety are derived equivalent. For nonabelian $G$, we consider a slightly larger class of almost balanced wall crossings. We produce derived equivalences for flops which excise a Grassmannian bundle over a smooth variety and replace it with the dual Grassmannian bundle, recovering recent work of Will Donovan and Ed Segal[10,11].

Finally, in Section 5 we investigate applications of Theorem 2.5 beyond smooth quotients $X/G$. We identify a criterion under which Property (L+) holds for a KN stratification, and apply it to hyperkähler reductions. We also explain how Morita theory[4] recovers derived Kirwan surjectivity for certain complete intersections and derived categories of singularities (equivalently categories of matrix factorizations) “for free” from the smooth case.

The inspiration for Theorem 1.1 were the grade restriction rules for the category of boundary conditions for B-branes of Landau-Ginzburg models studied by Hori, Herbst, and Page,[14] as interpreted mathematically by Segal[21]. The essential idea of splitting was present in that paper, but the analysis was only carried out for a linear action of $\mathbb{C}^*$, and the category $G_w$ was identified in an ad-hoc way. The main contribution of this paper is showing that the splitting can be globalized and applies to arbitrary $X/G$ as a categorification of Kirwan surjectivity, and that the categories $G_w$ arise...
naturally via the semiorthogonal decompositions to be described in the next section.

1.1. **Author’s note.** I would like to thank my PhD adviser Constantin Teleman for introducing me to his work [22], and for his support and useful comments throughout this project. I would like to thank Daniel Pomerleano for many enlightening conversations, and for explaining how to recover derived categories of singularities using Morita theory. I’d like to thank Anatoly Preygel for useful conversations about derived algebraic geometry and for carefully reviewing section 5. Finally, I’d like to thank Yujiro Kawamata for suggesting that I apply my methods to hyperkähler reduction and flops of Grassmannian bundles, and Kentaro Hori for carefully reviewing my work and discovering some mistakes in the first version of this paper.

The problems studied in this paper overlap greatly with the work [5], although the projects were independently conceived and carried out. I learned about [5] at the January 2012 Conference on Homological Mirror Symmetry at the University of Miami, where the authors presented a method for constructing equivalences between categories of matrix factorizations of toric LG models. In the finished version of their paper, they also treat the general VGIT for smooth quotients $X/G$, and present several new applications.

Here we work in slightly more generality and emphasize the categorification of Kirwan surjectivity, as well as some applications to hyperkähler quotients. We hope that the different perspectives brought to bear on the subject will be useful in elucidating further questions.

1.2. **Notation.** If $\mathcal{X}$ is a stack, $\mathcal{D}^b(\mathcal{X})$ will denote the bounded derived category of coherent sheaves on $\mathcal{X}$, $\text{Perf}(\mathcal{X})$ will denote the subcategory of perfect complexes.

2. **The main theorem**

In this section we will review the properties of the KN stratification and lay out the proof of Theorem 1.1.

2.1. **Equivariant stratifications in GIT.** First let us recall the construction of the KN stratification of a projective-over-affine variety $X \subset \mathbb{P}^n \times \mathbb{A}^m$ invariant with respect to the action of a reductive group $G$. We let $\mathcal{L} := \mathcal{O}_X(1)$ with a chosen linearization. We fix a $G$-invariant inner product on $\mathfrak{g}$ such that $|\lambda| > 0$ for all nontrivial one-parameter subgroups.

Let $\mathcal{P}$ denote the set of pairs $(\lambda, Z)$ where $\lambda$ is a 1-PS of $G$ and $Z$ is a connected component of $X^\lambda$, modulo the action of $G$. The set of $Z$ appearing in such pairs is finite up to the action of $G$. Define the numerical invariant

$$\mu(\lambda, Z) = \frac{1}{|\lambda|} \text{weight}_\lambda \mathcal{L}|_Z \in \mathbb{R}$$

One constructs the KN stratification iteratively by selecting a $(\lambda_\alpha, Z_\alpha) \in \mathcal{P}$ which maximizes $\mu$ among those $(\lambda, Z)$ for which $Z$ is not contained in
the previously defined strata. One defines the open subset \( Z_\alpha^o \subset Z_\alpha \) not intersecting any higher strata, the attracting set

\[
Y_\alpha^o := \{ x \in X \mid \lim_{t \to 0} \lambda_\alpha(t) \cdot x \in Z_\alpha^o \},
\]

and the new strata \( S_\alpha = G \cdot Y_\alpha^o \). The strata are ordered by the value of the numerical invariant \( \mu \), and it is a non-trivial fact that \( \bar{S}_\alpha \subset S_\alpha \cup \bigcup_{\mu_\beta > \mu_\alpha} S_\beta \). It is evident that the stratification of \( \mathbb{P}^n \times A^m \) induces the stratification of \( X \).

We define the parabolic subgroup \( P_\alpha \subset G \) of all \( p \in G \) such that \( \lambda_\alpha(t)p\lambda_\alpha(t)^{-1} \) has a limit as \( t \to 0 \). In addition we will define \( L_\alpha \subset P_\alpha \) to be the commutant of \( \lambda_\alpha \), it is a Levi component of \( P_\alpha \), so we have the semidirect product sequence

\[
1 \longrightarrow U_\alpha \longrightarrow P_\alpha \longrightarrow L_\alpha \longrightarrow 1
\]

where \( U_\alpha \subset P_\alpha \) is the unipotent radical. The locally closed subvariety \( S_\alpha \) has some special properties with respect to \( \lambda_\alpha \) (see [16], [12] and the references therein):

(S1) By definition there is an open subvariety \( Z_\alpha^o \subset X^\lambda \) fixed such that \( Y_\alpha^o \) is the attracting set \( (1) \) of \( Z_\alpha^o \) under the flow of \( \lambda_\alpha \). The variety \( Z_\alpha^o \) is \( L_\alpha \) equivariant, and \( Y_\alpha^o \) is \( P_\alpha \) equivariant. The map \( \pi : x \mapsto \lim_{t \to 0} \lambda_\alpha(t) \cdot x \) is algebraic and affine, and it is \( P_\alpha \)-equivariant if we let \( P_\alpha \) act on \( Z_\alpha^o \) via the quotient map \( P_\alpha \to L_\alpha \). Thus \( Y_\alpha^o = \text{Spec}_{\mathcal{O}_{Z_\alpha^o}}(\mathcal{A}) \) where \( \mathcal{A} = \mathcal{O}_{Z_\alpha^o} \oplus \bigoplus_{i < 0} \mathcal{A}_i \) is a coherently generated \( P_\alpha \)-equivariant \( \mathcal{O}_{Z_\alpha^o} \) algebra, nonpositively graded with respect to the weights of \( \lambda_\alpha \).

(S2) \( Y_\alpha^o \subset X \) is invariant under \( P_\alpha \) and the canonical map \( G \times_{P_\alpha} Y_\alpha^o \to G \cdot Y_\alpha^o =: S_\alpha \) is an isomorphism.

(S3) Property (S1) implies that the conormal sheaf \( \mathcal{N}_{S_\alpha/X} = \mathcal{I}_{S_\alpha}/\mathcal{I}_{S_\alpha}^2 \) restricted to \( Z_\alpha^o \) has positive weights with respect to \( \lambda_\alpha \).

Note that properties (S1) and (S3) hold for any subvariety which is the attracting set of some \( Z \subset X^\lambda \), so (S2) is the only property essential to the strata arising in GIT theory. Note also that when \( G \) is abelian, \( P_\alpha = G \) and \( Y_\alpha^o = S_\alpha \) for all \( \alpha \), which simplifies the description of the stratification.

Due to the iterative construction of the KN stratification, it will suffice to analyze a single closed stratum \( S \subset X \). We will simplify notation by dropping the index \( \alpha \) everywhere, and calling \( Z = Z_\alpha^o \) and \( Y = Y_\alpha^o \).
Definition 2.1 (KN stratification). Let \( X \) be a quasiprojective variety with the action of a reductive group \( G \). A closed Kirwan-Ness (KN) stratum is a closed subvariety \( S \subset X \) such that there is a \( \lambda \) and an open-and-closed subvariety \( Z \subset X^\lambda \) satisfying properties (S1)-(S3). We will introduce standard names for the morphisms

\[
\begin{array}{ccc}
Z & \xrightarrow{\sigma} & Y \\
\xleftarrow{\pi} & & \xrightarrow{j} \\
& S & \subset X
\end{array}
\]  

(3)

If \( X \) is not smooth along \( Z \), we make the technical hypothesis that there is a \( G \)-equivariant closed immersion \( X \subset X' \) and a KN stratum \( S' \subset X' \) such that \( S \) is a union of connected components of \( S' \cap X \) and \( X' \) is in a neighborhood of \( Z' \).

Let \( X^u \subset X \) be a closed equivariant subvariety. A stratification \( X^u = \bigcup_{\alpha} S_{\alpha} \) indexed by a partially ordered set \( I \) will be called a KN stratification if \( S_{\beta} \subset X - \bigcup_{\alpha > \beta} S_{\alpha} \) is a KN stratum for all \( \beta \).

Remark 2.2. The technical hypothesis is only used for the construction of Koszul systems in Section 3.3. It is automatically satisfied for the GIT stratification of a projective-over-affine variety.

We denote the open complement \( V = X - S \). We will use the notation \( X, S, \) and \( Y \) to denote the stack quotient of these schemes by \( G \). Property (S2) implies that as stacks the natural map \( Y/P \to S/G \) is an equivalence, and we will identify the category of \( G \)-equivariant quasicoherent sheaves on \( S \) with the category of \( P \)-equivariant quasicoherent sheaves on \( Y \) under the restriction functor. We will also use \( j \) to denote \( Y/P \to X/G \).

A KN stratum has a particularly nice structure when \( X \) is smooth along \( Z \). In this case \( Z \) must also be smooth, and \( Y \) is a locally trivial bundle of affine spaces over \( Z \). By (S2), \( S \) is smooth and hence \( S \subset X \) is a regular embedding.

When \( X \) is not smooth, \( Z, Y, \) and \( S \) can all be singular. In order to extend our analysis beyond the smooth setting, we will need two additional properties

\( (A) \) \( \pi : Y \to Z \) is a locally trivial bundle of affine spaces

\( (L+) \) The derived restriction of the relative cotangent complex \( L^r S/X \) has nonnegative weights w.r.t. \( \lambda \).

which do not automatically hold for a KN stratum. We will use the construction of the cotangent complex in characteristic 0 as discussed in ??.

Note that when \( X \) is smooth along \( Z \), \( L^r S/X \simeq \mathcal{R}S/X [1] \) is locally free on \( S \), so Property (L+) follows from (S3).

Example 2.3. Let \( X \subset \mathbb{P}^n \) be a projective variety with homogeneous coordinate ring \( A \). The affine cone \( \text{Spec} \ A \) has \( \mathbb{G}_m \) action given by the nonnegative grading of \( A \) and the unstable locus is \( Z = Y = S = \text{the cone point} \). \( \mathcal{O}_S \) can be resolved as a semi-free graded dg-algebra over \( A \), \( (A[x_1, x_2, \ldots], d) \to \)
$\mathcal{O}_S$ with generators of positive weight. Thus $\mathbf{L}_{S/Z}^i = \mathcal{O}_S \otimes \Omega^1_{[x_1, \ldots]/A}$ has positive weights. The Property (A) is automatic. In this case Theorem 2.5 is essentially Serre’s theorem on the derived category of a projective variety.

**Example 2.4.** Consider the graded polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f)$ where the $x_i$ have positive degrees and the $y_i$ have negative degrees and $f$ is a homogeneous polynomial such that $f(0) = 0$. This corresponds to a linear action of $\mathbb{G}_m$ on an equivariant hypersurface $X_f$ in the affine space $\mathbb{A}^n_x \times \mathbb{A}^m_y$. Assume that we have chosen the linearization such that $S = \{0\} \times \mathbb{A}^m_y \cap X_f$. One can compute

$$L_{S/X_f}^i \begin{cases} (\mathcal{O}_S x_1 \oplus \cdots \oplus \mathcal{O}_S x_n)[1], & \text{if } f \notin (x_1, \ldots, x_n) \\ (\mathcal{O}_S f \rightarrow \mathcal{O}_S dx_1 \oplus \cdots \oplus \mathcal{O}_S dx_n)[1] & \text{if } f \in (x_1, \ldots, x_n) \end{cases}$$

Thus $S \subset X_f$ satisfies Property (L+) iff either $\deg f \geq 0$, in which case $f \in (x_1, \ldots, x_n)$, or if $\deg f < 0$ but $f \notin (x_1, \ldots, x_n)$. Furthermore, Property (A) amounts to $S$ being an affine space, which happens iff $\deg f \geq 0$ so that $S = \mathbb{A}^m_y$, or $\deg f = -1$ with a nontrivial linear term in the $y_i$. Note in particular that in order for $X_f$ to satisfy these properties with respect to the stratum of the the opposite linearization, then we are left with only two possibilities: either $\deg f = 0$ or $\deg f = \pm 1$ with nontrivial linear terms. This illustrates the non-vacuousness of Properties (A) and (L+).

### 2.2. Statement and proof of the main theorem.

As the statement of Theorem 1.1 indicates, we will construct a splitting of $D^b(\mathcal{X}) \rightarrow D^b(\mathcal{X}^{ss})$ by identifying a subcategory $\mathcal{G}_w \subset D^b(\mathcal{X})$ that is mapped isomorphically onto $D^b(\mathcal{X}^{ss})$. In fact we will identify $\mathcal{G}_w$ as the middle factor in a large semiorthogonal decomposition of $D^b(\mathcal{X})$.

We denote a *semiorthogonal decomposition* of a triangulated category $\mathcal{D}$ by full triangulated subcategories $\mathcal{A}_i$ as $\mathcal{D} = \langle \mathcal{A}_n, \ldots, \mathcal{A}_1 \rangle$.\[6\] This means that all morphisms from objects in $\mathcal{A}_i$ to objects in $\mathcal{A}_j$ are zero for $i < j$, and for any object of $E \in \mathcal{D}$ there is a sequence $0 \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_n \rightarrow E$ with $\text{Cone}(E_{i-1} \rightarrow E_i) \in \mathcal{A}_i$, which is necessarily unique and thus functorial.\[3\] In our applications $\mathcal{D}$ will always be a pre-triangulated category, in which case if $\mathcal{A}_i \subset \mathcal{D}$ are full pre-triangulated categories then we will abuse the notation $\mathcal{D} = \langle \mathcal{A}_n, \ldots, \mathcal{A}_1 \rangle$ to mean that there is a semiorthogonal decomposition of homotopy categories, in which case $\mathcal{D}$ is uniquely identified with the gluing of the $\mathcal{A}_i$.\[?\]

A *baric decomposition* is simply a filtration of a triangulated category $\mathcal{D}$ by right-admissible triangulated subcategories, i.e. a family of semiorthogonal decompositions $\mathcal{D} = \langle \mathcal{D}_{\leq w}, \mathcal{D}_{\geq w} \rangle$ such that $\mathcal{D}_{\geq w} \supset \mathcal{D}_{\geq w+1}$, and thus $\mathcal{D}_{\leq w} \subset$

---

\[3\]There are two additional equivalent ways to characterize a semiorthogonal decomposition: 1) the inclusion of the full subcategory $\mathcal{A}_i \subset \langle \mathcal{A}_i, \mathcal{A}_{i-1}, \ldots, \mathcal{A}_1 \rangle$ admits a left adjoint $\forall i$, or 2) the subcategory $\mathcal{A}_i \subset \langle \mathcal{A}_n, \ldots, \mathcal{A}_1 \rangle$ is right admissible $\forall i$. In some contexts one also requires that each $\mathcal{A}_i$ be admissible in $\mathcal{D}$, but we will not require this here. See [6] for further discussion.
\[ \mathcal{D}_{<w+1}, \text{for all } w. \] This notion was introduced and used construct 'staggered' \( t \)-structures on equivariant derived categories of coherent sheaves.\[^1\]

Although the connection with GIT was not explored in the original development of the theory, baric decompositions seem to be the natural structure arising on the derived category of the unstable locus in geometric invariant theory. The key to our proof will be to consider a single closed KN stratum \( \mathfrak{S} \subset \mathfrak{X} \) and construct baric decompositions of \( \mathcal{D}^b(\mathfrak{S}) \) in Proposition 3.14 and of \( \mathcal{D}^b(\mathfrak{X}) \), the bounded derived category of complexes of coherent sheaves on \( \mathfrak{X} \) whose homology is supported on a KN stratum \( \mathfrak{S} \), in Proposition 3.21. We will postpone a detailed analysis of the homological structure of a single KN stratum to Section 3 – here we apply the results of that section iteratively to a stratification with multiple KN strata.

