MAPPING STACKS AND CATEGORICAL NOTIONS OF PROPERNESS (DRAFT)

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Abstract. One fundamental consequence of a scheme being proper is that there is an algebraic space classifying maps from it to any other finite type scheme, and this result has been extended to proper stacks. We observe, however, that it also holds for many examples where the source is a geometric stack, such as a global quotient. In our investigation, we are lead naturally to certain properties of the derived category of a stack which guarantee that the mapping stack from it to any geometric finite type stack is algebraic. We develop methods for establishing these properties in a large class of examples. Along the way, we introduce a notion of projective morphism of algebraic stacks, and prove strong h-descent results which hold in the setting of derived algebraic geometry but not in classical algebraic geometry.

Contents

1. Introduction 2
1.1. Introduction to the introduction 2
1.2. Mapping out of stacks which are "proper enough" 3
1.3. Techniques for establishing (GE) and (L) 5
1.4. A long list of examples 7
1.5. Comparison with previous results 8
1.6. Notation and conventions 8
1.7. Author's note 9
2. Artin's criteria for mapping stacks 9
2.1. Weil restriction of affine stacks 11
2.2. Deformation theory of the mapping stack 12
2.3. Integrability via the Tannakian formalism 15
2.4. Derived representability from classical representability 19
2.5. Application: (pGE) and the moduli of perfect complexes 20
3. Perfect Grothendieck existence 23
3.1. Proving (pGE) via semiorthogonal decompositions 25
4. h-descent Theorems 29
4.1. Descent pattern for closed immersions 29
4.2. Descent pattern for the h-topology and for (GE) morphisms 33
4.3. h-descent theorems for APerf and Perf 35
4.4. h-descent theorems for geometric stacks 37
4.5. Closure of (GE) and (pGE) under proper maps 38
5. Cohomologically projective morphisms 39
5.1. Cohomological properness (CP) 39
5.2. Cohomologically ample (CA) systems 41
5.3. Examples of cohomologically projective morphisms 44
6. Establishing Grothendieck Existence and (LL) 45
6.1. Theorem on formal functions 45
6.2. Fully faithfulness from (CP) 49
6.3. Essential surjectivity from (CA) and (CP) 50
6.4. Methods of establishing (L) 52
Appendix A. BG for reductive G in finite characteristic 55
Appendix B. Recollections on quasi-coherent complexes on derived stacks 57
B.1. Useful subcategories of $QC(X)$ 57

B.2. Quasi-coherent pushforwards, and base-change 58
B.3. Enough coherent / perfect complexes 59
B.4. More pushforward and base-change 60
Appendix C. Recollections on formal completions 62
C.1. Almost perfect complexes and coherent sheaves 64

1. Introduction

1.1. **Introduction to the introduction.** In this paper we study some instances of the following questions:

(Q1) Suppose that $\pi\colon X \to S$ is a map of stacks, and that a sheaf $\mathscr{F}$ on X is representable by an X-stack. When is the sheaf $\pi_*\mathscr{F}$ on S representable by an S-stack?

(Q2) Suppose that X, Y are two S-stacks. When is the sheaf $\mathrm{Hom}_S(X, Y)$ on S representable by an S-stack?

These questions are closely related as we will remind the reader in subsubsection 2.0.1. There are various cases where a positive answer to (Q1) is known:

- The case of π a finite flat morphism and of $\mathscr{F}$ affine over X is the classical Weil restriction / restriction of scalar.
- If π is a flat, proper, finitely presented morphism of algebraic stacks – satisfying an extra covering condition – then Olsson proves the existence of Weil restriction. We give a more precise summary in subsection 1.5.

Notice, however, that the condition that π be proper is often unreasonably strong for algebraic stacks. In particular, it implies that the diagonal Δ_π is proper – so this diagonal cannot be affine, unless it is in fact finite! This rules out the case of $S/G \to S$ for G an affine S-group scheme that is not finite. The following counter-example shows that this restriction is generally needed:

Example 1.1. Take $S = \mathrm{Spec}\, R$ affine and G, H two flat affine algebraic groups over S. A straightforward argument implies that the representability of $\mathrm{Hom}_S(BG, BH)$ implies the representability of $\mathrm{Hom}_{gp/S}(G, H)$.

We will show that at least when S admits a map from a field k of characteristic 0, $\mathrm{Hom}_{gp/S}(\mathbf{G}_a, \mathbf{G}_m)$ is *not* representable. If it were, then every compatible family of $k[x]/x^n$-points would necessarily come from an $k[\![x]\!]$-point. Letting t denote the coordinate on $\mathbf{G}_a$, the compatible family of morphisms

$$\exp(tx)\colon \mathbf{G}_a \times \mathrm{Spec}\, k[x]/x^n \longrightarrow \mathbf{G}_m \times \mathrm{Spec}\, k[x]/x^n$$

does not extend to a morphism from $\mathbf{G}_a \times \mathrm{Spec}\, k[\![x]\!]$ since $\exp(tx)$ evidently does not lie inside

$$k[t] \otimes_k k[\![x]\!] \subset \varprojlim_n k[t] \otimes_k k[x]/x^n.$$

Nevertheless, if $G = \mathbf{G}_m$, or more generally is an S-group of multiplicative type, one can show that $\mathrm{Hom}_{gp/S}(G, H)$ is representable for any smooth group scheme H. The proof in SGA3 Exp. XI (Cor. 4.2) relies on Hopf algebras some combinatorial book-keeping. The appearance of Hopf algebra methods anticipates the alternate approach which we take, which is to use the Tannakian formalism to relate $\mathrm{Hom}_{gp}(G, H)$ to symmetric monoidal functors H-mod to G-mod and then making use of the simple nature of the category $\mathbb{G}_m$-mod.

In fact, the deeper result in SGA3 Exp. XXIV (Cor. 7.2.3) that $\mathrm{Hom}_{gp/S}(G, H)$ is representable for any reductive G and smooth H fits into our Tannakian picture as well (See Appendix A), although to recover that result in full generality one must use recent advances in the structure theory of reductive group schemes.