**Theorem 2.5** (derived Kirwan surjectivity). Let \( \mathfrak{X} = X/G \) be a stack quotient of a quasiprojective variety by a reductive group, let \( \mathfrak{X}^{ss} \subset \mathfrak{X} \) be an open substack, and let \( \{\mathfrak{S}_\alpha\}_{\alpha \in I} \) be a KN stratification (Definition 2.1) of \( \mathfrak{X}^u = \mathfrak{X} \setminus \mathfrak{X}^{ss} \). Assume that each \( \mathfrak{S}_\alpha \) satisfies Properties (A) and (L+). Define the integers

\[
\begin{align*}
\lambda_i &:= \text{weight}_{\lambda_i} \det(N_{S_i}X_i) \quad (4) \\
\end{align*}
\]

For each KN stratum, choose an integer \( w_i \in \mathbb{Z} \). We refer to Section 3.5 for the definition of the shriek pullback \( \sigma_i^! : \mathcal{D}^b(\mathfrak{X}) \to \mathcal{D}^b(\mathfrak{X}_i) \). Define the full subcategories of \( \mathcal{D}^b(\mathfrak{X}) \)

\[
\begin{align*}
\mathcal{D}^b_{\geq w}(\mathfrak{X}) &:= \{ F^* \in \mathcal{D}^b(\mathfrak{X}) \mid \forall i, \lambda_i \text{ weights of } \mathcal{H}^*(\sigma_i^* F^*) \text{ are } \geq w_i \} \\
\mathcal{D}^b_{< w}(\mathfrak{X}) &:= \{ F^* \in \mathcal{D}^b(\mathfrak{X}) \mid \forall i, \lambda_i \text{ weights of } \mathcal{H}^*(\sigma_i^* F^*) \text{ are } < w_i + \lambda_i \} \\
\mathcal{G}_w &:= \left\{ F^* \mid \forall i, \mathcal{H}^*(\sigma_i^* F^*) \text{ has weights } \geq w_i, \text{ and } \mathcal{H}^*(\sigma_i^* F^*) \text{ has weights } < w_i + \lambda_i \right\} \\
\end{align*}
\]

Then there are semiorthogonal decompositions

\[
\begin{align*}
\mathcal{D}^b_{\geq w}(\mathfrak{X}) &= \langle \mathcal{D}^b_{\geq w}(\mathfrak{X})_{< w}, \mathcal{D}^b_{\geq w}(\mathfrak{X})_{\geq w} \rangle \quad (5) \\
\mathcal{D}^b(\mathfrak{X}) &= \langle \mathcal{D}^b_{< w}(\mathfrak{X}), \mathcal{G}_w, \mathcal{D}^b_{\geq w}(\mathfrak{X})_{\geq w} \rangle \quad (6) \\
\end{align*}
\]

and the restriction functor \( \iota^* : G_w \to \mathcal{D}^b(\mathfrak{X}^{ss}) \) is an equivalence of categories. We have \( \text{Perf}_{\mathfrak{X}^u}(\mathfrak{X}_{\geq w}) \Rightarrow \mathcal{D}^b_{\geq w}(\mathfrak{X}_{\geq w}) \subset \mathcal{D}^b_{\geq w}(\mathfrak{X})_{\geq w+w} \).

If \( \mathfrak{X} \) is smooth in a neighborhood of \( \mathfrak{X}^u \), then Properties (A) and (L+) hold automatically, and we define

\[
\begin{align*}
\eta_i &:= \text{weight}_{\lambda_i} \det(N_{S_i}X^\vee) \\
&= \text{weight}_{\lambda_i} \det(N_{X^\vee}^\vee X) - \text{weight}_{\lambda_i} \det(g_{\lambda_i>0}) \quad (7) \\
\end{align*}
\]

Then we have alternate characterizations

\[
\begin{align*}
\mathcal{D}^b_{< w}(\mathfrak{X}) &:= \{ F^* \in \mathcal{D}^b(\mathfrak{X}) \mid \forall i, \lambda_i \text{ weights of } \mathcal{H}^*(\sigma_i^* F^*) \text{ are } < w_i + \eta_i \} \\
\mathcal{G}_w &:= \{ F^* \mid \forall i, \mathcal{H}^*(\sigma_i^* F^*) \text{ has weights in } [w_i, \eta_i] \text{ w.r.t. } \lambda_i \} \\
\end{align*}
\]
Proof. Choose a total ordering of $I$, $\alpha_0 > \alpha_1 > \cdots$ such that $\alpha_n$ is maximal in $I \setminus \{\alpha_0, \ldots, \alpha_{n-1}\}$, so that $\mathcal{G}_{\alpha_n}$ is closed in $\mathfrak{X} \setminus \mathcal{G}_{\alpha_n} \cup \cdots \cup \mathcal{G}_{\alpha_{n-1}}$. Introduce the notation $\mathcal{G}^n = \bigcup_{i<n} \mathcal{G}_{\alpha_i}$. $\mathcal{G}^n \subset \mathfrak{X}$ is closed and admits a KN stratification by the $n$ strata $\mathcal{G}_{\alpha_i}$ for $i < n$, so we will proceed by induction on $n$. The base case is Theorem 3.31.

Assume the theorem holds for $\mathcal{G}^n \subset \mathfrak{X}$, so $D^b(\mathfrak{X}) = \langle D^b_{\mathcal{G}^n}(\mathfrak{X})_{<q}, G^n_q, D^b_{\mathcal{G}^n}(\mathfrak{X})_{\geq q} \rangle$ and restriction maps $G^n_q$ isomorphically onto $D^b(\mathfrak{X} \setminus \mathcal{G}^n)$. $\mathcal{G}_{\alpha_n} \subset \mathfrak{X} \setminus \mathcal{G}^n$ is a closed KN stratum, so Theorem 3.31 gives a semiorthogonal decomposition of $G^n_q \simeq D^b(\mathfrak{X} \setminus \mathcal{G}^n)$ which we combine with the previous semiorthogonal decomposition

$$D^b(\mathfrak{X}) = \langle D^b_{\mathcal{G}^n}(\mathfrak{X})_{<q}, D^b_{\mathcal{G}_{\alpha_n}}(\mathfrak{X} \setminus \mathcal{G}^n)_{<q(\alpha)}, G^n_{q+1}, D^b_{\mathcal{G}_{\alpha_n}}(\mathfrak{X} \setminus \mathcal{G}^n)_{\geq q(\alpha)}, D^b_{\mathcal{G}^n}(\mathfrak{X})_{\geq q} \rangle$$

The first two pieces correspond precisely to $D^b_{\mathcal{G}_{\alpha_{n+1}}}(\mathfrak{X})_{<q}$ and the last two pieces correspond to $D^b_{\mathcal{G}_{\alpha_{n+1}}}(\mathfrak{X})_{\geq q}$. The theorem follows by induction.

Remark 2.6. The semiorthogonal decomposition in this theorem can be refined further using ideas of Kawamata[?], and Ballard, Favero, Katzarkov[5] (See Amplification 3.23 below for a discussion in this context).

Corollary 2.7. Let $Z$ be a quasiprojective scheme and $\mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}_i$ a coherently generated sheaf of algebras over $Z$, with $\mathcal{A}_0 = \mathcal{O}_Z$. Let $j : Z \hookrightarrow \text{Spec}(\mathcal{A})$ be the inclusion. There is an infinite semiorthogonal decomposition,

$$D^b(\text{gr} \mathcal{A}) = \langle \ldots, D^b(Z)_{w-1} : G_w, D^b(Z)_w, D^b(Z)_{w+1}, \ldots \rangle$$

where $D^b(Z)_w$ denotes the subcategory generated by $j_* D^b(Z) \otimes \mathcal{O}_X(-w)$, and

$$G_w = \left\{ F^* \in D^b(X/C^*) \left| \begin{array}{c} \mathcal{H}^*(j^* F^*) \text{ has weights } \geq w, \text{ and } \\ \mathcal{H}^*(j^! F^*) \text{ has weights } < w \end{array} \right. \right\}$$

and the restriction functor $G_w \to D^b(\text{Proj} \mathcal{A})$ is an equivalence.

2.3. Explicit constructions of the splitting, and Fourier-Mukai kernels. Given an $F^* \in D^b(\mathfrak{X}^{ss}$), one can extend it uniquely up to weak equivalence to a complex in $G_q$. Due to the inductive nature of Theorem 2.5, the extension can be complicated to construct. We will discuss a procedure for extending over a single stratum at the end of Section 3, and one must repeat this for every stratum of $\mathfrak{X}^{ss}$.

Fortunately, it suffices to directly construct a single universal extension. Consider the product $\mathfrak{X}^{ss} \times \mathfrak{X} = (\mathfrak{X}^{ss} \times X)/(G \times G)$, and the open substack $\mathfrak{X}^{ss} \times \mathfrak{X}^{ss}$ whose complement has the KN stratification $\mathfrak{X}^{ss} \times \mathcal{G}_n$. One can uniquely extend the diagonal $\mathcal{O}_{\mathfrak{X}^{ss} \times \mathfrak{X}^{ss}}$ to a sheaf $\mathcal{O}_{\Delta}$ in the subcategory $G_q$ with respect to this stratification. The Fourier-Mukai transform $D^b(\mathfrak{X}^{ss}) \to D^b(\mathfrak{X})$ with kernel $\mathcal{O}_{\Delta}$ has image in the subcategory $G_q$ and is the identity over $\mathfrak{X}^{ss}$. Thus for any $F^* \in D^b(\mathfrak{X}^{ss})$, $(p_2)_*(\mathcal{O}_{\Delta} \otimes p_1^*(F^*))$ is the unique extension of $F^*$ to $G_q$. 

3. Homological structures on the unstable strata

In this section we will study in detail the homological properties of a single closed KN stratum $\mathfrak{S} := S/G \subset X$ as defined in 2.1. We establish a multiplicative baric decomposition of $D^b(\mathfrak{S})$, and when $\mathfrak{S} \subset X$ satisfies Property (L+), we extend this to a multiplicative baric decomposition of $D^b_S(X)$, the derived category of complexes of coherent sheaves on $X$ whose restriction to $\mathfrak{U} = X - \mathfrak{S}$ is acyclic. Then we use these baric decompositions to construct our main semiorthogonal decompositions of $D^b(X)$.

Recall the structure of a KN stratum (3) and the associated parabolic subgroup (2). By Property (S1), $\mathfrak{S} := S/G \simeq Y/P$ via the $P$-equivariant inclusion $Y \subset S$, so we will identify quasicoherent sheaves on $\mathfrak{S}$ with $P$-equivariant quasicoherent $O_Y$-modules. Furthermore, we will let $P$ act on $Z$ via the projection $P \to L$. Again by Property (S1), we have $Y/P = \text{Spec}_Z(A)/P$, where $A$ is a coherently generated $O_Z$-algebra with $A_i = 0$ for $i > 0$, and $A_0 = O_Z$. Thus we have identified quasicoherent sheaves on $\mathfrak{S}$ with quasicoherent $A$-modules on $\mathfrak{Z}' := Z/P$.

**Remark 3.1.** The stack $\mathfrak{Z} := Z/L$ is perhaps more natural than the stack $\mathfrak{Z}'$. The projection $\pi : Y \to Z$ intertwines the respective $P$ and $L$ actions via $P \to L$, hence we get a projection $\mathfrak{S} \to \mathfrak{Z} := Z/L$. Unlike the map $\mathfrak{S} \to \mathfrak{Z}'$, this projection admits a section $Z/L \to Y/P$. In other words, the projection $A \to A_0 = O_Z$ is $\lambda(\mathbb{C}^*)$-equivariant, but not $P$-equivariant. We choose to work with $\mathfrak{Z}'$, however, because the map $\mathfrak{S} \to \mathfrak{Z}$ is not representable, so the description of quasicoherent $\mathfrak{S}$ modules in terms of “$\mathfrak{Z}$-modules with additional structure” is less straightforward.

We will use the phrase $O_Z$-module to denote a quasicoherent sheaf on the stack $\mathfrak{Z}' = Z/P$, assuming quasicoherence and $P$-equivariance unless otherwise specified. $\lambda$ fixes $Z$, so equivariant $O_Z$ modules have a natural grading by the weight spaces of $\lambda$, and we will use this grading often.

**Lemma 3.2.** For any $F \in \text{QCoh}(\mathfrak{Z}')$ and any $w \in \mathbb{Z}$, the submodule $F_{\geq w} := \sum_{i \geq w} F_i$ of sections of weight $\geq w$ with respect to $\lambda$ is $P$ equivariant.

**Proof.** $\mathbb{C}^*$ commutes with $L$, so $F_{\geq w}$ is an equivariant submodule with respect to the $L$ action. Because $U \subset P$ acts trivially on $Z$, the $U$-equivariant structure on $F$ is determined by a coaction $a : F \to k[U] \otimes F$ which is equivariant for the $\mathbb{C}^*$ action. We have

$$a(F_{\geq w}) \subset (k[U] \otimes F)_{\geq w} = \bigoplus_{i+j \geq w} k[U]_i \otimes F_j \subset k[U] \otimes F_{\geq w}$$

The last inclusion is due to the fact that $k[U]$ is non-positively graded, and it implies that $F_{\geq w}$ is equivariant with respect to the $U$ action as well. Because we have a semidirect product decomposition $P = UL$, it follows that $F_{\geq p}$ is an equivariant submodule with respect to the $P$ action. \qed

**Remark 3.3.** This lemma is a global version of the observation that for any $P$-module $M$, the subspace $M_{\geq w}$ with weights $\geq w$ with respect to $\lambda$ is
a $P$-submodule, which can be seen from the coaction $M \to k[P] \otimes M$ and the fact that $k[P]$ is nonnegatively graded with respect to $\lambda$.

It follows that any $F \in \mathsf{QCoh}(\mathcal{Z}')$ has a functorial factorization $F_{\geq w} \hookrightarrow F \twoheadrightarrow F_{< w}$. Note that as $\mathbb{C}^*\text{-equivariant}$ instead of $P\text{-equivariant} \mathcal{O}_Z$-modules there is a natural isomorphism $F \simeq F_{\geq w} \oplus F_{< w}$. Thus the functors $(\bullet)_{\geq w}$ and $(\bullet)_{< w}$ are exact, and that if $F$ is locally free, then $F_{\geq w}$ and $F_{< w}$ are locally free as well.

We define $\mathsf{QCoh}(\mathcal{Z}')_{\geq w}$ and $\mathsf{QCoh}(\mathcal{Z}')_{< w}$ to be the full subcategories of $\mathsf{QCoh}(\mathcal{Z}')$ consisting of sheaves supported in weight $\geq w$ and weight $< w$ respectively. They are both Serre subcategories, they are orthogonal to one another, $(\bullet)_{\geq w}$ is right adjoint to the inclusion $\mathsf{QCoh}(\mathcal{Z}')_{\geq w} \subset \mathsf{QCoh}(\mathcal{Z}')$, and $(\bullet)_{< w}$ is left adjoint to the inclusion $\mathsf{QCoh}(\mathcal{Z}')_{< w} \subset \mathsf{QCoh}(\mathcal{Z}')$.

**Lemma 3.4.** Any $F \in \mathsf{QCoh}(\mathcal{Z}')_{< w}$ admits an injective resolution $F \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ such that $\mathcal{I}^i \in \mathsf{QCoh}(\mathcal{Z}')_{< w}$. Likewise any $F \in \mathsf{Coh}(\mathcal{Z}')_{\geq w}$ admits a locally free resolution $\cdots \to E_1 \to E_0 \to F$ such that $E_i \in \mathsf{Coh}(\mathcal{Z}')_{\geq w}$.

**Proof.** First assume $F \in \mathsf{QCoh}(\mathcal{Z}')_{< w}$, and let $F \to \mathcal{I}^0$ be the injective hull of $F$. Then $\mathcal{I}^0_{\geq w} \cap F_{< w} = 0$, hence $\mathcal{I}^0_{\geq w} = 0$ because $\mathcal{I}^0$ is an essential extension of $F$. $\mathsf{QCoh}(\mathcal{Z}')_{< w}$ is a Serre subcategory, so $\mathcal{I}^0/F \in \mathsf{QCoh}(\mathcal{Z}')_{< w}$ as well, and we can inductively build an injective resolution with $\mathcal{I}^i \in \mathsf{QCoh}(\mathcal{Z}')_{< w}$.

Next assume $F \in \mathsf{Coh}(\mathcal{Z}')_{\geq w}$. Choose a surjection $E \to F$ where $E$ is locally free. Then $E_0 := E_{\geq w}$ is still locally free, and $E_{\geq w} \to F$ is still surjective. Because $\mathsf{Coh}(\mathcal{Z}')_{\geq w}$ is a Serre subcategory, $\ker(E_0 \to F) \in \mathsf{Coh}(\mathcal{Z}')_{\geq w}$ as well, so we can inductively build a locally free resolution with $E_i \in \mathsf{Coh}(\mathcal{Z}')_{\geq w}$.

We will use this lemma to study the subcategories of $\mathcal{D}^b(\mathcal{Z}')$ generated by $\mathsf{Coh}(\mathcal{Z}')_{\geq w}$ and $\mathsf{Coh}(\mathcal{Z}')_{< w}$. Define the full triangulated subcategories
\[
\mathcal{D}^b(\mathcal{Z}')_{\geq w} = \{ F^* \in \mathcal{D}^b(\mathcal{Z}') | \mathcal{H}^i(F^*) \in \mathsf{Coh}(\mathcal{Z}')_{\geq w} \}
\]
\[
\mathcal{D}^b(\mathcal{Z}')_{< w} = \{ F^* \in \mathcal{D}^b(\mathcal{Z}') | \mathcal{H}^i(F^*) \in \mathsf{Coh}(\mathcal{Z}')_{< w} \}
\]

For any complex $F^*$ we have the canonical short exact sequence
\[
0 \to F^*_{\geq w} \to F^* \to F^*_{< w} \to 0 \tag{8}
\]
If $F^* \in \mathcal{D}^b(\mathcal{Z}')_{\geq w}$ then the first arrow is a quasi-isomorphism, because $(\bullet)_{\geq w}$ is exact. Likewise for the second arrow if $F^* \in \mathcal{D}^b(\mathcal{Z}')_{< w}$. Thus $F^* \in \mathcal{D}^b(\mathcal{Z}')_{\geq w}$ iff it is quasi-isomorphic to a complex of sheaves in $\mathsf{Coh}(\mathcal{Z}')_{\geq w}$ and likewise for $\mathcal{D}^b(\mathcal{Z}')_{< w}$.

---

\footnote{The injective hull exists because $\mathsf{QCoh}(\mathcal{Z}')$ is cocomplete and taking filtered colimits is exact.}
Proposition 3.5. These subcategories constitute a baric decomposition $D^b(3') = \{D^b(3')_{<w}, D^b(3')_{\geq w}\}$. This baric decomposition is multiplicative in the sense that

$$\text{Perf}(3')_{\geq w} \otimes D^b(3')_{\geq v} \subset D^b(3')_{\geq v+w}.$$  

It is bounded, meaning that every object lies in $D_{\geq w} \cap D_{<v}$ for some $w,v$. The baric truncation functors, the adjoints of the inclusions $D_{\geq w}, D_{<w} \subset D^b(3')$, are exact.

Proof. If $A \in \text{Coh}(3')_{\geq w}$ and $B \in \text{Coh}(3')_{<w}$, then by Lemma 3.4 we resolve $B$ by injectives in $\text{Qcoh}(3')_{<w}$, and thus $R\text{Hom}(A,B) \simeq 0$. It follows that $D^b(3')_{\geq w}$ is left orthogonal to $D^b(3')_{<w}$. $\text{Qcoh}(3')_{\geq w}$ and $\text{Qcoh}(3')_{<w}$ are Serre subcategories, so $F^w_{\geq w} \in D^b(3')_{\geq w}$ and $F^w_{<w} \in D^b(3')_{<w}$ for any $F^w \in D^b(3')$. Thus the natural sequence (8) shows that we have a baric decomposition, and that the right and left truncation functors are the exact functors $(\bullet)_{\geq w}$ and $(\bullet)_{<w}$ respectively. Boundedness follows from the fact that coherent equivariant $O_Z$-modules must be supported in finitely many $\lambda$ weights. Multiplicativity is also straightforward to verify. \hfill $\square$

Remark 3.6. A completely analogous baric decomposition holds for $3$ as well. In fact, for $3$ the two factors are mutually orthogonal.

3.1. Quasicoherent sheaves on $\mathcal{S}$. The closed immersion $\sigma : Z \hookrightarrow Y$ is $L$ equivariant, hence it defines a map of stacks $\sigma : 3 \rightarrow \mathcal{S}$. Recall also that because $\pi : \mathcal{S} \rightarrow 3'$ is affine, the derived pushforward $R\pi_* = \pi_*$ is just the functor which forgets the $A$-module structure. Define the thick triangulated subcategories

$$D^b(\mathcal{S})_{\leq w} = \{F^w \in D^b(\mathcal{S}) | \pi_* F^w \in D^b(3')_{\leq w}\}$$
$$D^b(\mathcal{S})_{\geq w} = \{F^w \in D^b(\mathcal{S}) | L\sigma^* F^w \in D^-(3)_{\geq w}\}$$

In the rest of this subsection we will analyze these two categories and show that they constitute a multiplicative baric decomposition.

Complexes on $\mathcal{S}$ of the form $A \otimes E^*$, where each $E^i$ is a locally free sheaf on $3'$, will be of prime importance. Note that the differential $d^i : A \otimes E^i \rightarrow A \otimes E^{i+1}$ is not necessarily induced from a differential $E^i \rightarrow E^{i+1}$. However we observe

Lemma 3.7. If $E \in \text{Qcoh}(3)$, then $A \cdot (A \otimes E)_{\geq w} = A \otimes E_{\geq w}$, where the left side denotes the smallest $A$-submodule containing the $O_Z$-submodule $(A \otimes E)_{\geq w}$.

Proof. By definition the left hand side is the $A$-submodule generated by $\bigoplus_{i+j \geq w} A_i \otimes E_j$ and the left hand side is generated by $\bigoplus_{j \geq w} A_0 \otimes E_j \subset A \otimes E_{\geq w}$. These $O_Z$-submodules clearly generate the same $A$-submodule. \hfill $\square$

This guarantees that $\text{im} d^i \subset A \otimes E^{i+1}_{\geq w}$, so $A \otimes E^*_{\geq w}$ is a subcomplex, and $E_{\geq w}$ is a direct summand as a non-equivariant $O_Z$-module, so we have a
canonical short exact sequence of complexes in \( \text{QCoh}(\mathcal{S}) \)

\[
0 \to A \otimes E^*_{\geq w} \to A \otimes E^* \to A \otimes E^*_w \to 0
\]  

(9)

**Proposition 3.8.** \( F^* \in D^b(\mathcal{S})_{\geq w} \) iff it is quasi-isomorphic to a right-bounded complex of the sheaves of the form \( A \otimes E^i \) with \( E^i \in \text{Coh}(\mathcal{Z})_{\geq w} \) locally free.

First we observe the following extension of Nakayama’s lemma to the derived category

**Lemma 3.9 (Nakayama).** Let \( F^* \in D^- (\mathcal{S}) \) with coherent cohomology. If \( L\sigma^* F^* \simeq 0 \), then \( F^* \simeq 0 \).

**Proof.** The natural extension of Nakayama’s lemma to stacks is the statement that the support of a coherent sheaf is closed. In our setting this means that if \( G \in \text{Coh}(\mathcal{S}) \) and \( G \otimes \mathcal{O}_Z = 0 \) then \( G = 0 \), because \( \text{supp}(G) \cap Z = \emptyset \) and every nonempty closed substack of \( S \) intersects \( Z \) nontrivially.

If \( H^i(F^*) \) is the highest nonvanishing cohomology group of a right bounded complex, then \( H^i(L\sigma^* F^*) \simeq \sigma^* H^i(F^*) \). By Nakayama’s lemma \( \sigma^* H^i(F^*) = 0 \Rightarrow H^i(F^*) = 0 \), so we must have \( \sigma^* H^i(F^*) \neq 0 \) as well. \( \square \)

**Remark 3.10.** Note another consequence of Nakayama’s lemma: if \( F^* \) is a complex of locally free sheaves on \( \mathcal{S} \) and \( \mathcal{H}^i(F^* \otimes \mathcal{O}_Z) = 0 \), then \( \mathcal{H}^i(F^*) = 0 \), because the canonical map on stalks \( \mathcal{H}^i(F^*) \otimes k(z) \to \mathcal{H}^i(F^* \otimes k(z)) \) is an isomorphism if it surjective. In particular if \( E^i \in \text{Coh}(\mathcal{Z}) \) are locally free and \( \sigma^*(A \otimes E^*) = E^* \) has bounded cohomology, then \( A \otimes E^* \) has bounded cohomology as well.

**Proof of Proposition 3.8.** We assume that \( L\sigma^* F^* \in D^b(\mathcal{Z})_{\geq w} \). Choose a right bounded presentation by locally frees \( A \otimes E^* \simeq F^* \) and consider the canonical sequence (9).

Restricting to \( \mathcal{Z} \) gives a short exact sequence \( 0 \to E^*_{\geq w} \to E^* \to E^*_w \to 0 \). The first and second terms have homology in \( \text{Coh}(\mathcal{Z})_{\geq w} \), and the third has homology in \( \text{Coh}(\mathcal{Z})_{< w} \). These two categories are orthogonal, so it follows from the long exact homology sequence that \( E^*_w \) is acyclic. Thus by Nakayama’s lemma \( A \otimes E^*_w \) is acyclic and \( F^* \simeq A \otimes E^*_w \).

Using this characterization of \( D^b(\mathcal{S})_{\geq w} \) we have semiorthogonality

**Lemma 3.11.** If \( F^* \in D^- (\mathcal{S})_{\geq w} \) and \( G^* \in D^+(\mathcal{S})_{< w} \), then \( R\text{Hom}_{\mathcal{S}}(F^*, G^*) = 0 \).

**Proof.** By Proposition 3.8 if suffices to prove the claim for \( F^* = A \otimes E \) with \( E \in \text{Coh}(\mathcal{Z})_{\geq w} \) locally free. Then \( A \otimes E \simeq L\pi^* E \), and the derived adjunction gives \( R\text{Hom}_{\mathcal{S}}(L\pi^* E, F^*) \simeq R\text{Hom}_{\mathcal{Z}}(E, R\pi_* F^*) \). \( \pi \) is affine, so \( R\pi_* F^* \simeq \pi_* F^* \in D^+(\mathcal{Z})_{< w} \). The claim follows from the fact that \( \text{QCoh}(\mathcal{Z})_{\geq w} \) is left orthogonal to \( D^+(\mathcal{Z})_{< w} \). \( \square \)
Remark 3.12. The category of coherent $\mathcal{G}$ modules whose weights are $< w$ is a Serre subcategory of Coh($\mathcal{G}$) generating $\text{D}^b(\mathcal{G})_{<w}$, but there is no analogue for $\text{D}^b(\mathcal{G})_{\geq w}$. Consider for instance, when $G$ is abelian there is a short exact sequence $0 \rightarrow A_{<0} \rightarrow A \rightarrow O_Z \rightarrow 0$. This nontrivial extension shows that $R\text{Hom} \mathcal{G}(O_Z, A_{<0}) \neq 0$ even though $O_Z$ has nonnegative weights.

Every $F \in \text{Coh}(\mathcal{G})$ has a highest weight submodule as an equivariant $O_Z$-module $F_{\geq h} \neq 0$ where $F_{\geq w} = 0$ for $w > h$. Furthermore, because $A_{<0}$ has strictly negative weights the map $(F)_{\geq h} \rightarrow (F \otimes O_Z)_{\geq h}$ is an isomorphism of $L$-equivariant $O_Z$-modules. Using the notion of highest weight submodule we prove

Proposition 3.13. If $A \otimes E^*$ is a right-bounded complex with bounded cohomology, then $E^*_{\geq w} := (\sigma^*(A \otimes E^*))_{\geq w}$ has bounded cohomology and thus so does $A \otimes E^*_{\geq w}$ by Remark 3.10. If $A \otimes E^*$ is perfect, then so are $E^*_{\geq w}$ and $A \otimes E^*_{\geq w}$.

Proof. We define the subquotient $A \otimes E^*_{[a,b]} = A \otimes (E^*_{\geq w})_{<b}$ for any $\infty \leq a < b \leq \infty$, noting that the functors commute so order doesn’t matter. The generalization of the short exact sequence (9) for $a < b < c$ is

$$0 \rightarrow A \otimes E^*_{[b,c]} \rightarrow A \otimes E^*_{[a,c]} \rightarrow A \otimes E^*_{[a,b]} \rightarrow 0 \quad (10)$$

We will use this sequence to prove the claim by descending induction.

First we show that for $W$ sufficiently high, $A \otimes E^*_{\geq W} \simeq 0$. By Nakayama’s lemma and the fact that $A \otimes E^*$ has bounded cohomology, it suffices to show $(L\sigma^*F)_{\geq W} \simeq 0$ for any $F \in \text{Coh}(\mathcal{G})$, and this follows by constructing a resolution of $F$ by vector bundles whose weights are $\leq$ the highest weight of $F$.

Now assume that the claim is true for $A \otimes E^*_{\geq w+1}$. It follows from the sequence (9) that $A \otimes E^*_{<w+1}$ has bounded cohomology. The complex $E^*_{[w,w+1]}$ is precisely the highest weight space of $A \otimes E^*_{<w+1}$, and thus has bounded cohomology as well. Applying $\sigma^*$ to sequence (10) gives

$$0 \rightarrow E^*_{\geq w+1} \rightarrow E^*_{\geq w} \rightarrow E^*_{[w,w+1]} \rightarrow 0,$$

thus $E^*_{\geq w}$ has bounded cohomology and the result follows by induction.

The argument for perfect complexes similar to the previous paragraph. By induction $A \otimes E^*_{<w+1}$ is perfect, thus so is $\sigma^*(A \otimes E^*_{<w+1})$ and its highest weight space $E^*_{[w,w+1]}$. Because $E^*_{[w,w+1]}$ is concentrated in a single weight, the differential on $A \otimes E^*_{[w,w+1]}$ is induced from the differential on $E^*_{[w,w+1]}$, i.e. $A \otimes E^*_{[w,w+1]} = L\pi^*(E^*_{[w,w+1]})$. It follows that $A \otimes E^*_{[w,w+1]}$ is perfect, and thus so is $A \otimes E^*_{\geq w}$ by the exact sequence (10).

Proposition 3.14. The categories $\text{D}^b(\mathcal{G}) = \langle \text{D}^b(\mathcal{G})_{<w}, \text{D}^b(\mathcal{G})_{\geq w} \rangle$ constitute a multiplicative baric decomposition. This restricts to a multiplicative baric decomposition of Perf($\mathcal{G}$), which is bounded. If $Z \hookrightarrow Y$ has finite Tor dimension, for instance if Property (A) holds, then the baric decomposition on $\text{D}^b(\mathcal{G})$ is bounded as well.
Lemma 3.11 implies $D^b(\mathcal{G})_{\geq w}$ is left orthogonal to $G^* \in D^b(\mathcal{G})_{<w}$. In order to obtain left and right truncations for $F' \in D^b(\mathcal{G})$ we choose a presentation of the form $A \otimes E^*$ with $E^* \in \text{Coh}(\mathfrak{F})$ locally free. The canonical short exact sequence (9) gives an exact triangle $A \otimes E^*_{\geq w} \to F' \to A \otimes E^*_{<w} \to$. By Proposition 3.13 all three terms have bounded cohomology, thus our truncations are $\beta_{\geq w} F^* = A \otimes E^*_{\geq w}$ and $\beta_{< w} F^* = A \otimes E^*_{<w}$. If $F^* \in \text{Perf}(\mathcal{G})$, then by Proposition 3.13 so are $\beta_{\geq w} F^*$ and $\beta_{< w} F^*$.

The multiplicativity of $D^b(\mathcal{G})_{\geq w}$ follows from the fact that $D^b(\mathfrak{F})_{\geq w}$ is multiplicative and the fact that $L\varpi^*$ respects derived tensor products.

Every $M \in \text{Coh}(\mathcal{G})$ has a highest weight space, so $M \in D^b(\mathcal{G})_{<w}$ for some $w$. This implies that any $F^* \in D^b(\mathcal{G})$ lies in $D^b(\mathcal{G})_{<w}$ for some $w$. The analogous statement for $D^b(\mathcal{G})_{\geq w}$ is false in general, but if $F^* \in D^b(\mathcal{G})$ is such that $\sigma^* F^*$ is cohomologically bounded, then $F^* \in D^b(\mathcal{G})_{\geq w}$ for some $w$. The boundedness properties follow from this observation. □

Amplification 3.15. If Property (A) holds, then $\beta_{\geq w} F^*$ and $\beta_{< w} F^*$ can be computed from a presentation $F^* \simeq A \otimes E^*$ with $E^i \in \text{Coh}(\mathfrak{F})$ coherent but not necessarily locally free. Furthermore $L\pi^* = \pi^* : D^b(\mathfrak{F})_{w} \to D^b(\mathcal{G})_{w}$ is an equivalence, where $D^b(\mathcal{G})_{w} := D^b(\mathcal{G})_{\geq w} \cap D^b(\mathcal{G})_{< w+1}$ and likewise for $D^b(\mathfrak{F})_{w}$.