Notice that the Tannakian formalism imposes some restrictions on the *target*, not just the source, of our mapping stack: it must have a sufficiently rich theory of quasi-coherent complexes, eliminating examples like the proper stack BA for A an abelian variety. For this reason, we will often find it reasonable to restrict the target stack to be *geometric* (i.e., affine diagonal, very much not proper!). The following example shows that some restriction of this type is necessary:

Example 1.2. Let $\mathscr{X} = B\mathbb{G}_m$ and let $\mathscr{Z}$ be the stack of flat families of connected curves of arithmetic genus 1. Then $\mathscr{M} = \mathrm{Map}_{\mathrm{Spec}(k)}(B\mathbb{G}_m, \mathscr{Z})$ is not algebraic. Letting $I\mathscr{M}$ denote its inertia pre-stack (classifying a genus 1 curve with an automorphism), we will construct a diagram

$$\begin{array}{ccc} \mathrm{Spf}\, k[\![q]\!] & \longrightarrow & I\mathscr{M} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}\, k[\![q]\!] & \xrightarrow[t]{} & \mathscr{M} \end{array}$$

where the dotted arrow does not exist. If $\mathscr{M}$ were algebraic then $I\mathscr{M} \to \mathscr{M}$ would be algebraic, and so such a dotted arrow would have to exist!

We take the map t to be the classifying map for the Tate curve

$$\mathscr{E}_q \to \mathrm{Spec}\, k[\![q]\!]$$

degenerating a smooth elliptic curve to the nodal genus 1 curve. Consider the connected component of the automorphism group scheme of the Tate curve

$$\mathcal{A}^\circ = \mathrm{Aut}_{k[\![q]\!]}(\mathscr{E}_q)^\circ \subset \mathrm{Spec}\, k[\![q]\!] \times_M I\mathscr{M}.$$

The connected component of the special fiber is a $\mathbb{G}_m$, while the connected component of the generic fiber is an elliptic curve. By the rigidity of groups of multiplicative type, there is a compatible family of isomorphisms

$$\mathcal{A}^\circ \times_{\mathrm{Spec}\, k[\![q]\!]} \mathrm{Spec}\, k[\![q]\!]/q^n \simeq \mathbb{G}_m \times \mathrm{Spec}\, k[\![q]\!]/q^n$$

for each n. Thus, we obtain a map $\mathrm{Spf}\, k[\![q]\!] \to I\mathscr{M}$ fitting into the above diagram. We claim that there cannot exist a dotted lift: It would imply the existence of a map of group schemes over $k(\!(q)\!)$ from $\mathbb{G}_m$ to an elliptic curve – and such do not exist!

This is one central theme of this paper: That the Tannakian formalism allows us to give positive answers to (Q1) and (Q2) for a large class of morphisms π which are not proper, such as $BG \to \mathrm{Spec}\, k$ for G reductive, $\mathbf{A}^1/\mathbf{G}_m \to \mathrm{Spec}\, k$ (with $\mathbf{G}_m$ acting by a non-trivial character), among many others. In general, we have three main techniques for proving that something is "proper enough" for our purposes:

1. In many interesting cases (including both BG and $\mathbf{A}^1/\mathbf{G}_m$, one can construct an (infinite) semi-orthogonal decomposition of $\mathrm{Perf}(\mathscr{X})$.
2. In other cases (including both BG and $\mathbf{A}^1/\mathbf{G}_m$), one can ask that $\mathscr{X}$ have a family of vector bundles that behave a bit like the powers of an ample line bundle (in terms of generation and erasing Exts).
3. Many stacks $\mathscr{X}$ have a "Chow's lemma set up", by which we mean a representable proper surjective morphism $\mathscr{Y} \to \mathscr{X}$ from a stack $\mathscr{Y}$ which has a family of vector bundles as in (2).

The heart of the paper is dedicated to showing that these reasonable to verify conditions in fact suffices for our purposes. We give more precise formulation of these conditions, and precise statements of our results, in the following section.

Our methods use derived algebraic geometry, but even in the non-derived setting our results are novel. We have used derived methods for two reasons: first because we could obtain stronger results using these methods[1], and second because the existence of derived mapping stacks are essential in a forthcoming paper of the first author. In an attempt to make our results accessible to readers who are less familiar with derived algebraic geometry, we have written the statements of our main results in the introduction purely in terms of classical stacks and refer the reader to the corresponding theorems in the main text for the full and stronger statements in the derived setting.

1.2. **Mapping out of stacks which are "proper enough".** We introduce two properties formulated in terms of the derived category of a stack that will serve as substitutes for properness. The first property,(GE), is a derived version of the Grothendieck existence theorem, and the second, (L), is the existence of a certain adjoint functor. Our main theorem on mapping stacks states that when $\mathscr{X}$ satisfies these properties, and $\mathscr{Y}$ is a stack with affine diagonal, the mapping stack $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})$ is an algebraic stack with affine diagonal.

[1] Most notably: although the conclusion of (3) holds for classical stacks, the proof requires powerful descent results which only hold in the derived setting

The classical Grothendieck existence theorem states that when $\mathscr{X} \to \operatorname{Spec} R$ is a proper morphism and R is a Noetherian ring which is complete with respect to and ideal I, then the canonical restriction functor

$$\operatorname{Coh}(\mathscr{X}) \to \varprojlim \operatorname{Coh}(\mathscr{X} \times_S \operatorname{Spec}(R/I^n))$$

is an equivalence of categories. There are a variety of natural enhancements of this statement in the setting of derived algebraic geometry that one can call "Grothendieck existence theorems for $\mathscr{X}$." The the notion of the formal completion $\widehat{\mathscr{X}}$ of an algebraic stack $\mathscr{X}$ along a cocompact closed subset $Z \subset |\mathscr{X}|$ will play a central role, and we review its construction and properties in Appendix C. Rather than a sheaf of pro-algebras on a topos, we take the definition of $\widehat{\mathscr{X}}$ as a functor of points: maps to $\widehat{\mathscr{X}}$ are maps to $\mathscr{X}$ which factor set-theoretically through Z.

Suppose that $S = \operatorname{Spec} R$ where R is a Noetherian ring (or more generally a Noetherian derived ring) which is complete with respect to an some ideal I defining a closed subscheme of S, and let $\mathscr{X}$ be an S-stack. Let $\widehat{S} = \operatorname{Spf} R$, and consider the fiber product sheaf $\widehat{\mathscr{X}} = \mathscr{X} \times_S \widehat{S}$, which is the formal completion of $\mathscr{X}$ along the preimage of this closed subset. We will make repeated use of Proposition C.3, which provides a special tower of perfect R-algebras $\cdots \to R_1 \to R_0$ such that $\widehat{S} = \varinjlim \operatorname{Spec}(R_n)$ as prestacks. This allows us to express $\widehat{\mathscr{X}} = \varinjlim \mathscr{X}_n$ as prestacks, where $\mathscr{X}_n := \mathscr{X} \times_S \operatorname{Spec} R_n$ (See Remark C.4).

Recall that for a classical algebraic stack $\mathscr{X}$, $\operatorname{Perf}(\mathscr{X})$ denotes the category of complexes which are locally equivalent to finite complexes of locally free modules, and in the Noetherian case $\operatorname{APerf}(\mathscr{X})$ is a natural ∞-categorical enhancement of the right bounded derived category $D^-_{coh}(\mathscr{X})$. Likewise when $\mathscr{X}$ is a classical stack, $QC(\mathscr{X})$ is the natural ∞-categorical enhancement of the unbounded derived category $D_{qcoh}(\mathscr{X})$. In the derived setting one can define the corresponding categories for any derived algebraic stack and in fact any functor of points – in particular these categories are naturally defined for $\widehat{\mathscr{X}}$ (See Appendix B for details about these constructions and their properties). More concretely, we have canonical equivalences $QC(\widehat{X}) \simeq \varprojlim_n QC(\mathscr{X}_n)$, and likewise for $\operatorname{APerf}(\widehat{\mathscr{X}})$ and $\operatorname{Perf}(\widehat{\mathscr{X}})$. We will consider the following analogs of the Grothendieck existence theorem:

$(\mathrm{pGE})_R$ The pullback functor induces an equivalence of ∞-categories

$$\operatorname{Perf}(\mathscr{X}) \longrightarrow \operatorname{Perf}(\widehat{\mathscr{X}}) \simeq \varprojlim_n \operatorname{Perf}(\mathscr{X}_n)$$

$(\mathrm{GE})_R$ The pullback functor induces an equivalence of ∞-categories

$$\operatorname{APerf}(\mathscr{X}) \longrightarrow \operatorname{APerf}(\widehat{\mathscr{X}}) \simeq \varprojlim_n \operatorname{APerf}(\mathscr{X}_n)$$

Remark 1.3. In Section 6 we will discuss several other versions of the Grothendieck existence theorem, including the classical one, which applies to the category of coherent sheaves $\operatorname{Coh}(\mathscr{X})$. In particular we will see that under reasonable hypotheses $(\mathrm{GE})_R$ is equivalent to the classical Grothendieck existence theorem (Lemma 6.9).

Usually $(\mathrm{GE})_R$ or $(\mathrm{pGE})_R$ are regarded as major theorems, proven under the hypothesis that $\mathscr{X}$ is proper over S. We will be considering non-proper $\mathscr{X}$, so we shall instead regard the Grothendieck existence theorem as a *property* of a morphism, which will partially take the place of "properness" for many applications.

Definition 1.4. Let $f\colon \mathscr{X} \to \mathscr{S}$ be a morphism of Noetherian (derived) algebraic stacks. We say that f satisfies (GE) (respectively (pGE)) if for every morphism from a Noetherian affine scheme $S = \operatorname{Spec} R \to \mathscr{S}$, we have that $\mathscr{X} \times_{\mathscr{S}} S$ satisfies $(\mathrm{GE})_R$ (respectively $(\mathrm{pGE})_R$).

Another categorical property enjoyed by a flat and proper morphism of schemes $f : X \to S$ is that the pullback functor $f^* : D_{qc}(S) \to D_{qc}(X)$ admits a left adjoint f_+: for perfect complexes we have $f_+(F) \simeq (f_*(F^\vee))^\vee$, and we define f_+ by approximating an arbitrary complex of quasicoherent sheaves by perfect complexes. The second property we will need is a natural ∞-categorical enhancement of this fact.

Definition 1.5. For any Noetherian affine derived scheme $\operatorname{Spec} R$ and any R-stack $f : \mathscr{X} \to \operatorname{Spec} R$, we say that $\mathscr{X}$ satisfies $(\mathrm{L})_R$ if the pullback functor $f^* : QC(\operatorname{Spec} R) \to QC(\mathscr{X}_T)$ admits a left adjoint f_+. If $f : \mathscr{X} \to \mathscr{S}$ is a morphism of stacks, we say that f satisfies (L) if for any Noetherian affine derived scheme $\operatorname{Spec} R \to \mathscr{S}$, the base change $\mathscr{X} \times_{\mathscr{S}} \operatorname{Spec} R \to \operatorname{Spec} R$ satisfies $(\mathrm{L})_R$.

We will discuss methods for establishing (GE) and (L) and many examples of algebraic stacks which satisfy these two properties, but first we state the main theorem which motivates these definitions.

Theorem 1.6 (Theorem 2.1)**.** *Let $S = \operatorname{Spec} A$ where A is a Noetherian Grothendieck ring (i.e. a G-ring), and let $\mathscr{Y}$ be a locally finitely presented algebraic stack over S whose diagonal $\mathscr{Y} \to \mathscr{Y} \times_S \mathscr{Y}$ is affine. Let $\pi : \mathscr{X} \to S$ be a flat S-stack which is locally Noetherian and satisfies (L), and assume that either of the following hypotheses hold:*

(1) $\mathscr{X}$ and $\mathscr{Y}$ are perfect, and $\mathscr{X}$ satisfies (pGE) and (CD); or
(2) $\mathscr{X}$ satisfies (GE).

Then the classical mapping stack $\underline{\operatorname{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$, as well as its natural derived enhancement $\underline{\operatorname{Map}}_S(\mathscr{X}, \mathscr{Y})$, is algebraic and locally finitely presented over S with an affine diagonal.

Remark 1.7. See the statement of Theorem 2.1 for slightly more general hypotheses in the case when $\mathscr{X}$ satisfies (pGE).

We prove this theorem by applying Artin's Representability Criterion. A large part of the proof – establishing the existence of a cotangent complex, etc. – depends only on (L). The proof under hypotheses (1) and (2) bifurcates only in proving the *integrability* condition: For a complete local ring R over $\mathscr{S}$, any family of maps over $\operatorname{Spf} R$ must come from a family of maps over $\operatorname{Spec} R$. Here we imitate Lurie's argument in [?DAG-XIV] and use variants of the Tannakian formalism.

Another application of (pGE) and (L) is the algebraicity of the moduli of perfect complexes on $\mathscr{X}$. Following Lieblich [?Lieblich], we introduce a notion of a family of universally gluable perfect complexes on a flat morphism. We define a moduli functor $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq 1}$ of such objects in Definition 2.25.