Proof. If $\pi : Y \to Z$ is flat and $E \in \text{Coh}(\mathfrak{F})$, then $A \otimes E \in D^b(\mathcal{G})_{\geq w}$ iff $E \in \text{Coh}(\mathfrak{F})_{\geq w}$ and likewise for $< w$. Thus

$$A \otimes E_{\geq w} \to F^* \to A \otimes E_{<w} \to$$

is the exact triangle defining the baric truncations of $F^*$. In fact for any coherent $A$-module $M$ there is a coherent $E \in \text{Coh}(\mathfrak{F})$ and a surjection $A \otimes E \to M$ which is an isomorphism on highest weight subsheaves, and one can use this fact to construct a presentation of this form in which $E_{\geq w} = 0$ for $i \ll 0$. So in fact $\beta_{\geq w} F^*$ is equivalent to a finite complex of the form $A \otimes E_{\geq w}$.

For the second claim, fully faithfulness of $L\pi^*$ follows formally from the fact that $\pi$ admits the section $\sigma : \mathfrak{F} = Z/L \to \mathcal{G}$. Essential surjectivity follows from the first part of the proposition. □

Our final observation is that the components of the baric decomposition of $D^b(\mathcal{G})$ and $D^b(\mathfrak{F})$ can be characterized pointwise over $Z$. We let $\mathbb{C}^*$ act on $Y$ via $\lambda$ and consider the flat morphism of stacks $Y/\mathbb{C}^* \to Y/P$. This gives a pullback (forgetful) functor $D(\mathcal{G}) \to D(Y/\mathbb{C}^*)$. Given a point $p : * \to Z$, one can compose this forgetful functor with the pullback and shriek-pullback to get functors $p^* : D^-(\mathcal{G}) \to D^-(*/\mathbb{C}^*)$ and $p^! : D^+(\mathcal{G}) \to D^+(*/\mathbb{C}^*)$. By abuse of notation we denote the analogous functors $p^* : D^-(\mathfrak{F}) \to D^-(*/\mathbb{C}^*)$ and $p^! : D^+(\mathfrak{F}) \to D^+(*/\mathbb{C}^*)$, so that $p^! = p^! \varpi^!$ and $p^* = p^* \varpi^!$.

Lemma 3.16. A complex $F^* \in D^-(\mathfrak{F})$ lies in $D^-(\mathfrak{F})_{\geq, < w}$ iff $p^* F^* \in D^-(*/\mathbb{C}^*)_{\geq, < w}$ for all $p : * \to Z$. Dually, a complex $F^* \in D^+(\mathfrak{F})$ lies in $D^+(\mathfrak{F})_{\geq, < w}$ iff $p^! F^* \in D^+(*/\mathbb{C}^*)_{\geq, < w}$ for all $p$.
Proof. It suffices to work over $\mathbb{Z}/\mathbb{C}^*$. Because every quasicoherent sheaf functorially splits into $\lambda$ eigensheaves, $p^*(F^*)_{\geq, <w} = (p^* F^*)_{\geq, <w}$ and $p^!(F^*)_{\geq, <w} = (p^! F^*)_{\geq, <w}$. The claim for $F^* \in \mathcal{D}^{-}(\mathcal{S})$ now follows by applying derived Nakayama’s Lemma to $F^*_{\geq, <w}$. Likewise the claim for $F^* \in \mathcal{D}^+(\mathcal{S})$ follows from the Serre dual statement of derived Nakayama’s Lemma, namely that $F^* \in \mathcal{D}^+(\mathbb{Z})$ is acyclic iff $p^* F^* = \text{Hom}(p_* \mathcal{C}, F^*)$ is acyclic for all $p$ (note that we only need Nakayama’s Lemma in the non-equivariant setting).

Corollary 3.17. The subcategories $\mathcal{D}^- (\mathcal{S})_{\geq, w}, \mathcal{D}(\mathcal{S})_{< w} \subset \mathcal{D}(\mathcal{S})$ are characterized by their images in $\mathcal{D}(\mathbb{Y}/\mathbb{C}^*)$. If we consider all points $p : * \hookrightarrow \mathbb{Z}$.

- $F^* \in \mathcal{D}^- (\mathcal{S})$ lies in $\mathcal{D}^- (\mathcal{S})_{\geq, w}$ iff $p^* F^* \in \mathcal{D}^- (\mathbb{Y}/\mathbb{C}^*)_{\geq, w}, \forall p$
- $F^* \in \mathcal{D}^b (\mathcal{S})$ lies in $\mathcal{D}^b (\mathcal{S})_{< w}$ iff $p^* F^* \in \mathcal{D}^+ (\mathbb{Y}/\mathbb{C}^*)_{< w}, \forall p$

Dually, if $\pi : \mathbb{Y} \to \mathbb{Z}$ is a bundle of affine spaces with determinant weight $a$, then $\mathcal{D}^b (\mathcal{S})_{\geq, w}$ are characterized by the weights of $\sigma F^*$. We have

- $F^* \in \mathcal{D}^+ (\mathcal{S})$ lies in $\mathcal{D}^+ (\mathcal{S})_{< w}$ iff $p^* F^* \in \mathcal{D}^+ (\mathbb{Y}/\mathbb{C}^*)_{< w+a}, \forall p$
- $F^* \in \mathcal{D}^b (\mathcal{S})$ lies in $\mathcal{D}^b (\mathcal{S})_{\geq, w}$ iff $p^* F^* \in \mathcal{D}^+ (\mathbb{Y}/\mathbb{C}^*)_{\geq, w+a}, \forall p$

Proof. The first claim is immediate from the definitions of $\mathcal{D}^- (\mathcal{S})_{\geq, w}$ and $\mathcal{D}(\mathcal{S})_{< w}$, so for the remainder of the proof we can work in the category $\mathcal{D}(\mathbb{Y}/\mathbb{C}^*)$. From Proposition 3.14 and the discussion preceding it, we know that $\mathcal{D}^- (\mathcal{S})_{\geq, w}$ and $\mathcal{D}(\mathcal{S})_{< w}$ are characterized by $\sigma F^* \in \mathcal{D}^- (\mathbb{Z}/\mathbb{C}^*)$. The claim now follows from Lemma 3.16.

Now assume that $\pi : \mathbb{Y} \to \mathbb{Z}$ is a bundle of affine spaces. Locally over $\mathbb{Z}$, $\sigma F^* \simeq \sigma F^*(-a)[d]$, where $d$ is the fiber dimension of $\pi$ and $a$ is the weight of $\lambda$ on $\det(\mathcal{H}_{\mathbb{Z}/\mathbb{Y}})$, so the weights of $\sigma F^*$ are simply shifts of the weights of $\sigma F^*$. If $F^* \in \mathcal{D}(\mathcal{S})$ the claim again follows from Lemma 3.16. By definition an unbounded object $F^* \in \mathcal{D}(\mathcal{S})$ lies in $\mathcal{D}(\mathcal{S})_{< w}$ iff $\mathcal{H}^i (F^*) \in \mathcal{QCoh}(\mathcal{S})_{< w}$ for all $i$, i.e. iff $\pi \leq n F^* \in \mathcal{D}(\mathcal{S})_{< w}$ for all $n$. One can prove the claim for $\mathcal{D}(\mathcal{S})_{< w}$ by writing $F^* = \lim_{\to} \pi \leq n F^*$ and that each homology sheaf of $\sigma F^* = \lim_{\to} \sigma \pi \leq n F^*$ stabilizes after some finite $n$.

3.2. The cotangent complex and Property (L+). We review the construction of the cotangent complex and prove the main implication of the positivity Property (L+):

Lemma 3.18. If $\mathcal{S} \to \mathcal{X}$ satisfies Property (L+) and $F^* \in \mathcal{D}(\mathcal{S})_{\geq, w}$, then $Lj^* j_* F^* \in \mathcal{D}^- (\mathcal{S})_{\geq, w}$ as well.

We can inductively construct a cofibrant replacement $\mathcal{O}_\mathcal{S}$ as an $\mathcal{O}_\mathcal{X}$ module: a surjective weak equivalence $\mathcal{B}^* \to \mathcal{O}_\mathcal{S}$ from a sheaf of dg-$\mathcal{O}_\mathcal{X}$ algebras with $\mathcal{B}^* \simeq (\mathcal{S}(E^*), d)$, where $\mathcal{S}(E^*)$ is the free graded commutative sheaf of algebras on the graded sheaf of $\mathcal{O}_\mathcal{X}$-modules $E^*$ with $E^i$ locally free and $E^i = 0$ for $i \geq 0$. Note that the differential is uniquely determined by its restriction to $E^*$, and letting $e$ be a local section of $E^*$ we decompose $d(e) = d_{-1}(e) + d_0(e) + \cdots$ where $d_i(e) \in \mathcal{S}^{i+1}(E^*)$. 

The $\mathcal{B}^*$-module of Kähler differentials is

$$\mathcal{B}^* \xrightarrow{\delta} \Omega_{\mathcal{B}^*/\mathcal{O}_X}^1 = \mathcal{S}(E^*) \otimes_{\mathcal{O}_X} E^*$$

with the universal closed degree 0 derivation defined by $\delta(e) = 1 \otimes e$. The differential on $\Omega_{\mathcal{B}^*/\mathcal{O}_X}^1$ is uniquely determined by its commutation with $\delta$

$$d(1 \otimes e) = \delta(de) = 1 \otimes d_0(e) + \delta(d_1(e) + \cdots)$$

By definition

$$L^*(\mathcal{S} \hookrightarrow \mathcal{X}) := \mathcal{O}_\mathcal{S} \otimes_{\mathcal{B}^*} \Omega_{\mathcal{B}^*/\mathcal{O}_X}^1 \simeq \mathcal{O}_\mathcal{S} \otimes E^*$$

where the differential is the restriction of $d_0$.

**Proof of Lemma 3.18.** First we prove the claim for $\mathcal{O}_\mathcal{S}$. Note that $\mathcal{B}^* \to \mathcal{O}_\mathcal{S}$, in addition to a cofibrant replacement of $dg\mathcal{O}_X$-algebras, is a left bounded resolution of $\mathcal{O}_\mathcal{S}$ by locally frees. Thus $Lj^*j_*\mathcal{O}_\mathcal{S} = \mathcal{O}_\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{B}^* = \mathcal{S}(E^*)_\mathcal{S}$ with differential $d(e) = d_0(e) + d_1(e) + \cdots$. The term $d_{-1}$ in the differential vanishes when restricting to $\mathcal{S}$. Restricting further to $\mathcal{S}$, we have a deformation of complexes of $\mathcal{O}_\mathcal{S}$ modules over $\mathcal{A}^1$

$$F^*_t := (\mathcal{S}(E^*)_3|_3, d_0 + td_1 + t^2d_2 + \cdots)$$

which is trivial over $\mathcal{A}^1 - \{0\}$. The claim of the lemma is that $(F^*_t)_{t<0} \sim 0$. Setting $t = 0$, the differential becomes the differential of the cotangent complex, so $F^*_0 = \mathcal{S}(\mathcal{L}_\mathcal{S}/\mathcal{X})|_3$. By hypothesis $\mathcal{L}_\mathcal{S}/\mathcal{X} \to (\mathcal{L}_\mathcal{S}/\mathcal{X})_{\geq 0}$ is a weak equivalence, so $\mathcal{S}(\mathcal{L}_\mathcal{S}/\mathcal{X}|_3) \to \mathcal{S}(\mathcal{L}_\mathcal{S}/\mathcal{X}|_3)_{\geq 0}$ is a weak equivalence with a complex of locally frees generated in nonnegative weights. Thus $(F^*_0|_3)_{t<0} \sim 0$. By semicontinuity it follows that $(F^*_t)_{t<0} = 0$ for all $t \in \mathcal{A}^1$, and the lemma follows for $\mathcal{O}_\mathcal{S}$.

Now we consider arbitrary $F^* \in \mathcal{D}^b(\mathcal{S})$. Let $\mathcal{O}_\mathcal{S} := \mathcal{S}(E^*)_\mathcal{S} = Lj^*j_*\mathcal{O}_\mathcal{S}$ denote the derived self intersection. $\mathcal{O}_\mathcal{S}$ is a summand of $\mathcal{O}_\mathcal{S}$ as an $\mathcal{O}_\mathcal{S}$ module, and we have already established that $\mathcal{O}_\mathcal{S} \in \mathcal{D}^b(\mathcal{S})_{\geq 0}$, so $E^* \in \mathcal{D}^b(\mathcal{S})_{\geq w}$ iff $\mathcal{O}_\mathcal{S} \otimes E^* \in \mathcal{D}^b(\mathcal{S})_{\geq w}$. The proof of the lemma follows from this and the projection formula.

$$Lj^*j_*(\mathcal{O}_\mathcal{S} \otimes F^*) = Lj^*(j_*\mathcal{O}_\mathcal{S} \otimes L j_* F^*) = \mathcal{O}_\mathcal{S} \otimes L j_* j_* F^*$$

\[\square\]

3.3. **Koszul systems and cohomology with supports.** We recall some properties of the right derived functor of the subsheaf with supports functor $R\Gamma_{\mathcal{O}_\mathcal{S}}(\bullet)$. It can be defined by the exact triangle $R\Gamma_{\mathcal{O}_\mathcal{S}}(F^*) \to F^* \to i_* (F^*|_\mathcal{S})$, and it is the right adjoint of the inclusion $\mathcal{D}(\mathcal{S}) \subset \mathcal{D}(\mathcal{X})$. It is evident from this exact triangle that if $F^* \in \mathcal{D}^b(\mathcal{X})$, then $R\Gamma_{\mathcal{O}_\mathcal{S}}(F^*)$ is still bounded, but no longer has coherent cohomology. On the other hand the formula

$$R\Gamma_{\mathcal{O}_\mathcal{S}}(F^*) = \lim_{\to} \mathcal{H}om(\mathcal{O}_\mathcal{X}/\mathcal{T}_E, F^*)$$

shows that the subsheaf with supports is a limit of coherent complexes.
We will use a more general method of computing the subsheaf with supports similar to the Koszul complexes which can be used in the affine case.

**Lemma 3.19.** Let \( \mathcal{X} = X/G \) with \( X \) quasiprojective and \( G \) reductive, and let \( S \subset X \) be a KN stratum. Then there is a direct system \( K_i^* \to K_{i+1}^* \to \cdots \) in \( \text{Perf}(\mathcal{X})^{[0,N]} \) along with compatible maps \( K_i^* \to \mathcal{O}_\mathcal{X} \) such that

1. \( \mathcal{H}^*(K_i^*) \) is supported on \( \mathcal{S} \)
2. \( \lim (K_i^* \otimes F^*) \to F^* \) induces an isomorphism \( \lim (K_i^* \otimes F^*) \simeq R\Gamma_{\mathcal{S}}(F^*) \).
3. \( \text{Cone}(K_i^* \to K_{i+1}^*) \in D^b(3)_{<w_i} \) where \( w_i \to -\infty \) as \( i \to \infty \).

We will call such a direct system a Koszul system for \( S \subset \mathcal{X} \).

**Proof.** First assume \( \mathcal{X} \) is smooth in a neighborhood of \( S \). Then \( \mathcal{O}_\mathcal{X}/T_{\mathcal{S}} \) is perfect, so the above formula implies that the duals \( K_i^* = (\mathcal{O}_\mathcal{X}/T_{\mathcal{S}})^\vee \) satisfy properties (1) and (2) with \( K_i^* \to \mathcal{O}_\mathcal{X} \) the dual of the map \( \mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{X}/T_{\mathcal{S}} \).

We compute the mapping cone

\[
\text{Cone}(K_i^* \to K_{i+1}^*) = (T_{\mathcal{S}}/T_{\mathcal{S}}^{i+1})^\vee = (j_* (S^i(\mathfrak{N}_{\mathcal{S}/\mathcal{X}})))^\vee
\]

Where the last equality uses the smoothness of \( \mathcal{X} \). Because Property (L+) is automatic for smooth \( \mathcal{X} \), it follows from Lemma 3.18 that \( Lj_* j_* (S^i(\mathfrak{N}_{\mathcal{S}/\mathcal{X}})) \in D^b(\mathcal{S}) \cap \mathcal{X} \), hence \( \text{Cone}(K_i^* \to K_{i+1}^*) \) has weights \( \leq -i \), and the third property follows.