Proposition 1.8 (Corollary 2.28)**.** *Let $\pi : \mathscr{X} \to S$ be a flat morphism satisfying (pGE) and (L). Then the moduli $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq 1}$ of universally gluable perfect complexes on $\mathscr{X}$ is a locally finitely presented algebraic stack.*

The proof also amounts to verifying Artin's criterion for this moduli functor. The key observation is that the integrability property for the moduli functor $\underline{\operatorname{Perf}}(\mathscr{X}/S)$ is expressed exactly by (pGE). In fact we formulate a moduli functor $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n}$ for any n which is an n-stack, and we verify the derived Artin's criterion for these moduli functors. Thus modulo a version of Artin's criterion for higher stacks, which we do not prove, the moduli functor $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n}$ will be an algebraic n-stack.

1.3. Techniques for establishing (GE) and (L).

We dedicate the majority of this paper to developing techniques for proving that a morphism $\mathscr{X} \to \mathscr{S}$ satisfies (GE) or (pGE). Note that (GE) implies (pGE) by identifying $\operatorname{Perf}(\mathscr{X})$ as the dualizable objects of $\operatorname{APerf}(\mathscr{X})$. Our discussion starts with (pGE), whose methods of proof are somewhat simpler, before developing the more general techniques used to establish (GE).

When $\mathscr{X}$ is a perfect stack (See Appendix B), $(\text{pGE})_R$ has the pleasant feature of being purely a property of the R-linear ∞-category $\operatorname{Perf}(\mathscr{X})$, whereas the formulation of $(\text{GE})_R$ depends on the geometry of $\mathscr{X}$. In nice cases, this lets us prove (pGE) using purely categorical decompositions e.g., via semi-orthogonal decompositions:

Theorem 1.9 (Corollary 3.14)**.** *Suppose that $S = \operatorname{Spec} R$ for some Noetherian derived ring R, and $\pi : \mathscr{X} \to S$ is a perfect S stack such that the $\pi_*(\operatorname{Perf}(\mathscr{X})) \subset \operatorname{APerf}(S)$ (which follows from (CD) and $(\text{CP})_R$). Assume that $\operatorname{Perf}(\mathscr{X})$ has an R-linear semi-orthogonal decomposition*

$$\operatorname{Perf}(\mathscr{X}) = \langle \mathcal{A}_i ; i \in I \rangle$$

such that each subcategory $\mathcal{A}_i$ is fully dualizable (i.e., smooth and proper) over $\operatorname{Perf}(R)$. Then, $\mathscr{X} \to \mathscr{S}$ satisfies (pGE).

As discussed at the start of the introduction, such examples include (but are not limited to) $B\mathbf{G}_m$, $\mathbf{A}^1/\mathbf{G}_m$, and BG for linearly reductive groups G. For instance, one can easily deduce (pGE) for quotient stacks of the form V/G where V is a linear representation of a reductive G for which $k[V]^G$ is finite dimensional (See Example 3.16, Corollary 3.17, and the following discussion for more examples). We can also show that if

$\mathscr{X} \to \mathscr{Y}$ is a morphism of perfect stacks which is representable by smooth and proper Deligne-Mumford stacks, and $\mathscr{Y}$ satisfies (pGE), then $\mathscr{X}$ satisfies (pGE) as well.

Our techniques for establishing (GE) come in two flavors. The simplest to state is a descent result which we believe is of independent interest:

Theorem 1.10 (Theorem 4.12)**.** *The presheaf* $\mathrm{APerf}(-)$ *satisfies derived h-descent. (In particular, it has flat descent and descent for proper surjections.)*

This allows us to prove:

Corollary 1.11 (Corollary 4.18)**.** *Suppose R is a Noetherian derived ring, and $I \subset \pi_0 R$ an ideal. Let $S = \operatorname{Spec} R$ and suppose $\mathscr{X}$ is a Noetherian geometric S-stack and that $\pi \colon \mathscr{X}' \to \mathscr{X}$ is a proper surjective morphism such that $\mathscr{X}'$ satisfies (GE) and $\mathscr{X}' \times_{\mathscr{X}} \mathscr{X}'$ satisfies (CP). Then $\mathscr{X}$ satisfies (GE).*

Since (GE) and (CP) are known for proper algebraic spaces by [?DAG-XII] this criterion applies to any stack admitting a proper representable (by a relative algebraic space) morphism from a proper algebraic space.

Our second main technique is by generalizing the classical proof of Grothendieck extistence for projective varieties to a broad class of stacks, including many global quotient stacks in characteristic zero (or more generally for quotient stacks with finite global dimension). To this end, in Section 5 we introduce the notion of a "cohomologically projective" morphism in analogy to a projective morphism. The definition is composed of three components, which we synopsize here:

Definition 1.12. We say that $\pi \colon \mathscr{X} \to \mathscr{S}$ is *cohomologically projective* if it satisfies three conditions (CD), (CP), and (CA);

(1) (CD) requires that π be of universally bounded cohomological dimension when base changed along any $\operatorname{Spec} R \to \mathscr{S}$;
(2) (CP) ("coherent pushforwards") requires that, after base change along any $\operatorname{Spec} R \to S$, each $H_i \circ \pi_*$ preserves coherence;
(3) (CA) ("cohomological ampleness") requires that there be a system $\{V_\alpha\}$ of vector bundles on $\mathscr{X}$ that, after base change along any $\operatorname{Spec} R \to S$, that generate and let one erase-Exts in a suitable way (made precise in Definition 5.6).

The definition of (CA) is rigged so that, in the presence of (CD) and (CP), it is compatible with composition and – for a fixed system of vector bundles – admits a fibral criterion. We have:

Theorem 1.13 (Theorem 6.10)**.** *If f is cohomologically projective, then it satisfies (GE).*

By comparison, (L), is somewhat simpler to establish. We discuss two methods in subsection 6.4 which are analogous to the methods for (GE).

Proposition 1.14 (See Proposition 6.11)**.** *If $\mathscr{X}$ and $\mathscr{S}$ are geometric stacks, and $f : \mathscr{X} \to \mathscr{S}$ is a perfect morphism of finite Tor-dimension satisfying (CD), then f satisfies (L) if and only if it satisfies (CP).*

The property (CP) is often easier to verify in examples. In particular this implies that any flat, cohomologically projective, perfect morphism satisfies (L). In addition we have

Proposition 1.15 (Proposition 6.15)**.** *Let $\mathscr{Y} \xrightarrow{f} \mathscr{X} \xrightarrow{g} \mathscr{S}$ be locally finitely presented morphisms of algebraic stacks where f is surjective and g is flat. If either of the following holds:*

- *f is flat, and f and $f \circ g$ satisfy (L), or*
- *All of the stacks are Noetherian qc.qs., f satisfies (GE), and every level of the Cech nerve $\mathscr{Y}_n := \mathscr{Y} \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \mathscr{Y}$ satisfies (L) over $\mathscr{S}$,*

then g satisfies (L).