If \( X \) is not smooth in a neighborhood of \( S \), then by hypothesis we have have a \( G \)-equivariant closed immersion \( \phi : X \hookrightarrow X' \) and closed KN stratum \( S' \subset X' \) such that \( S \) is a connected component of \( S' \cap X \) and \( X' \) is smooth in a neighborhood of \( S' \). Then we let \( K_i^* \in \text{Perf}(\mathcal{X}) \) be the restriction of \( L\phi^* (\mathcal{O}_\mathcal{X}/T_{\mathcal{S}})^\vee \). These \( K_i^* \) still satisfy the third property. Consider the canonical morphism \( \lim (K_i^* \otimes F^*) \to R\Gamma_{\mathcal{S} \cap \mathcal{X}}^\vee F^* \). Its push forward \( \lim \phi_*(K_i^* \otimes F^*) \to \phi_* R\Gamma_{\mathcal{S} \cap \mathcal{X}}(F^*) = R\Gamma_{\mathcal{S} \cap \mathcal{X}} \phi_* F^* \) is an isomorphism, hence the \( K_i^* \) form a Koszul system for \( \mathcal{S} \cap \mathcal{X} \). Because \( \mathcal{S} \) is a connected component of \( \mathcal{S} \cap \mathcal{X} \), the complexes \( R\Gamma_{\mathcal{S}} K_i^* \) form a Koszul system for \( \mathcal{S} \).

We note an alternative definition of a Koszul system, which will be useful below.

**Lemma 3.20.** Property (3) of a Koszul system is equivalent to the property that for all \( w \),

\[
\text{Cone}(K_i^* \to \mathcal{O}_\mathcal{X}) |_{3} \in D^b(3)_{<w} \text{ for all } i \gg 0
\]

**Proof.** First, by the octahedral axiom we have an exact triangle

\[
\text{Cone}(K_i^* \to \mathcal{O}_\mathcal{X})[-1] \to \text{Cone}(K_{i+1}^* \to \mathcal{O}_\mathcal{X})[-1] \to \text{Cone}(K_i^* \to K_{i+1}^*) \rightarrow
\]

So the property stated in this Lemma implies property (3) of the definition of a Koszul system.

Conversely, let \( K_i^* \) be a Koszul system for \( \mathcal{S} \subset \mathcal{X} \). For any \( F^* \in D^+(\mathcal{X}) \),

\[
j^! F^* \simeq j^! \mathcal{G}^! F^*
\]

so

\[
\sigma^! F^* \simeq \sigma^! \mathcal{G}^! F^* \simeq \lim \sigma^! (K_i^* \otimes F^*) \simeq \lim \sigma^! K_i^* |_3 \otimes \sigma^! F^*
\]
where we have used compactness of $\mathcal{O}_3$ as an object of $D^+(\mathcal{X})$ (which follows from the analogous statement for schemes proved in Section 6.3 of [20]) in order to commute $\sigma^i$ with the direct limit computing $\Gamma_S F^*$.

Now let $\omega^* \in D^b(\mathcal{X})$ be a dualizing complex, which by definition means that $\omega$ is a dualizing complex in $D^b(\mathcal{X})$ after forgetting the $G$ action (see [2] for a discussion of dualizing complexes for stacks). Then $\sigma^i \omega^*$ is a dualizing complex on $\mathcal{X}$, and its restriction to $Z/C^*$ is again a dualizing complex. Any dualizing complex on $Z/C^*$ must be concentrated in a single weight, so $\sigma^i \omega^* \in D^b(\mathcal{X})$ for some weight $N \in \mathbb{Z}$.

Now the formula above says that $\sigma^i \omega^* = \lim(K_i^{*} \otimes \sigma^i \omega^*)$. If we assume that Cone($K_i^{*} \to K_i^{*}|_{\omega^*}$) $\in D^b(\mathcal{X})$, where $w_i \to -\infty$ as $i \to \infty$, then for any $v$ the canonical map $(K_i^{*} \otimes \sigma^i \omega^*) \to \lim(K_i^{*} \otimes \sigma^i \omega^*)$ is an isomorphism for $i \gg 0$. In particular for any fixed $v < N$ we have $\sigma^i \omega^* (\sigma^i \omega^*) \geq v \cong (K_i^{*} \otimes \sigma^i \omega^*) \otimes \sigma^i \omega^*$ for all $i \gg 0$. Thus the map $K_i^{*} \otimes \sigma^i \omega^* \to \mathcal{O}_3 \otimes \mathcal{O}_3$ is an isomorphism for $i \gg 0$, hence Cone($K_i^{*} \to \mathcal{O}_3|_{\omega^*}$) $\in D^b(\mathcal{X})$ for $i \gg 0$. 

3.4. Quasicoherent sheaves with support on $\mathcal{S}$, and the quantization theorem. We turn to the derived category $D^b(\mathcal{S})$ of coherent sheaves on $\mathcal{X}$ with support on $\mathcal{S}$. We will extend the baric decomposition of $D^b(\mathcal{S})$ to a baric decomposition of $D^b(\mathcal{X})$. Using this baric decomposition we will prove a generalization of the quantization commutes with reduction theorem, one of the results which motivated this work.

**Proposition 3.21.** Let $\mathcal{S} \subset \mathcal{X}$ be a KN stratum satisfying Property (L+). There is a unique multiplicative baric decomposition $D^b(\mathcal{S}) = (D^b(\mathcal{S})_{\geq w}, D^b(\mathcal{S})_{< w})$ such that

$$j_*(D^b(\mathcal{S})_{\geq w}) \subset D^b(\mathcal{X})_{\geq w} \text{ and } j_*(D^b(\mathcal{S})_{< w}) \subset D^b(\mathcal{X})_{< w}$$

It is described explicitly by

$$D^b(\mathcal{S})_{\geq w} = \{F^* \in D^b(\mathcal{S})|Rj^* F^* \in D^+(\mathcal{S})_{< w}\}$$

$$D^b(\mathcal{S})_{< w} = \{F^* \in D^b(\mathcal{S})|Lj^* F^* \in D^-(\mathcal{S})_{\geq w}\}$$

When Property (A) holds, this baric decomposition is bounded.

**Proof.** Let $D^b(\mathcal{X})_{\geq w}$ and $D^b(\mathcal{X})_{< w}$ be the triangulated subcategories generated by $j_*(D^b(\mathcal{S})_{\geq w})$ and $j_*(D^b(\mathcal{S})_{< w})$ respectively. By Lemma 3.18, $Lj^* j_*(D^b(\mathcal{S})_{\geq w}) \subset D^-(\mathcal{S})_{\geq w}$, and so $D^b(\mathcal{X})_{\geq w}$ is right orthogonal to $D^b(\mathcal{X})_{< w}$ as a consequence of Lemma 3.11.

Next we must show $D^b(\mathcal{X}) = D^b(\mathcal{S})_{\geq w} \ast D^b(\mathcal{S})_{< w}$, where the $\mathcal{A} \ast \mathcal{B}$ denotes the full subcategory consisting of $F$ admitting triangles $A \to F \to B \to -$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

If $\mathcal{A}$ and $\mathcal{B}$ are triangulated subcategories, and $\mathcal{B} \subset \mathcal{A}^\perp$, then the subcategory $\mathcal{A} \ast \mathcal{B}$ is triangulated as well. Furthermore, for any $F \in D^b(\mathcal{S})$ we have the exact triangle $j_\ast \beta_{\geq w} F \to j_\ast F \to j_\ast \beta_{< w} F \to -$ so $D^b(\mathcal{X})_{\geq w} \ast D^b(\mathcal{X})_{< w}$ is
a triangulated subcategory containing $j_*(D^b(\mathcal{G}))$, and so $D^b_\mathcal{G}(\mathcal{X}) = D^b_\mathcal{G}(\mathcal{X})_{\geq w}^* \cap D^b_\mathcal{G}(\mathcal{X})_{< w}^*$ as desired.

Now that we have shown that $D^b_\mathcal{G}(\mathcal{X}) = \langle D^b_\mathcal{G}(\mathcal{X})_{\geq w}^*, D^b_\mathcal{G}(\mathcal{X})_{< w}^* \rangle$, we can characterize each $D^b_\mathcal{G}(\mathcal{X})_{\geq w}^*, D^b_\mathcal{G}(\mathcal{X})_{< w}^*$ as the orthogonal of the other. The adjunctions $L_j^* \dashv j_*$ and $j_* \dashv Rj^*$ give the a posteriori characterizations in the last statement of the proposition.

\[ \square \]

**Remark 3.22.** Note that $\text{Perf}(\mathcal{X}) \subset \bigcup_w D^b_\mathcal{G}(\mathcal{X})_{\geq w} \cap \bigcup_v D^b_\mathcal{G}(\mathcal{X})_{< v}$.

The following is an extension to our setting of an observation which appeared in [5], following ideas of Kawamata[?]. There the authors described semiorthogonal factors appearing under VGIT in terms of the quotient $Z/L'$.

**Amplification 3.23.** Define $D^b_\mathcal{G}(\mathcal{X})_w := D^b_\mathcal{G}(\mathcal{X})_{\geq w} \cap D^b_\mathcal{G}(\mathcal{X})_{< w+1}$. If the weights of $L_{S/X}$ are strictly positive, then $j_* : D^b(\mathcal{G})_w \to D^b_\mathcal{G}(\mathcal{X})_w$ is an equivalence with inverse $\beta_{< w+1} \circ L_j^*(F^*)$.

**Corollary 3.24.** If $L_{S/X}^*$ has strictly positive weights, then the baric decomposition of Proposition 3.21 can be refined to an infinite semiorthogonal decomposition

$$D^b_\mathcal{G}(\mathcal{X}) = \langle \ldots, D^b(3)_w, D^b(3)_{w+1}, D^b(3)_{w+2}, \ldots \rangle$$

where factors are the essential images of the fully faithful embeddings $j_*\pi^* : D^b(3)_w \to D^b_\mathcal{G}(\mathcal{X})$.

Finally we will use the baric decomposition of Proposition 3.21 to generalize a Theorem of Teleman [22], which was one of the motivations for this paper.

**Definition 3.25.** We define the thick triangulated subcategories of $D^b(\mathcal{X})$

$$D^b(\mathcal{X})_{\geq w} := \{ F^* \in D^b(\mathcal{X}) | L_j^* F^* \in D^-(\mathcal{G})_{\geq w} \}$$

$$D^b(\mathcal{X})_{< v} := \{ F^* \in D^b(\mathcal{X}) | Rj^* F^* \in D^+(\mathcal{G})_{< v} \}$$

**Theorem 3.26 (Quantization Theorem).** Let $F^* \in D^b(\mathcal{X})_{\geq w}$ and $G^* \in D^b(\mathcal{X})_{< v}^*$ with $w \geq v$, then the restriction map

$$R \text{Hom}_\mathcal{X}(F^*, G^*) \to R \text{Hom}_\mathcal{G}(F^*|_\mathcal{Y}, G^*|_\mathcal{Y})$$

is an isomorphism.

**Proof.** This is equivalent to the vanishing of $R \Gamma_\mathcal{G}(R \text{Hom}_\mathcal{X}(F^*, G^*))$. By the formula $Rj^* \text{Hom}_\mathcal{X}(F^*, G^*) \simeq \text{Hom}_\mathcal{G}(Lj^* F^*, Rj^* G^*)$ it suffices to prove the case where $F^* = \mathcal{O}_\mathcal{X}$, i.e. showing that $R \Gamma_\mathcal{G}(G^*) = 0$ whenever $Rj^* G^* \in D^+(\mathcal{G})_{<0}$.

From Property (S3) we have a system $K_1 \to K_2 \to \cdots$ of perfect complexes in $D^b_\mathcal{G}(\mathcal{X})_{\leq 0}$ such that $R \Gamma_\mathcal{G}(G^*) = \varprojlim R \Gamma(K^*_i \otimes G^*)$ so it suffices to show the vanishing of each term in the limit. We have $j^*(K^*_i \otimes G^*) = j^*(K^*_i) \otimes j^* G^*$, so $K^*_i \otimes G^* \in D^b_\mathcal{G}(\mathcal{X})_{< 0}$. The category $D^b_\mathcal{G}(\mathcal{X})_{< 0}$ is generated...
by objects of the form \( j_\ast F \) with \( F \in D^b(\mathfrak{S})_{<0} \), and thus \( R\Gamma(F^\ast) \) for all \( F^\ast \in D^b_{\mathfrak{S}}(\mathfrak{X})_{<0} \).

\[ \square \]

3.5. **Semiorthogonal decomposition of** \( D^b(\mathfrak{X}) \). In this section we construct the semiorthogonal decomposition of \( D^b(\mathfrak{X}) \) used to prove the derived Kirwan surjectivity theorem. When \( Y \) is a bundle of affine spaces over \( Z \), we construct right adjoints for each of the inclusions \( D^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \subset D^b(\mathfrak{X})_{\geq w} \subset D^b(\mathfrak{X}) \).

We prove this in two steps. First we define a full subcategory \( D^b(\mathfrak{X})_{\text{fin}} \subset D^b(\mathfrak{X}) \) of complexes whose weights along \( \mathfrak{S} \) are bounded and construct a semiorthogonal decomposition of this category. Then we prove that \( D^b(\mathfrak{X})_{\text{fin}} = D^b(\mathfrak{X}) \) when \( Y \) is a bundle of affine spaces over \( Z \).

**Definition 3.27.** Define the full triangulated subcategories

\[
D^b(\mathfrak{X})_{\text{fin}} := \bigcup_v \left(D^b(\mathfrak{X})_{\geq v} \cap D^b(\mathfrak{X})_{<v}\right)
\]

\[
D^b(\mathfrak{X})_{\geq w} := D^b(\mathfrak{X})_{\geq w} \cap D^b(\mathfrak{X})_{\text{fin}}, \quad D^b(\mathfrak{X})_{< w} := D^b(\mathfrak{X})_{< w} \cap D^b(\mathfrak{X})_{\text{fin}}
\]

By Remark 3.26, any object of \( D^b_{\mathfrak{S}}(\mathfrak{X}) \) lies in \( D^b_{\mathfrak{S}}(\mathfrak{X})_{< w} \) for some \( w \). Thus \( D^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \subset D^b(\mathfrak{X})_{\text{fin}} \) for all \( w \).

**Proposition 3.28.** Let \( F^\ast \in D^b(\mathfrak{X})_{\text{fin}} \) and let \( K_i^\ast \) be a Koszul system for \( \mathfrak{S} \subset \mathfrak{X} \). Then for sufficiently large \( i \) the canonical map

\[
\beta_{\geq w}(K_i^\ast \otimes F^\ast) \to \beta_{\geq w}(K_{i+1}^\ast \otimes F^\ast)
\]

is an equivalence. The functor

\[
\beta_{\geq w}\mathfrak{S}(F^\ast) := \lim_{\rightarrow i} \beta_{\geq w}(K_i^\ast \otimes F^\ast)
\]

(11)

is well-defined and is a right adjoint to the inclusions \( D^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \subset D^b(\mathfrak{X})_{\geq w} \) and \( D^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \subset D^b(\mathfrak{X})_{\text{fin}} \).

**Proof.** By hypothesis the \( C_i^\ast := \text{Cone}(K_i^\ast \to K_{i+1}^\ast) \) is a perfect complex in \( D^b_{\mathfrak{S}}(\mathfrak{X})_{< w_i} \), where \( w_i \to -\infty \) as \( i \to \infty \). Because \( F^\ast \in D^b(X)_{\text{fin}} \), we have \( F^\ast \in D^b(\mathfrak{X})_{< N} \) for some \( N \), so if \( w_i + N < w \) we have \( C_i^\ast \otimes F^\ast \in D^b_{\mathfrak{S}}(\mathfrak{X})_{< w} \) and

\[
\text{Cone} \left( \beta_{\geq w}(K_i^\ast \otimes F^\ast) \to \beta_{\geq w}(K_{i+1}^\ast \otimes F^\ast) \right) = \beta_{\geq w}(C_i^\ast \otimes F^\ast) = 0
\]

Thus the direct system \( \beta_{\geq w}(K_i^\ast \otimes F^\ast) \) stabilizes, and the expression (11) defines a functor \( D^b(\mathfrak{X})_{\text{fin}} \to D^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \).

The fact that \( \beta_{\geq w}R\Gamma \) is the right adjoint of the inclusion follows from the fact that elements of \( D^b_{\mathfrak{S}}(\mathfrak{X}) \) are compact in \( D^b_{\mathfrak{S}}(\mathfrak{X}) \). For \( G^\ast \in D^b_{\mathfrak{S}}(\mathfrak{X})_{\geq w} \) we compute

\[
R\text{Hom}(G^\ast, \beta_{\geq w}\mathfrak{S}(F^\ast)) = \lim_{\rightarrow i} R\text{Hom}(G^\ast, K_i^\ast \otimes F^\ast) = R\text{Hom}(G^\ast, F^\ast)
\]

\[ \square \]
The right orthogonal to $D^b_{G}(\mathfrak{X})_{\geq w}$ can be determined a posteriori from the fact that $D^b_{G}(\mathfrak{X})_{\geq w}$ is generated by $j_* D^b(\mathcal{O})_{\geq w}$. Proposition 3.28 gives semiorthogonal decompositions

$$D^b(\mathfrak{X})_{\geq w}^{\text{fin}} = \langle G_w, D^b_{G}(\mathfrak{X})_{\geq w} \rangle \quad D^b(\mathfrak{X})^{\text{fin}} = \langle D^b(\mathfrak{X})_{< w}^{\text{fin}}, D^b_{G}(\mathfrak{X})_{\geq w} \rangle$$

where $G_w := D^b(\mathfrak{X})_{\geq w}^{\text{fin}} \cap D^b(\mathfrak{X})_{< w}^{\text{fin}}$. What remains is to show that $D^b(\mathfrak{X})_{\geq w}^{\text{fin}} \subset D^b(\mathfrak{X})^{\text{fin}}$ is right admissible.