This proposition will allow us to establish (L) for large classes of examples.

1.4. **A long list of examples.** There are many pleasant examples of cohomologically projective morphisms, including:

Proposition 1.16 (Proposition 5.14, Proposition 5.17)**.** *Let $f : \mathscr{X} \to S$ be a stack over a base scheme S. If either*

- *$S = \operatorname{Spec}(k)$ for a field k, $\mathscr{X} \simeq X/G$ where G is a linearly reductive algebraic group with a linearizable action on projective-over-affine k-scheme X, and $\Gamma(X, \mathcal{O}_X)^G$ is finite dimensional, or*
- *$\mathscr{X}$ has enough vector bundles and admits a good moduli space which is projective over S* [?Alper]*.*

Then f is cohomologically projective, hence f satisfies (GE).

Under either of the hypotheses above, the global section functor is continuous and $\mathscr{X}$ has enough vector bundles, so the stack $\mathscr{X}$ is perfect. Hence if f is flat, it will follow from Proposition 6.11 that f satisfies (L) as well.

Beyond the cohomologically projective case, we obtain many more examples via a set up which is analogous to the output of Chow's lemma:

Corollary 1.17. *Consider morphisms of Noetherian algebraic stacks*

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{f} & \mathscr{Y} \\ & {\scriptstyle g\circ f}\searrow & \downarrow{\scriptstyle g} \\ & & \mathscr{S} \end{array}$$

where $g \circ f$ is cohomologically projective and f is representable, proper, and surjective. Then g satisfies (GE). If either f and $g \circ f$ are flat, or g and every morphism $\mathscr{X} \times_{\mathscr{Y}} \cdots \times_{\mathscr{Y}} \mathscr{X} \to \mathscr{S}$ are flat, then g satisfies (L).

Proof. This is an immediate consequence of Corollary 4.18 and Proposition 6.15. □

Example 1.18. By Olsson's proper coverings Theorem [?OlssonProper], and proper Artin stack over a Noetherian base scheme S admits a representable proper surjection from a projective S-scheme. Thus we recover (GE) for all proper Artin S-stacks. If $S = \operatorname{Spec} k$ where k is a field, so that every stack is flat over S, then it follows that every proper Artin S-stack satisfies (L) as well.

Example 1.19. Let $\mathbb{G}_m$ act on $\mathbf{P}^1$, and let X be the nodal curve obtained by identifying the two fixed points of $\mathbf{P}^1$. Then $\mathbf{P}^1/\mathbb{G}_m \to X/\mathbb{G}_m$ is a finite morphism, and $\mathbf{P}^1/\mathbb{G}_m$ is cohomologically projective as well as all of the fiber products $\mathbf{P}^1 \times_X \cdots \times_X \mathbf{P}^1/\mathbb{G}_m$, hence $X/\mathbb{G}_m$ satisfies (GE) and (L).

The argument in this example applies to any non-normal projective variety with an action of a linearly reductive group G which is not linearizable.

Example 1.20. Let G be a smooth group scheme over a perfect field k and $1 \to N \to G \to A \to 1$ is the factorization given by Chevalley's theorem, i.e. N is a connected affine group and A is a finite extension of an Abelain variety. This is an example of a Chow set up, because the fiber of $BN \to BG$ is A. If N is linearly reductive, then the corollary above implies that BG satisfies (GE) and (L).

Example 1.21. Any Deligne-Mumford stack over S which admits a finite flat surjective morphism from a flat proper algebraic space over S satisfies (GE) and (L). This is the hypothesis on the source in Olsson's paper on mapping stacks [?Ol06].

Example 1.22. Let $\mathscr{X}$ be an algebraic S-stack which admits a good moduli space $\phi : \mathscr{X} \to X$, where X is proper over S. Choosing a projective S-scheme X' with a surjection $X' \to X$, the base change $\mathscr{X}' := \mathscr{X} \times_X X' \to X'$ is a good moduli space morphism [?Alper]. Hence $\mathscr{X}'$ is a cohomologically projective S-stack, and $\mathscr{X}' \to \mathscr{X}$ is representable and proper. It follows that $\mathscr{X} \to S$ satisfies (GE). If $\mathscr{X}$ is perfect and flat over S, then it also satisfies (L).

We know of one other class of examples of stacks satisfying (GE) and (L) which do not come from a Chow set up as above. If k is a field of positive characteristic and G is a reductive k-group, then we show in Appendix A that BG satisfies (GE) and (L).

1.5. **Comparison with previous results.** The idea of using Artin's criteria to prove the algebraicity of mapping stacks is certainly not new. Martin Olsson has shown in [?Ol06, Theorem 1.1] that if S is an algebraic space and $\mathscr{X}$ and $\mathscr{Y}$ are finitely presented separated Artin stacks over S with finite diagonals, and if

- X is flat and proper over S, and
- locally in the fppf topology on S, there exists a finite and finitely presented flat surjection $Z \to \mathscr{X}$ from an algebraic space Z,

then the fibered category $\mathrm{Map}_S(\mathscr{X}, \mathscr{Y})$ is an Artin stack locally of finite presentation over S with separated and quasi-compact diagonal. If $\mathscr{Y}$ is a Deligne-Mumford stack (resp., algebraic space), then $\mathrm{Map}_S(\mathscr{X}, \mathscr{Y})$ is also a Deligne-Mumford stack (resp., algebraic space).

Olsson uses the local existence of a finite flat surjection $Z \to \mathscr{X}$ to reduce the algebraicity of mapping stacks to a restriction of scalars statement for algebraic spaces: the Weil restriction $f_* \mathcal{F}$ of a stack $\mathcal{F}/X$ along a proper, finitely presented, and flat morphism of algebraic spaces $f : X \to Y$ is algebraic and locally of finite presentation with quasicompact and separated diagonal, provided that $\mathcal{F}$ is a separated Artin stack, locally of finite presentation over X with quasi-compact and separated diagonal [?Ol06, Theorem 1.5].

In the direction of weakening hypotheses on the source, Aoki has proven that if $\mathscr{X}/S$ is proper, and $\mathscr{Y}$ is locally finite presentation and $\mathscr{Y}$ is separated or $\mathscr{Y} = B\mathbb{G}_m$ [?Ao06a, ?Ao06b]. Similarly, Lieblich [?Lieblich, Section 2.3] has shown that if S is an excellent algebraic space and $\mathscr{X} \to S$ is a flat[2] proper Artin stack of finite presentation, then $\mathrm{Map}_S(\mathscr{X}, \mathscr{Y})$ is algebraic and locally of finite presentation whenever $\mathscr{Y} = Z/G$ is a global quotient stack with G a flat linear algebraic group scheme over S and Z separated and finite presentation over S.

Our main theorem extends the previous results on Hom-stacks in two ways. First, we allow arbitrary geometric targets $\mathscr{Y}/S$ (which is a slight generalization of the global quotients considered by Lieblich). More importantly, as discussed above, our result applies to situations where the source $\mathscr{X}/S$ is non-separated, such as when $\mathscr{X}$ is a global quotient stack.

1.6. **Notation and conventions.** Unless we explicitly state otherwise, all of our categories will be ∞-categories, and we will work in the setting of derived algebraic geometry. For readers with a background in classical rather than derived algebraic geometry, we have tried to make the paper readable by substituting classical schemes and stacks into all of the proofs – the main warning is that fiber products denote derived rather than classical fiber products, which makes things like the base change formula work more cleanly. In fact, most of our statements admit proofs in the classical context which do not require derived algebraic geometry (the notable exception being the h-descent results in Section 4), but we have worked in greater generality because one of the main motivating applications requires mapping stacks into derived stacks.

The model for ∞-categories we have in mind is that of quasi-categories [?HigherTopos], and our model for the ∞-category of ∞-groupoids or "spaces" will be Kan simplicial sets, which we denote $\mathcal{S}$. The reader may freely substitute their favorite models for each. We will take our ∞-category of "commutative algebras" to be simplicial commutative rings, and we denote this CAlg.[3] In characteristic 0 we can equivalently work with connective *dg*-algebras, and our discussion and all of our results apply in this context as well, so we will use the ambiguous phrase *derived ring* to denote either of these things.

We define a *pre-stack* to be any functor $F : \mathrm{CAlg} \to \mathcal{S}$, and a stack is a functor which is local for the étale topology on CAlg. We will say that a stack F is a 1*-stack* if for every connective ring (i.e. $R \simeq \pi_0 R$), the ∞-groupoid $F(R)$ is equivalent to the nerve of a classical groupoid (i.e. is 1-truncated). Thus a 1-stack has an underlying classical stack by restricting F to connective algebras. We use the phrase *algebraic stack* to denote a derived 1-stack which admits a surjective morphism $U \to F$ such that U is a disjoint union of affine derived schemes and the morphism is relatively representable by smooth derived algebraic spaces. F is locally Noetherian if these can be taken to be Noetherian derived rings, F is quasi-compact and quasi-separated (qc.qs.) if U can be taken to be affine and $U \to F$ is relatively representable by qc.qs. algebraic spaces.[4] An

[2]The results [?Lieblich, Section 2.3], specifically Lemma 2.3.1, seem to be missing the hypothesis that Y is flat over S.

[3]The key technical result Proposition C.3 does not seem to hold in the ∞-category of E_∞-algebras.

[4]This is a special case of the notion of ∞-quasi-compact which is an inductive and relative notion: Every map of affine schemes is ∞-quasi-compact; a map of functors is ∞-quasi-compact if and only if its base-change to every affine scheme is so; and a higher stack $\mathscr{X}/\operatorname{Spec} R$ is ∞-quasi-compact if it admits an affine atlas $U = \operatorname{Spec} A \to \mathscr{X}$ such that $U \times_{\mathscr{X}} U/\operatorname{Spec} R$ is

algebraic stack is *geometric* if there is an affine derived scheme U and a surjection $U \to F$ which is relatively representable by smooth derived affine schemes.

In addition to working in a derived setting, we depart from the usual algebro-geometric literature in some potentially confusing notational conventions.

Notation 1.23. Let us point out the main such offenses:

(1) We think of our t-structures as *homologically indexed*, and write H_i for H^{-i} and e.g., $\tau_{\leq i}$ for $\tau^{\geq -i}$ and $\mathcal{C}_{\leq i}$ for $\mathcal{C}^{\geq -i}$. In particular, "bounded above" will mean *homologically bounded above* (i.e., lying in $\mathcal{C}_{\leq i}$ for some i).
(2) We implicitly work with ∞-categorical enhancements of various triangulated categories of sheaves. We review this material in Appendix B.
(3) The symbols f_*, f^*, etc. will, unless otherwise stated, denote the functors *of ∞-categories*, as explained in Appendix B. We do not include extra decorations to indicate that they are "derived," and will instead sometimes write e.g., $H_0 \circ f_*$ for the functor on abelian categories. (The one exception is global sections: We will write $R\Gamma$ for global sections, as a reminder that this is global sections of sheaves of spectra and not spaces.)
(4) If $\mathcal{C}$ is an ∞-category, we will use the three symbols Map, Hom, and RHom with distinct meaning: We use Map for a mapping *simplicial set* in an ∞-category; we will use $\underline{\mathrm{Map}}$ to denote other unstable enrichments (e.g., for an enrichment in groupoids, sheaves of groupoids, of sheaves of spaces). We use RHom to denote any stable enrichment (e.g., in spectra, in chain complexes, or in complexes of sheaves) in case $\mathcal{C}$ is stable. Finally, we use Hom to denote the maps in the homotopy category of $\mathcal{C}$.

For example:

Example 1.24. Suppose that $R_\bullet$ is a simplicial commutative ring and let $\mathcal{C} = R_\bullet$-mod – by which we mean the stable ∞-category of left $N(R_\bullet)$-modules in chain complexes. (Here, $N(R_\bullet)$ is the normalized chain complex of $R_\bullet$.) Then,

(1) $\mathcal{C}_{>0}$ is the unstable ∞-category of $N(R_\bullet)$-modules in homologically positive degrees – i.e., Dold-Kan provides an equivalence of $\mathcal{C}_{>0}$ with the ∞-category of simplicial R-modules;
(2) If $M_\bullet, N_\bullet$ are simplicial $R_\bullet$-modules, then $\mathrm{Map}_R(M, N)$ is the simplicial set of maps from a (cofibrant replacement of) M to a (fibrant replacement) of N; meanwhile, $\mathrm{RHom}_R(M, N)$ is the $N(R_\bullet)$-module with $(\mathrm{RHom}_R(M, N))_i$ the degree i morphims from (a replacement of) $N(M_\bullet)$ to (a replacement of) $N(N_\bullet)$; and finally, $\mathrm{Hom}_R(M, N) = \mathrm{Ext}^0_R(M, N)$ is the set of maps in the derived category of $N(R_\bullet)$-modules.