**Proposition 3.29.** The inclusion of the subcategory $D^b(\mathfrak{X})_{\geq w}^{\text{fin}} \subset D^b(\mathfrak{X})^{\text{fin}}$ admits a right adjoint $\beta_{\geq w}(-)$ defined by the exact triangle

$$\beta_{\geq w} F^* \rightarrow F^* \rightarrow \beta_{< w}((K_i^*)^! \otimes F^*) \rightarrow \quad \text{for } i \gg 0$$

**Proof.** First note that for $F^* \in D^b(\mathfrak{X})^{\text{fin}}$ and for $i \gg 0$, $\text{Cone}(K_i^* \rightarrow K_{i+1}^*)^! \otimes F^* \in D^b(\mathfrak{X})_{\geq w}$. It follows that the inverse system $(K_i^*)^! \otimes F^*$ stabilizes, as in Proposition 3.28.

If we define $\beta_{\geq w} F^*$ as above and consider the composition $F^* \rightarrow (K_i^*)^! \otimes F^* \rightarrow \beta_{< w}((K_i^*)^! \otimes F^*) \rightarrow$, the octahedral axiom gives a triangle

$$\text{Cone}(F^* \rightarrow (K_i^*)^! \otimes F^*) \rightarrow \beta_{\geq w} F^*[1] \rightarrow \beta_{\geq w}((K_i^*)^! \otimes F^*)$$

Thus for $i \gg 0$, $\beta_{\geq w} F^* \in D^b(\mathfrak{X})_{\geq w}^{\text{fin}}$ and is right orthogonal to $D^b_{G}(\mathfrak{X})_{< w}$. It follows that $\beta_{\geq w} F^*$ is functorial in $F^*$ and is right adjoint to the inclusion $D^b(\mathfrak{X})_{\geq w}^{\text{fin}} \subset D^b(\mathfrak{X})^{\text{fin}}$. 

Combining Propositions 3.28 and 3.29, we have a semiorthogonal decomposition

$$D^b(\mathfrak{X})^{\text{fin}} = \langle D^b_{G}(\mathfrak{X})_{< w}^{\text{fin}}, G_w, D^b_{G}(\mathfrak{X})_{\geq w} \rangle$$

(12) where the restriction functor $G_w \rightarrow D^b(\mathfrak{O})$ is fully faithful by theorem 3.26. Recall that our goal is to use the semiorthogonal decomposition (12) as follows: any $F^* \in D^b(\mathfrak{O})$ extends to $D^b(\mathfrak{X})$, then using (12) one can find an element of $G_w$ restricting to $F^*$, hence $i^* : G_w \rightarrow D^b(\mathfrak{O})$ is an equivalence of categories. Unfortunately, in order for this argument to work, we need $D^b(\mathfrak{X}) = D^b(\mathfrak{X})^{\text{fin}}$. In the rest of this section we show that when $\pi : Y \rightarrow Z$ is a bundle of affine spaces, $D^b(\mathfrak{X})^{\text{fin}} = D^b(\mathfrak{X})$.

**Lemma 3.30.** Suppose that $\pi : Y \rightarrow Z$ is a bundle of affine spaces, then $D^b(\mathfrak{X})^{\text{fin}} = D^b(\mathfrak{X})$.

**Proof.** First we show that any $F^* \in D^b(\mathfrak{X})_{\geq w}$ for some $w$. Let $P^*$ be a perfect complex in $D^b_{G}(\mathfrak{X})$ whose support contains $\mathfrak{X}$ – for instance any object in the Koszul system constructed in Lemma 3.19 will suffice. We know then $P^* \otimes F^* \in D^b_{G}(\mathfrak{X})_{\geq a}$ for some $a$ by Remark ??, so $\sigma^*(P^* \otimes F^*) = \sigma^* P^* \otimes \sigma^* F^* \in D^b_{G}(\mathfrak{X})_{\geq a}$.

Because $P^*$ is perfect, $\sigma^* P^* \in D^b(\mathfrak{X})_{< q}$ for some $q$. It suffices to forget the action of $L$ on $\mathfrak{X}$ and work in the derived category of $Z/\mathbb{C}^*$. Let $p : * \rightarrow Z$ be a point, then $p^*(\sigma^* P^* \otimes \sigma^* F^*) = p^* P^* \otimes_k p^* F^*$ has weight $\geq a$. However $p^* P^*$ is non-zero by hypothesis and is equivalent in $D^b(\ast / \mathbb{C}^*)$ to a direct sum
of shifts $k(w)[d]$ with $w > -q$, so this implies that $p^*F^* \in D^b(\mathbb{H}/\mathbb{C}^*)_{a-q}$. This holds for every point in $Z$, so by Lemma 3.16, $\sigma^*F^* \in D^b(\mathbb{C}^*)_{a-q}$. Thus $F^* \in D^b(\mathbb{X})_{a-q}$.

By Lemma 3.16 we have that $F^* \in D^b(\mathbb{X})_{w}$ iff $\sigma^*F^* \in D^+(\mathbb{C}^*)_{w+a}$. By the same argument above, we can assume $\sigma^*(P^* \otimes F^*) = \sigma^*P^* \otimes \sigma^*F^* \in D^+(\mathbb{C}^*)_{N}$ for some $N$. For any $p : * \to Z$, we have $p^!(\sigma^*P^* \otimes \sigma^*F^*) = p^*P^* \otimes k^!\sigma^!F^*$, so by Lemma 3.16 we have $F^* \in D^b(\mathbb{X})_{N-q}$ where $q$ is the highest weight in $\sigma^*P^*$.

Now that we have identified $D^b(\mathbb{X})_{\text{fin}} = D^b(\mathbb{X})$ in this case, we collect the main results of this section in the following

**Theorem 3.31.** Let $\mathcal{S} \subset \mathfrak{X}$ be a closed KN stratum (Definition 2.1) satisfying Properties (L+) and (A). Let $G_w = D^b(\mathbb{X})_{\geq w} \cap D^b(\mathbb{X})_{< w}$, then

$$G_w = \left\{ F^* \in D^b(\mathbb{X}) \mid \sigma^*F^* \text{ supported in weights } \geq w, \text{ and } \sigma^!F^* \text{ supported in weights } < w + a \right\}$$

where $a$ is the weight of $\det(\mathfrak{M}_{Z/Y})$. There are semiorthogonal decompositions

$$D^b(\mathbb{X}) = (D^b(\mathbb{S})_{< w}, G_w, D^b(\mathbb{S})_{\geq w})$$

And the restriction functor $i^* : D^b(\mathfrak{X}) \to D^b(\mathfrak{Y})$ induces an equivalence $G_w \simeq D^b(\mathfrak{Y})$, where $\mathfrak{Y} = \mathfrak{X} - \mathcal{S}$.

**Proof.** Because $\pi : Y \to Z$ is a bundle of affine spaces, Lemma 3.30 states that $D^b(\mathbb{X})_{\text{fin}} = D^b(\mathbb{X})$, and Lemma 3.16 implies that $D^b(\mathbb{X})_{< w} = \{ F^* | \sigma^!F^* \in D^+(\mathbb{C}^*)_{w+a} \}$. As noted above, the existence of the semiorthogonal decomposition follows formally from the adjoint functors constructed in Propositions 3.28 and 3.29.

The fully faithfulness of $i^* : G_w \to D^b(\mathfrak{Y})$ is Theorem 3.26. Any $F^* \in D^b(\mathfrak{Y})$ admits a lift to $D^b(\mathbb{X})$, and the component of this lift lying in $G_w$ under the semiorthogonal decomposition also restricts to $F^*$, hence $i^*$ essential surjectivity follows.

Now let $X$ be smooth in a neighborhood of $Z$. Passing to an open subset containing $Z$, we can assume that $X$ is smooth. Recall that in this case $S, Y$, and $Z$ are smooth, and the equivariant canonical bundle $\omega_X := (\Lambda^{top} g) \otimes (\Lambda^{top} \Omega^1_X)$ is a dualizing bundle on $\mathfrak{X}$ and defines the Serre duality functor $D_X(\bullet) = R\text{Hom}(\bullet, \omega_X[\text{vdim } \mathfrak{X}])$, and likewise for $\mathfrak{S}$ and $\mathfrak{Z}$. The canonical bundles are related by $j^!\omega_X \simeq \omega_\mathfrak{S}[-\text{codim}(S, X)]$ and $\sigma^!\omega_\mathfrak{S} \simeq \omega_\mathfrak{Z}[-\text{codim}(Z, S)]$.

Using the fact that $\omega_\mathfrak{Z}$ has weight 0, so $D_\mathfrak{Z}(D^b(\mathfrak{Z})_{\geq w}) = D^b(\mathfrak{Z})_{w+1}$, and the fact that $D_\mathfrak{S}\sigma^*F^* \simeq \sigma^!D_X$ and likewise for $\mathfrak{S}$, we have

$$D_\mathfrak{S}(D^b(\mathfrak{S})_{\geq w}) = D^b(\mathfrak{S})_{a+1-w}, \text{ and } D_\mathfrak{X}(D^b(\mathfrak{X})_{\geq w}) = D^b(\mathfrak{X})_{a+1-w}$$

where $a$ is the weight of $\lambda$ on $\omega_\mathfrak{S}|_{\mathfrak{Z}}$. 

Furthermore any $F_q \in D^b(X)$ is perfect, so $j_! F_q \simeq j_!(O_X) \otimes j^* F^\ast \simeq \det(\mathcal{N}_S/X)^\vee \otimes j^* F^\ast [-\operatorname{codim}(S,X)]$. If we let $\eta$ denote the weight of $\lambda$ on $\det \mathcal{N}_S/X$, then this implies that

$$D^b(X)_{<w} = \{ F^\ast | \sigma^* F^\ast \text{ supported in weights } < w + \eta \}$$

Using this we can reformulate Theorem 3.31 as

**Corollary 3.32.** Let $S \subset X$ be a KN stratum such that $X$ is smooth in a neighborhood of $S$. Let $G_w = D^b(X)_{\geq w} \cap D^b(X)_{<w}$, then

$$G_w = \{ F^\ast \in D^b(X) | \sigma^* F^\ast \text{ supported in weights } [w, w + \eta] \}$$

where $\eta$ is the weight of $\det(\Omega^N_S/X)$. There are semiorthogonal decompositions

$$D^b(X) = \langle D^b(S)(X)_{<w}, G_w, D^b(S)(X)_{\geq w} \rangle$$

And the restriction functor $i^* : D^b(X) \to D^b(S)$ induces an equivalence $G_w \simeq D^b(S)$.

One can explicitly define the inverse using the functors $\beta_{\geq w}$ and $\beta_{<w}$ on $D^b(S)(X)$. Given $F^\ast \in D^b(S)$, choose a complex $\tilde{F}^\ast \in D^b(X)$ such that $\tilde{F}^\ast |_{\mathcal{S}} \simeq F^\ast$. Now for $N \gg 0$ take the mapping cone

$$\beta_{\geq w} R\text{Hom}_X(O_X/I^N_S, \tilde{F}^\ast) = \beta_{\geq w} R\Gamma_S \tilde{F}^\ast \to \tilde{F}^\ast \to G^\ast \to$$

So $G^\ast \in D^b(X)_{<w}$. By Serre duality the left adjoint of the inclusion $D^b(S)(X)_{<w} \subset D^b(X)_{<w}$ is $D_X \beta_{\geq 1-w} R\Gamma_S D_X$, and this functor can be simplified using Lemma 3.28. We form the exact triangle

$$\tilde{G}^\ast \to G^\ast \to \beta_{<w}(G^\ast \otimes^L O_X/I^N_S)$$

and $\tilde{G}^\ast \in G_w$ is the unique object in $G_w$ mapping to $F^\ast$.

4. **Derived equivalences and variation of GIT**

We apply Theorem 2.5 to the derived categories of birational varieties obtained by a variation of GIT quotient. First we study the case where $G = \mathbb{C}^\ast$, in which the KN stratification is particularly easy to describe. Next we generalize this analysis to arbitrary variations of GIT, one consequence of which is the observation that if a smooth projective-over-affine variety $X$ is equivariantly Calabi-Yau for the action of a torus, then the GIT quotients of any two generic linearizations are derived equivalent.

A normal projective variety $X$ with linearized $\mathbb{C}^\ast$ action is sometimes referred to as a birational cobordism between $X//_L G$ and $X//_{L'} G$ where $L(m)$ denotes the twist of $L$ by the character $t \mapsto t^m$. A priori this seems like a highly restrictive type of VGIT, but by Thaddeus’ master space construction[23], any two spaces that are related by a general VGIT are related by a birational cobordism. We also have the weak converse due to Hu & Keel:
Figure 1. Schematic diagram for the fixed loci \( Z_\alpha \). \( S_\alpha \) is the ascending or descending manifold of \( Z_\alpha \) depending on the sign of \( \mu_\alpha \). As the moment fiber varies, the unstable strata \( S_\alpha \) flip over the critical sets \( Z_\alpha \).

**Theorem 4.1** (Hu & Keel). Let \( Y_1 \) and \( Y_2 \) be two birational projective varieties, then there is a birational cobordism \( X/\mathbb{C}^* \) between \( Y_1 \) and \( Y_2 \). If \( Y_1 \) and \( Y_2 \) are smooth, then by equivariant resolution of singularities \( X \) can be chosen to be smooth.

The GIT stratification for \( G = \mathbb{C}^* \) is very simple. If \( L \) is chosen so that the GIT quotient is an orbifold, then the \( Z_\alpha \) are the connected components of the fixed locus \( X^G \), and \( S_\alpha \) is either the ascending or descending manifold of \( Z_\alpha \), depending on the weight of \( L \) along \( Z_\alpha \).

We will denote the tautological choice of 1-PS as \( \lambda^+ \), and we refer to “the weights” of a coherent sheaf at point in \( X^G \) as the weights with respect to this 1-PS. We define \( \mu_\alpha \in \mathbb{Z} \) to be the weight of \( L|_{Z_\alpha} \). If \( \mu_\alpha > 0 \) (respectively \( \mu_\alpha < 0 \)) then the maximal destabilizing 1-PS of \( Z_\alpha \) is \( \lambda^+ \) (respectively \( \lambda^- \)). Thus we have

\[
S_\alpha = \left\{ x \in X \mid \lim_{t \to 0} t \cdot x \in Z_\alpha \text{ if } \mu_\alpha > 0 \right. \\
\left. \lim_{t \to 0} t^{-1} \cdot x \in Z_\alpha \text{ if } \mu_\alpha < 0 \right\}
\]

Next observe the weight decomposition under \( \lambda^+ \)

\[
\Omega^1_X|_{Z_\alpha} \simeq \Omega^1_{Z_\alpha} \oplus \mathcal{N}^+ \oplus \mathcal{N}^-
\]

Then \( \Omega^1_{S_{\alpha}}|_{Z_\alpha} = \Omega^1_{Z_\alpha} \oplus \mathcal{N}^- \) if \( \mu_\alpha > 0 \) and \( \Omega^1_{S_{\alpha}}|_{Z_\alpha} = \Omega^1_{Z_\alpha} \oplus \mathcal{N}^+ \) if \( \mu_\alpha < 0 \), so we have

\[
\eta_\alpha = \begin{cases} 
\text{weight of } \det \mathcal{N}^+|_{Z_\alpha} & \text{if } \mu_\alpha > 0 \\
\text{weight of } \det \mathcal{N}^-|_{Z_\alpha} & \text{if } \mu_\alpha < 0 
\end{cases}
\]

There is a parallel interpretation of this in the symplectic category. A sufficiently large power of \( L \) induces a equivariant projective embedding and thus a moment map \( \mu : X \to \mathbb{R} \) for the action of \( S^1 \subset \mathbb{C}^* \). The semistable locus is the orbit of the zero fiber \( X^{ss} = G \cdot \mu^{-1}(0) \). The reason for the collision of notation is that the fixed loci \( Z_\alpha \) are precisely the critical points of \( \mu \), and the number \( \mu_\alpha \) is the value of the moment map on the critical set \( Z_\alpha \).
Varying the linearization \( L(r) \) by twisting by the character \( t \mapsto t^{-r} \) corresponds to shifting the moment map by \(-r\), so the new zero fiber corresponds to what was previously the fiber \( \mu^{-1}(r) \). For non-critical moment fibers the GIT quotient will be a DM stack, and the critical values of \( r \) are those for which \( \mu_\alpha = \text{weight of } L(r) |_{Z_\alpha} = 0 \) for some \( \alpha \).