For any prestack $\mathscr{X}$, we let $QC(\mathscr{X})$ denote the stable ∞-category of quasicoherent sheaves on $\mathscr{X}$, whose construction and basic properties we recall in Appendix B. We will write $\mathrm{Coh}(-)$ for the *ordinary* 1-category of coherent modules. More generally, for each $n \geq 0$ we will introduce

$$\mathrm{Coh}^n(\mathscr{X}) := \varprojlim_{\mathrm{Spec}\, A \to \mathscr{X}} \mathrm{Coh}^n(\mathrm{Spec}\, A) \quad \text{where} \quad \mathrm{Coh}^n(\mathrm{Spec}\, A) := (A\text{-mod}^{cn}_{\leq n})^c$$

where as shown $\mathrm{Coh}^n(\mathrm{Spec}\, A)$ denotes the compact objects inside of the n-category of connective, n-truncated A-modules. (We will write $\mathrm{Coh}(\mathscr{X})$ in place of $\mathrm{Coh}^0(\mathscr{X})$.) In particular, if $n = 0$ then $\mathrm{Coh}^0(\mathrm{Spec}\, A)$ denotes the compact objects of the abelian category $(A\text{-mod})^\heartsuit$ – identifying this with modules over the discrete algebra $\pi_0 A$, we identify $\mathrm{Coh}^0(\mathrm{Spec}\, A)$ with the ordinary category of finitely-presented $\pi_0 A$-modules.

1.7. Author's note.

2. Artin's criteria for mapping stacks

In this section we use the categorical properties (GE),(L), and (pGE) to verify Artin's representability criteria for mapping stacks. Recall that for derived S-stacks $\mathscr{X}$ and $\mathscr{Y}$, the mapping stack is defined by the functor of points

$$\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y}) : T \mapsto \mathrm{Map}_S(\mathscr{X} \times_S T, \mathscr{Y}).$$

∞-quasi-compact. If $\mathscr{X}$ is an n-stack for some finite n, then this is really a finitary condition since high enough diagonals of $\mathscr{X}$ are isomorphisms.

where T is a derived affine scheme. When $\mathscr{X}$, S, and T are classical, $\mathrm{Map}_S(\mathscr{X}_T, \mathscr{Y})$ is 1-truncated, so we can regard the restriction to classical T, $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$, as a classical stack in groupoids. We shall prove the following:

Theorem 2.1. *Let $S = \operatorname{Spec} A$ where A is a Noetherian (classical) Grothendieck ring, and let $\mathscr{Y}$ be a derived 1-stack locally almost of finite presentation over S whose diagonal $\mathscr{Y} \to \mathscr{Y} \times_S \mathscr{Y}$ is affine. Let $\pi : \mathscr{X} \to S$ be a flat S-stack which is locally Noetherian and satisfies (L), and assume that either of the following hypotheses hold:*

(1) $\mathscr{X}$ satisfies (pGE) and the functor in (GE) is fully faithful, and $\mathscr{Y}_{\operatorname{Spec} R}$ is perfectly generated for any complete local Noetherian A-algebra R; or
(2) $\mathscr{X}$ satisfies (GE).

Then the classical mapping stack $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$, as well as its natural derived enhancement $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})$, is algebraic and locally almost of finite presentation over S with an affine diagonal.

Remark 2.2. If $\mathscr{X}$ satisfies (pGE), then the fully faithfulness of the functor in (GE) is an equivalence follows with mild hypotheses. See Remark 2.18.

Throughout this section we will let $S = \operatorname{Spec} A$ as in Theorem 2.1. Let us recall Artin's representability criteria [?Artin, Corollary 5.2] for algebraic stacks, which says that a (classical) stack (in groupoids) $\mathcal{F}$ over the big étale site of S-schemes is algebraic if it has the properties:

i) the functor $\mathcal{F}(T)$ is limit preserving [locally almost of finite presentation]
ii) the diagonal $\mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable by algebraic spaces,
iii) $\mathcal{F}$ admits an obstruction theory with certain properties and satisfies the Rim-Schlessinger condition [admits a cotangent complex, infinitesimally cohesive, and nilcomplete]
iv) for a complete Noetherian ring R, the groupoid $\mathcal{F}(\operatorname{Spec} R)$ is equivalent to $\mathcal{F}(\operatorname{Spf} R)$ [integrable]

where we have used square brackets to indicate the terminology for the analogous concepts in the setting of derived algebraic geometry [?DAG-XIV]. The derived versions are slightly stronger – in particular they apply in the context where we regard $\mathcal{F}$ as a functor on derived rings in addition to classical rings.

As of the writing of this paper, we are not aware of a reference for Artin's representability criterion for derived geometric stacks in the literature. Thus our approach will be to verify the derived form or Artin's axioms for $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})$, show that they imply the classical form of Artin's axioms and thus $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$ is algebraic, and finally bootstrap this to show that the derived mapping stack is algebraic in Theorem 2.22. This will occupy Sections 5.1-5.4 below.

First we note that $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})$ is automatically a sheaf on the big fppf site of S-schemes because $\mathscr{Y}$ is (See the proof of [?DAG-XIV, Proposition 3.3.5], and thus $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$ is an étale sheaf. Furthermore the functor $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$ is limit preserving in the sense that if $T = \varprojlim T_i$ is an inverse limit in the category of classical affine schemes, then $\varinjlim \underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})(T_i) \to \underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})(T)$ is an equivalence. This reduces immediately to the corresponding property for the Weil restriction $\pi_* \mathcal{F}$ for an $\mathcal{F}$ which is algebraic and locally of finite presentation (in the classical sense) over $\mathscr{X}$, which follows from [?LMB00, Proposition 4.18].