Say that as \( r \) increases it crosses a critical value for which \( \mu_\alpha = 0 \). The maximal destabilizing 1-PS \( \lambda_\alpha \) flips from \( \lambda^+ \) to \( \lambda^- \), and the unstable stratum \( S_\alpha \) flips from the ascending manifold of \( Z_\alpha \) to the descending manifold of \( Z_\alpha \). In the decomposition (13), the normal bundle of \( S_\alpha \) changes from \( N^+ \) to \( N^- \), so applying \( \det \) to (13) and taking the weight gives

\[
\text{weight of } \omega_X |_{Z_\alpha} = \eta_\alpha - \eta'_\alpha
\]

Thus if \( \omega_X \) has weight 0 along \( Z_\alpha \), the integer \( \eta_\alpha \) does not change as we cross the wall. The grade restriction window of Theorem 2.5 has the same width for the GIT quotient on either side of the wall, and it follows that the two GIT quotients are derived equivalent because they are identified with the same subcategory \( G_q \) of the equivariant derived category \( D^b(X/G) \). We summarize this with the following

**Proposition 4.2.** Let \( L \) be a critical linearization of \( X/\mathbb{C}^* \), and assume that \( Z_\alpha \) is the only critical set for which \( \mu_\alpha = 0 \). Let \( a \) be the weight of \( \omega_X |_{Z_\alpha} \), and let \( \epsilon > 0 \) be a small rational number.

1. If \( a > 0 \), then there is a fully faithful embedding
   \[
   D^b(X//L(\epsilon)G) \subseteq D^b(X//L(-\epsilon)G)
   \]
2. If \( a = 0 \), then there is an equivalence
   \[
   D^b(X//L(\epsilon)G) \simeq D^b(X//L(-\epsilon)G)
   \]
3. If \( a < 0 \), then there is a fully faithful embedding
   \[
   D^b(X//L(-\epsilon)G) \subseteq D^b(X//L(\epsilon)G)
   \]

The analytic local model for a birational cobordism is the following

**Example 4.3.** Let \( Z \) be a smooth variety and let \( N = \bigoplus N_\iota \) be a \( \mathbb{Z} \)-graded locally free sheaf on \( Z \) with \( N_0 = 0 \). Let \( X \) be the total of \( N \) – it has a \( \mathbb{C}^* \) action induced by the grading. Because the only fixed locus is \( Z \) the underlying line bundle of the linearization is irrelevant, so we take the linearization \( \mathcal{O}_X(r) \).

If \( r > 0 \) then the unstable locus is \( N_- \subset X \) where \( N_- \) is the sum of negative weight spaces of \( N \), and if \( r < 0 \) then the unstable locus is \( N_+ \) (we are abusing notation slightly by using the same notation for the sheaf and its total space). We will borrow the notation of Thaddeus [23] and write

\[
X/\pm = (X \setminus N_\pm)/\mathbb{C}^*.
\]

Inside \( X/\pm \) we have \( N_\pm /\pm \simeq \mathbb{P}(N_\pm) \), where we are still working with quotient stacks, so the notation \( \mathbb{P}(N_\pm) \) denotes the weighted projective bundle associated to the graded locally free sheaf \( N_\pm \). If \( \pi_\pm : \mathbb{P}(N_\pm) \to Z \) is the
projection, then \( X/\pm \) is the total space of the vector bundle \( \pi^*_{\pm}N_{\pm}(-1) \). We have the common resolution

\[
\mathcal{O}_{\pi(N_-) \times \pi(N_+)}(-1, -1) \to \pi^*_{\pm}N_{\pm}(-1) \]

Let \( \pi : X \to Z \) be the projection, then the canonical bundle is \( \omega_X = \pi^*(\omega_Z \otimes \det(N_+)^v \otimes \det(N_-)^v) \), so the weight of \( \omega_X|_Z \) is \( \sum i \text{ rank}(N_i) \). In the special case of a flop, Proposition 4.2 says

if \( \sum i \text{ rank}(N_i) = 0 \), then \( \text{D}^b(\pi^*_{\pm}N_{\pm}(-1)) \simeq \text{D}^b(\pi^*_{\pm}N_{\pm}(-1)) \)

4.1. **General variation of GIT quotient.** We will generalize the analysis of a birational cobordism to an arbitrary variation of GIT quotient. Until this point we have taken the KN stratification as given, but now we must recall its definition and basic properties as described in [12].

Let \( \text{NS}^G(X)_{\mathbb{R}} \) denote the group of equivariant line bundles up to homological equivalence, tensored with \( \mathbb{R} \). For any \( L \in \text{NS}^G(X)_{\mathbb{R}} \) one defines a stability function on \( X \)

\[ M^L(x) := \max \left\{ \frac{\text{weight}_\lambda L_y}{|\lambda|} \mid \lambda \text{ s.t. } y = \lim_{t \to 0} \lambda(t) \cdot x \text{ exists} \right\} \]

\( M^L(\bullet) \) is upper semi-continuous, and \( M^L(x) \) is lower convex and thus continuous on \( \text{NS}^G(X)_{\mathbb{R}} \) for a fixed \( x \). A point \( x \in X \) is semistable if \( M^L(x) \leq 0 \), stable if \( M^L(x) < 0 \), strictly semistable if \( M^L(x) = 0 \) and unstable if \( M^L(x) > 0 \).

The \( G \)-ample cone \( \mathcal{C}^G(X) \subset \text{NS}^G(X)_{\mathbb{R}} \) has a finite decomposition into convex conical chambers separated by hyperplanes – the interior of a chamber is where \( M^L(x) \neq 0 \) for all \( x \in X \), so \( \mathcal{X}^{ss}(\mathcal{L}) = \mathcal{X}^{s}(\mathcal{L}) \). We will be focus on a single wall-crossing: \( \mathcal{L}_0 \) will be a \( G \)-ample line bundle lying on a wall such that for \( \epsilon \) sufficiently small \( \mathcal{L}_{\pm} := \mathcal{L}_0 \pm \epsilon \mathcal{L}' \) both lie in the interior of chambers.

By continuity of the function \( M^L(\bullet) \) on \( \text{NS}^G(X)_{\mathbb{R}} \), all of the stable and unstable points of \( \mathcal{X}^{s}(\mathcal{L}_0) \) will remain so for \( \mathcal{L}_{\pm} \). Only points in the strictly semistable locus, \( \mathcal{X}^{sss}(\mathcal{L}_0) = \{ x \in \mathcal{X} | M^L(x) = 0 \} \subset \mathcal{X} \), change from being stable to unstable as one crosses the wall.

In fact \( \mathcal{X}^{ss}(\mathcal{L}_0) \) is a union of KN strata for \( \mathcal{X}^{ss}(\mathcal{L}_+) \), and symmetrically it can be written as a union of KN strata for \( \mathcal{X}^{ss}(\mathcal{L}_-).[12] \) Thus we can write \( \mathcal{X}^{ss}(\mathcal{L}_0) \) in two ways

\[ \mathcal{X}^{ss}(\mathcal{L}_0) = \mathcal{G}_1^\pm \cup \cdots \cup \mathcal{G}_{m_\pm}^\pm \cup \mathcal{X}^{ss}(\mathcal{L}_{\pm}) \]  \( \tag{16} \)

Where \( \mathcal{G}_i^\pm \) are the KN strata of \( \mathcal{X}^{ss}(\mathcal{L}_{\pm}) \) lying in \( \mathcal{X}^{ss}(\mathcal{L}_0) \).

**Definition 4.4.** A wall crossing \( \mathcal{L}_{\pm} = \mathcal{L}_0 \pm \epsilon \mathcal{L}' \) will be called balanced if \( m_+ = m_- \) and \( \mathcal{G}^+_i = \mathcal{G}^-_i \) under the decomposition (16).
By the construction of the strata outlined above, there is a finite collection of locally closed \( Z_i \subset X \) and one parameter subgroups \( \lambda_i \) fixing \( Z_i \) such that \( G\cdot Z_i / G \) are simultaneously the attractors for the KN strata of both \( X^{ss}(L_{\pm}) \) and such that the \( \lambda_i^{\pm 1} \) are the maximal destabilizing 1-PS’s.

**Proposition 4.5.** Let a reductive \( G \) act on a projective-over-affine variety \( X \). Let \( L_0 \) be a \( G \)-ample line bundle on a wall, and define \( L_{\pm} = L_0 \pm \epsilon L' \) for some other line bundle \( L' \). Assume that

- for \( \epsilon \) sufficiently small, \( X^{ss}(L_{\pm}) = X^{s}(L_{\pm}) \neq \emptyset \),
- the wall crossing \( L_{\pm} \) is balanced, and
- for all \( Z_i \) in \( X^{ss}(L_0) \), \( (\omega_X)|_{Z_i} \) has weight 0 with respect to \( \lambda_i \)

then \( D^b(X^{ss}(L_{+})) \cong D^b(X^{ss}(L_{-})) \).

**Remark 4.6.** Full embeddings analogous to those of Proposition 4.2 apply when the weights of \( (\omega_X)|_{Z_i} \) with respect to \( \lambda_i \) are either all negative or all positive.

**Proof.** The proof is an immediate application of Theorem 2.5 to the open substack \( X^s(L_{+}) \subset X^{ss}(L_0) \) whose complement admits the KN stratification (16). Because the wall crossing is balanced, \( Z_i^+ = Z_i^- \) and \( \lambda_i^- (t) = \lambda_i^+ (t^{-1}) \), and the condition on \( \omega_X \) implies that \( \eta_i^+ = \eta_i^- \). So Theorem 2.5 identifies the category \( G_q \subset D^b(X^{ss}(L_0)) \) with both \( D^b(X^s(L_+)) \) and \( D^b(X^s(L_-)) \). \( \Box \)

**Example 4.7.** Dolgachev and Hu study wall crossings which they call truly faithful, meaning that the identity component of the stabilizer of a point with closed orbit in \( X^{ss}(L_0) \) is \( \mathbb{C}^* \). They show that every truly faithful wall is balanced.[12, Lemma 4.2.3]

Dolgachev and Hu also show that for the action of a torus \( T \), there are no codimension 0 walls and all codimension 1 walls are truly faithful. Thus any two chambers in \( C^F(X) \) can be connected by a finite sequence of balanced wall crossings, and we have

**Corollary 4.8.** Let \( X \) be a projective-over-affine variety with an action of a torus \( T \). Assume \( X \) is equivariantly Calabi-Yau in the sense that \( \omega_X \cong \mathcal{O}_X \) as an equivariant \( \mathcal{O}_X \)-module. If \( L_0 \) and \( L_1 \) are \( G \)-ample line bundles such that \( X^s(L_i) = X^{ss}(L_i) \), then \( D^b(X^s(L_0)) \cong D^b(X^s(L_1)) \).

A compact projective manifold with a non-trivial \( \mathbb{C}^* \) action is never equivariantly Calabi-Yau, but Corollary 4.8 applies to a large class of non compact examples. The simplest are linear representations \( V \) of \( T \) such that \( \text{det} \ V \) is trivial. More generally we have

**Example 4.9.** Let \( T \) act on a smooth projective Fano variety \( X \), and let \( \mathcal{E} \) be an equivariant ample locally free sheaf such that \( \text{det} \ \mathcal{E} \cong \omega_X^\vee \). Then the total space of the dual vector bundle \( Y = \text{Spec}_X(S^* \mathcal{E}) \) is equivariantly Calabi-Yau and the canonical map \( Y \rightarrow \text{Spec}(\Gamma(X, S^* \mathcal{E})) \) is projective, so \( Y \) is projective over affine and by Corollary 4.8 any two generic GIT quotients \( Y//T \) are derived equivalent.
When $G$ is non-abelian, the chamber structure of $C^G(X)$ can be more complicated. There can be walls of codimension 0, meaning open regions in the interior of $C^G(X)$ where $X^s \neq X^{ss}$, and not all walls are truly faithful.[12] Still, there are examples where derived Kirwan surjectivity can give derived equivalences under wall crossings which are not balanced.

**Definition 4.10.** A wall crossing $\mathcal{L}_\pm = \mathcal{L}_0 \pm \epsilon \mathcal{L}'$ will be called *almost balanced* if $m_+ = m_-$ and under the decomposition (16), one can choose maximal destabilizers such that $\lambda_i^- = (\lambda_i^+)^{-1}$ and $cl(Z_i^+) = cl(Z_i^-)$.

In an almost balanced wall crossing for which $\omega_X|_{Z_i}$ has weight 0 for all $i$, we have the following general principal for establishing a derived equivalence:

**Ansatz 4.11.** For some $w$ and $w'$, $G^+_w = G^-_{w'}$ as subcategories of $D^b(X^{ss}(\mathcal{L}_0)/G)$, where $G^\pm_*$ is the category identified with $D^b(X^{ss}(\mathcal{L}_\pm)/G)$ under restriction.

For instance, one can recover a result of Segal & Donnovan[7]:

**Example 4.12** (Grassmannian flop). Choose $k < N$ and let $V$ be a $k$-dimensional vector space. Consider the action of $G = GL(V)$ on $X = T^* \text{Hom}(V, \mathbb{C}^N) = \text{Hom}(V, \mathbb{C}^N) \times \text{Hom}(\mathbb{C}^N, V)$. A 1-PS $\lambda : \mathbb{C}^* \to G$ corresponds to a choice of weight decomposition $V \simeq \bigoplus V_\alpha$ under $\lambda$. A point $(a, b)$ has a limit under $\lambda$ iff

$$V_{>0} \subset \ker(a) \quad \text{and} \quad \text{im}(b) \subset V_{\geq 0}$$

in which case the limit $(a_0, b_0)$ is the projection onto $V_0 \subset V$. There are only two nontrivial characters up to rational equivalence, $\det^\pm$. A point $(a, b)$ is semistable iff any 1-PS for which $\lambda(t) \cdot (a, b)$ has a limit as $t \to 0$ has nonpositive pairing with the chosen character.

In order to determine the stratification, it suffices to fix a maximal torus of $GL(V)$, i.e. and isomorphism $V \simeq \mathbb{C}^k$, and to consider diagonal one parameter subgroups $(t^{w_1}, \ldots, t^{w_k})$ with $w_1 \leq \cdots \leq w_k$. If we linearize with respect to det, then the KN stratification is

$$\lambda_i = (0, \ldots, 0, 1, \ldots, 1) \text{ with } i \text{ zeros}$$

$$Z_i = \left\{ \begin{bmatrix} [ \square | 0 ] \end{bmatrix}, \begin{bmatrix} * \end{bmatrix} \right\} \text{ with } * \in M_{i \times N}, \quad \text{and } \square \in M_{N \times i} \text{ full rank}$$

$$Y_i = \left\{ \begin{bmatrix} [ \square | 0 ] \end{bmatrix}, b \right\} \text{ with } b \in M_{k \times N} \text{ arbitrary}, \quad \text{and } \square \in M_{N \times i} \text{ full rank}$$

$$S_i = \{(a, b) | b \text{ arbitrary, rank } a = i\}$$

So $(a, b) \in X$ is semistable iff $a$ is injective. If instead we linearize with respect to $\det^{-1}$, then $(a, b)$ is semistable iff $b$ is surjective, the $\lambda_i$ flip, and the critical loci $Z_i$ are the same except that the role of $\square$ and $*$ reverse. So this is an almost balanced wall crossing with $\mathcal{L}_0 = \mathcal{O}_X$ and $\mathcal{L}' = \mathcal{O}_X(\det)$.

Let $G(k, N)$ be the Grassmannian parametrizing $k$-dimensional subspaces $V \subset \mathbb{C}^N$, and let $0 \to U(k, N) \to \mathcal{O}^N \to Q(k, N) \to 0$ be the tautological sequence of vector bundles on $G(k, N)$. Then $X^{ss}(\det)$ is the total space...
of $U(k, N)^N$, and $\mathcal{X}^{ss}(\det^{-1})$ is the total space of $(\mathcal{Q}(N - k, N)^\vee)^N$ over $\mathbb{G}(N - k, N)$.

In order to verify that $G^+_w = G^-_{w'}$ for some $w'$, one observes that the representations of $GL_k$ which form the Kapranov exceptional collection\[7\] lie in the weight windows for $G_0^+ \simeq \mathbb{D}(\mathcal{X}^{ss}(\det)) = \mathbb{D}(U(k, N)^N)$. Because $U(k, N)^N$ is a vector bundle over $G(k, N)$, these objects generate the derived category. One then verifies that these object lie in the weight windows for $\mathcal{X}^{ss}(\det^{-1})$ and generate this category for the same reason. Thus by verifying Ansatz 4.11 we have established an equivalence of derived categories

$$\mathbb{D}(U(k, N)^N) \simeq \mathbb{D}((\mathcal{Q}(N - k, N)^\vee)^N)$$

The astute reader will observe that these two varieties are in fact isomorphic, but the derived equivalences we have constructed are natural in the sense that they generalize to families. Specifically, if $E$ is an $N$-dimensional vector bundle over a smooth variety $Y$, then the two GIT quotients of the total space of $\text{Hom}(\mathcal{O}_Y \otimes V, E) \oplus \text{Hom}(E, \mathcal{O}_Y \otimes V)$ by $GL(V)$ will have equivalent derived categories.

The key to verifying Ansatz 4.11 in this example was simple geometry of the GIT quotients $\mathcal{X}^{ss}(\det^\pm)$ and the fact that we have explicit generators for the derived category of each. With a more detailed analysis, one can verify Ansatz 4.11 for many more examples of balanced wall crossings, and we will describe this in a future paper.

**Remark 4.13.** This example is similar to the generalized Mukai flops of [9]. The difference is that we are not restricting to the hyperkähler moment fiber $\{ba = 0\}$. The surjectivity theorem cannot be applied directly to the GIT quotient of this singular variety, but in the next section we will explore some applications to abelian hyperkähler reduction.

### 5. Applications to complete intersections: matrix factorizations and hyperkähler reductions

In the example of a projective variety, where we identified $\mathbb{D}(Y)$ with a full subcategory of the derived category of finitely generated graded modules over the homogeneous coordinate ring of $Y$, the point of the affine cone satisfied Property (L+) “for free.” In more complicated examples, the cotangent positivity property (L+) can be difficult to verify.

Here we discuss several techniques for extending derived Kirwan surjectivity for stacks $X/G$ where $X$ is a local complete intersection. First we provide a geometric criterion for Property (L+) to hold, which allows us to apply Theorem 2.5 to some hyperkähler quotients. We also discuss two different approaches to derived Kirwan surjectivity for LCI quotients, using morita theory and derived categories of singularities.

#### 5.1. A criterion for Property (L+) and non-abelian Hyperkähler reduction

In this section we study a particular setting in which Property
We consider the intermediate variety on in positive weights implies that \( s \) vanishing locus of the map \( Y(V_\Omega) \) so it suffices to consider the later.

Let \( X' \) be a smooth quasiprojective variety with an action of a reductive \( G \), and let \( S' = G \cdot Y' \subset X' \) be a closed KN stratum (Definition 2.1). Because \( X' \) is smooth, \( Y' \) is a \( P \)-equivariant bundle of affine spaces over \( Z' \). Let \( V \) be a linear representation of \( G \), and \( s : X' \to V \) and equivariant map. Alternatively, we think of \( s \) as an invariant global section of the locally free sheaf \( O_{X'} \otimes V \). We define \( X = s^{-1}(0) \) and \( S = S' \cap X \), and likewise for \( Y \) and \( Z \).

Note that if we decompose \( V = V_+ \oplus V_0 \oplus V_- \) under the weights of \( \lambda \), then \( \Gamma(\mathcal{S'}, \mathcal{O}_{\mathcal{E'}} \otimes V_-) = 0 \), so \( s|_{\mathcal{E'}} \) is a section of \( \mathcal{O}_{\mathcal{E'}} \otimes V_0 \oplus V_+ \).

**Lemma 5.1.** If for all \( z \in Z \subset Z' \), \( (ds)_z : T_zX \to V \) is surjective in positive weights w.r.t. \( \lambda \), then

\[
(\sigma^*L_{\mathcal{E}}^*)_{<0} \simeq [\mathcal{O}_3 \otimes V_+^{\vee} \xrightarrow{(ds)_z^{\vee}} (\Omega_{\mathcal{Y}}|_Z)_{<0}]
\]

and is thus a locally free sheaf concentrated in cohomological degree 0.

**Proof.** First of all note that from the inclusion \( \sigma : \mathcal{S} \hookrightarrow \mathcal{E} \) we have

\[
(\sigma^*L_{\mathcal{E}}^*)_{<0} \to (L^*_{\mathcal{S}})_{<0} \to (L^*_{\mathcal{S}/\mathcal{E}})_{<0} \twoheadrightarrow
\]

The cotangent complex \( L^*_{\mathcal{S}/\mathcal{E}} \) is supported in weight 0 because \( \lambda \) acts trivially on \( Z \), so the middle term vanishes, and we get \( (\sigma^*L_{\mathcal{E}}^*)_{<0} \simeq (L^*_{\mathcal{S}/\mathcal{E}})_{<0}[-1] \), so it suffices to consider the later.

By definition \( Y \) is the zero fiber of \( s : Y' \to V_0 \oplus V_+ \). Denote by \( s_0 \) the section of \( V_0 \) induced by the projection of \( P \)-modules \( V_+ \oplus V_0 \to V_0 \). We consider the intermediate variety \( Y \subset Y_0 \coloneqq s_0^{-1}(0) \subset Y' \). Note that \( Y = \pi^{-1}(Z) \), where \( \pi : Y' \to Z' \) is the projection.

Note that \( Y_0 \to Z \) is a bundle of affine spaces with section \( \sigma \), in particular \( \mathcal{S} \subset \mathcal{E}_0 \) is a regular embedding with conormal bundle \( (\Omega_{\mathcal{Y}}^1|_Z)_{<0} = (\Omega_{\mathcal{Y}}^1|_{\mathcal{E}_0})_{<0} \). Furthermore, on \( Y_0 \) the section \( s_0 \) vanishes by construction, so \( Y \subset Y_0 \), which by definition is the vanishing locus of \( s|_{Y_0} \), is actually the vanishing locus of the map \( s_+ : Y_0 \to V_+ \). The surjectivity of \( (ds)_z \) for \( z \in Z \) in positive weights implies that \( s_+^{-1}(0) \) has expected codimension in every fiber over \( Z \) and thus \( \mathcal{S} \subset \mathcal{E}_0 \) is a regular embedding with conormal bundle \( \mathcal{O}_{\mathcal{S}} \otimes V_+^{\vee} \).

It now follows from the canonical triangle for \( \mathcal{S} \subset \mathcal{S} \subset \mathcal{E}_0 \) that

\[
L^*_{\mathcal{S}/\mathcal{E}} \simeq \text{Cone}(\sigma^*L_{\mathcal{E}/\mathcal{S}_0} \to L^*_{\mathcal{S}/\mathcal{E}_0}) \simeq [\mathcal{O}_3 \otimes V_+^{\vee} \xrightarrow{(ds)_z} (\Omega_{\mathcal{Y}}^1|_{\mathcal{E}_0})_{<0}]
\]

with terms concentrated in cohomological degree \(-2\) and \(-1\). The result follows. \( \square \)

**Proposition 5.2.** Let \( X' \) be a smooth quasiprojective variety with reductive \( G \) action, and let \( Z' \subset S' \subset X' \) be a KN stratum. Let \( s : X' \to V \) be an equivariant map to a representation of \( G \).
Define $X = s^{-1}(0)$, $S = S' \cap X$, and $Z = Z' \cap X$, and assume that $X$ has codimension $\dim V$. If for all $z \in Z$, $(ds)_z : T_z X' \to V$ is surjective in positive weights w.r.t. $\lambda$, then Property (L+) holds for $S/G \hookrightarrow X/G$.

**Proof.** We will use Lemma 5.1 to compute the relative cotangent complex $(\sigma^* L_{S/X})_{<0}$. We consider the canonical diagram

\[
\begin{array}{ccc}
O_S \otimes V^\vee & \rightarrow & \Omega^1_{X'}|_S \\
\downarrow & & \downarrow \\
\sigma^* L'_{S/X} & \rightarrow & L'_v \rightarrow L'_{vX} \rightarrow 0
\end{array}
\]

where the bottom row is an exact triangle. □

Now let $(M, \omega)$ be an algebraic symplectic manifold with a Hamiltonian $G$ action, i.e. there is a $G$-equivariant algebraic map $\mu : M \to \mathfrak{g}^\vee$ satisfying $d(\xi, \mu) = -\omega(\partial_\xi, \bullet) \in \Gamma(M, \Omega^1_M)$, where $\partial_\xi$ is the vector field corresponding to $\xi \in \mathfrak{g}$.

For any point $x \in M$, we have an exact sequence

\[0 \to \text{Lie } G_x \to \mathfrak{g} \xrightarrow{d\mu} T^*_x M \to T_x (G \cdot x)^\perp \to 0 \tag{17}\]

Showing that $X := \mu^{-1}(0)$ is regular at any point with finite stabilizer groups. Thus if the set such points is dense in $X$, then $X \subset M$ is a complete intersection cut out by $\mu$. Thus we have

**Proposition 5.3.** Let $(M, \omega)$ be a projective-over-affine algebraic symplectic manifold with a Hamiltonian action of the reductive group $G$, and let $X = \mu^{-1}(0) \subset M$. If $X^s$ is dense in $X$, then Property (L+) holds for the GIT stratification of $X$.

**Example 5.4** (stratified Mukai flop). We return to $M := \text{Hom}(V, \mathbb{C}^N) \times \text{Hom}(\mathbb{C}^N, V)$. In Example 4.12 we considered the GIT stratification for the action of $GL(V)$, but this group action is also algebraic Hamiltonian with moment map $\mu(a, b) = ba \in \mathfrak{gl}(V)$. The stratification of $X = \mu^{-1}(0)$ is induced by the stratification of $M$. Thus the $Y_i$ in $X$ consist of

\[Y_i = \left\{ \begin{bmatrix} a_1 & 0 \\ b_1 \\ b_2 \end{bmatrix} \right\} \text{ with } b_1 a_1 = 0, b_2 a_1 = 0, \text{ and } a_1 \in M_{N \times i} \text{ full rank} \}
\]

and $Z_i \subset Y_i$ are those points where $b_2 = 0$. Note that over a point in $Z_i$, the condition $b_2 a_1 = 0$ is linear in the fiber, and so $Y_i \rightarrow Z_i$ satisfies Property (A).

The GIT quotient $X^{ss}/GL(V)$ is the cotangent bundle $T^* G(k, N)$. Property (A) holds in this example, and Property (L+) holds by Proposition 5.2, so Theorem 2.5 gives a fully faithful embedding $D^b(T^* G(k, N)) \subset D^b(X/GL(V))$ for any choice of integers $w_i$. The derived category $D^b(T^* G(k, N))$
has been intensely studied by Cautis, Kamnitzer, and Licata from the perspective of categorical $\mathfrak{sl}_2$ actions. We will discuss the connection between their results and derived Kirwan surjectivity in future work.

5.2. Extending the main theorem using Morita theory. In this section I remark that Theorem 1.1 extends to complete intersections in a smooth $X/G$ for purely formal reasons, where by complete intersection I mean one defined by global invariant functions on $X/G$.

In this section I will use derived Morita theory ([4],[15]), and so I will switch to a notation more common in that subject. $QC(X)$ will denote the unbounded derived category of quasicoherent sheaves on a perfect stack $X$, and $Perf(X)$ will denote the category of perfect complexes, i.e. the compact objects of $QC(X)$. All of the stacks we use are global quotients of quasiprojective varieties, so $Perf(X)$ are just the objects of $QC(X)$ which are equivalent to a complex of vector bundles.

Now let $X = X/G$ as in the rest of this paper. Assume we have a map $f : X \to B$ where $B$ is a quasiprojective scheme. The restriction $i^* : Perf(X) \to Perf(X^{ss})$ is a dg-$\otimes$ functor, and in particular it is a functor of module categories over the monoidal dg-category $Perf(B)^\otimes$.

The subcategory $G_q$ used to construct the splitting in Theorem 1.1 is defined using conditions on the weights of various 1-PS’s of the isotropy groups of $X$, so tensoring by a vector bundle $f^*V$ from $B$ preserves the subcategory $G_q$. It follows that the splitting constructed in Theorem 1.1 is a splitting as modules over $Perf(B)$. Thus for any point $b \in B$ we have a split surjection

$$Fun_{Perf(B)}(Perf({b}), Perf(X)) \to Fun_{Perf(B)}(Perf({b}), Perf(X^{ss}))$$

Using Morita theory, both functor categories correspond to full subcategories of $QC((\bullet)_b)$, where $(\bullet)_b$ denotes the derived fiber $(\bullet) \times_B \{b\}$. Explicitly, $Fun_{Perf(B)}(Perf({b}), Perf(X))$ is equivalent to the full dg-subcategory of $QC((X)_b)$ consisting of complexes of sheaves whose pushforward to $X$ is perfect. Because $X$ is smooth, and $O(X)_b$ is coherent over $O_X$, this is precisely the derived category of coherent sheaves $D^b(Coh((X)_b))$. The same analysis applied to the tensor product $Perf({b}) \otimes_{Perf(B)} Perf(X)$ yields a splitting for the category of perfect complexes.

Corollary 5.5. Given a map $f : X \to B$ and a point $b \in B$, the splitting of Theorem 1.1 induces splittings of the natural restriction functors

$$D^b(Coh((X)_b)) \to D^b(Coh((X^{ss})_b))$$

$$Perf((X)_b) \to Perf((X^{ss})_b)$$

In the particular case of a complete intersection one has $B = \mathbb{A}^r$, $b = 0 \in B$, and the derived fiber agrees with the non-derived fiber.
As a special case of Corollary 5.5, one obtains equivalences of categories of matrix factorizations in the form of derived categories of singularities. Namely, if \( W : \mathcal{X} \to \mathbb{C} \) is a function, a “potential” in the language of mirror symmetry, then the category of matrix factorizations corresponding to \( W \) is

\[
\text{MF}(\mathcal{X}, W) \simeq \text{D}^b_{\text{sing}}(W^{-1}(0)) = \text{D}^b(\text{Coh}(W^{-1}(0)))/\text{Perf}(W^{-1}(0))
\]

From Corollary 5.5 the restriction functor \( \text{MF}(\mathcal{X}, W) \to \text{MF}(\mathcal{X}, W) \) splits. In particular, if two GIT quotients \( \text{Perf}(\mathcal{X}^{ss}(\mathcal{L}_1)) \) and \( \text{Perf}(\mathcal{X}^{ss}(\mathcal{L}_2)) \) can be identified with the same subcategory of \( \text{Perf}(\mathcal{X}) \) as in Proposition 4.2, then the corresponding subcategories of matrix factorizations are equivalent

\[
\text{MF}(\mathcal{X}^{ss}(\mathcal{L}_1), W|_{\mathcal{X}^{ss}(\mathcal{L}_1)}) \simeq \text{MF}(\mathcal{X}^{ss}(\mathcal{L}_2), W|_{\mathcal{X}^{ss}(\mathcal{L}_2)})
\]

Corollary 5.5 also applies to the context of hyperkähler reduction. Let \( T \) be a torus, or any group whose connected component is a torus, and consider a Hamiltonian action of \( T \) on a hyperkähler variety \( X \) with algebraic moment map \( \mu : X/T \to \mathfrak{t}^\vee \). One forms the hyperkähler quotient by choosing a linearization on \( X/T \) and defining \( X///T = \mu^{-1}(0) \cap X^{ss} \). Thus we are in the setting of Corollary 5.5.

**Corollary 5.6.** Let \( T \) be an extension of a finite group by a torus. Let \( T \) act on a hyperkähler variety \( X \) with algebraic moment map \( \mu : X \to \mathfrak{t}^\vee \). Then the restriction functors

\[
\text{D}(\text{Coh}(\mu^{-1}(0)/T)) \to \text{D}(\text{Coh}(\mu^{-1}(0)^{ss}/T))
\]

\[
\text{Perf}(\text{Coh}(\mu^{-1}(0)/T)) \to \text{Perf}(\text{Coh}(\mu^{-1}(0)^{ss}/T))
\]

both split.

This splitting does not not give as direct a relationship between \( \text{D}^b(X/T) \) and \( \text{D}^b(X///T) \) as Theorem 2.5 does for the usual GIT quotient, but it is enough for some applications, for instance

**Corollary 5.7.** Let \( X \) be a projective-over-affine hyperkähler variety with a Hamiltonian action of a torus \( T \). Then the hyperkähler quotients with respect to any two generic linearization \( \mathcal{L}_1, \mathcal{L}_2 \) are derived equivalent.

**Proof.** By Corollary 4.8 all \( \mathcal{X}^{ss}(\mathcal{L}) \) for generic \( \mathcal{L} \) will be derived equivalent. In particular there is a finite sequence of wall crossings \( \text{Perf}(\mathcal{X}^{ss}(\mathcal{L}_+)) \to \text{Perf}(\mathcal{X}^{ss}(\mathcal{L}_-)) \) identifying each GIT quotient with the same subcategory. By Corollary 5.6 these splittings descend to \( \mu^{-1}(0) \), giving equivalences of both \( \text{D}^b(\text{Coh}(\bullet)) \) and \( \text{Perf}(\bullet) \) for the hyperkähler reductions.

**References**