2.0.1. *Mapping stacks and Weil restriction.* We consider a geometric morphism $f : \mathscr{X} \to \mathscr{Y}$. One defines the Weil restriction of a stack $\mathfrak{F}/\mathscr{X}$ along f by

$$f_* \mathfrak{F}(T) := \mathrm{Map}_{\mathscr{X}}(T \times_{\mathscr{Y}} \mathscr{X}, \mathfrak{F})$$

Both the mapping stack and the Weil restriction are stacks with respect to the fppf topology. Whether each of these stacks is algebraic and locally of finite presentation is related by the following

Lemma 2.3. *Let $f : \mathscr{X} \to \mathscr{S}$ be a morphism of stacks, and let P be a property of morphisms of stacks, thought of as a relative property of stacks. We consider the following*

(1) For all $\mathscr{Y}/\mathscr{S}$ with property P, the mapping stack $\mathrm{Map}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y})/\mathscr{S}$ has property P.
(2) For all $\mathfrak{F}/\mathscr{X}$ with property P, the Weil restriction $f_* \mathfrak{F}/\mathscr{S}$ has property P.

If property P is stable under base change, then $(1) \Rightarrow (2)$. *If in addition P is such that h and $h \circ g$ having P implies g has P, and if f has P, then* $(2) \Rightarrow (1)$.

Proof. On the one hand, we have an isomorphism of stacks

$$\mathrm{Map}_{\mathscr{Y}}(\mathscr{X}, \mathscr{Z}) \simeq f_*(f^*(\mathscr{Z}))$$

On the other hand, we can re-express the Weil restriction as the fiber product

$$\begin{array}{ccc} f_*(\mathfrak{F}) & \longrightarrow & \mathrm{Map}_{\mathscr{Y}}(\mathscr{X}, \mathfrak{F}) \\ \downarrow & & \downarrow \\ \mathscr{Y} & \longrightarrow & \mathrm{Map}_{\mathscr{Y}}(\mathscr{X}, \mathscr{X}) \end{array}$$

Where $\mathfrak{F}$ is regarded as a stack over $\mathscr{Y}$, the right arrow is composition with the morphism $\mathfrak{F} \to \mathscr{X}$, and the bottom arrow corresponds to the identity morphism. □

We apply the lemma primarily in the following

Corollary 2.4. *Let $f : \mathscr{X} \to \mathscr{S}$ be morphism of stacks. If $\mathrm{Map}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y})$ is geometric and l.f.p. for all geometric, l.f.p. $\mathscr{Y}/\mathscr{S}$, then $f_*\mathfrak{F}$ is geometric and l.f.p. for all geometric, l.f.p. $\mathfrak{F}/\mathscr{X}$. If f itself is geometric and l.f.p. then the converse is true.*

We record the following lemma, which allows us to prove the algebraicity of mapping stacks by restricting to the case where $\mathscr{X}$ and $\mathscr{Y}$ are over an affine base.

Lemma 2.5. *In order to show Weil restriction along morphisms $\mathscr{X} \to S$ where S is an algebraic space, it suffices to show Weil restriction for such morphisms where $S = \mathrm{Spec}(A)$ is affine.*

Proof. This is [?Ol06], Lemma 3.2. □

2.1. **Weil restriction of affine stacks.** We will show that the diagonal of $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$ is representable by affine schemes provided $\mathscr{X}$ is flat and satisfies (L) over S and the diagonal of $\mathscr{Y}$ is affine. Hence we shall verify condition (1) of [?Artin, Corollary 5.2]. Lemma 2.3 shows that it suffices to prove that $(\pi_*\mathcal{F})^{cl}$ has affine diagonal when $\mathcal{F} \to \mathscr{X}$ is relatively geometric. Furthermore, $(\pi_*\mathcal{F})^{cl} \simeq \pi_*(\mathcal{F}^{cl})$ in our situation and $\mathcal{F}^{cl}$ is geometric over $\mathcal{F}$ because passing to classical stacks respects limits. Thus we will work entirely with classical stacks (although see Remark 2.8).

First we consider the category $\mathrm{Alg}(\mathcal{O}_{\mathscr{X}})$ of quasicoherent $\mathcal{O}_{\mathscr{X}}$ algebras over $\mathscr{X}$, and likewise for $\mathcal{O}_S$. The pullback $\pi^* : QC(S)^{\heartsuit} \to QC(\mathscr{X})^{\heartsuit}$ descends to a functor $\pi^* : \mathrm{Alg}(\mathcal{O}_S) \to \mathrm{Alg}(\mathcal{O}_{\mathscr{X}})$. We let $\mathrm{Alg}^{l.f.p.}(\bullet)$ denote the full subcategories of locally finitely presented algebras. Note that any quasicoherent sheaf of algebras A is a union of its coherent subsheaves, and so A is l.f.p. if and only if it is coherently generated in the sense that it admits a surjection $S^\bullet(F) \twoheadrightarrow A$ for some coherent sheaf F.

Lemma 2.6. *Let $\pi : \mathscr{X} \to S$ be a flat morphism satisfying (L). Then $\pi^* : \mathrm{Alg}(\mathcal{O}_S) \to \mathrm{Alg}(\mathcal{O}_{\mathscr{X}})$ admits a left adjoint π_+^{alg}. Furthermore π_+^{alg} takes l.f.p. $\mathcal{O}_{\mathscr{X}}$-algebras to l.f.p. $\mathcal{O}_S$-algebras.*

Proof. Let A be a quasicoherent $\mathcal{O}_{\mathscr{X}}$-algebra, and let B be a quasicoherent $\mathcal{O}_S$-algebra. Choosing a presentation for A amounts to finding a quasicoherent sheaf F_0 which generates A as an algebra and a quasicoherent sheaf F_1 which generates the kernel of $S^\bullet(F_0) \to A$ as an ideal. It follows that

$$A = \mathrm{coeq}(S^\bullet(F_1) \rightrightarrows S^\bullet(F_0))$$

where one homomorphism is the augmentation followed by the inclusion $S^\bullet(F_1) \to \mathcal{O}_{\mathscr{X}} \to S^\bullet(F_0)$ and the other homomorphism is induced by the inclusion $F_1 \subset S^\bullet(F_0)$. The colimit is computed in $\mathrm{Alg}(\mathcal{O}_{\mathscr{X}})$. Finally, if A is l.f.p., then one can choose F_0 and F_1 to be coherent.

Because A is a colimit of algebras of the form $S^\bullet(F)$, it suffices to show that the adjoint exists on these algebras. We have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_{\mathscr{X}})}(S^\bullet(F), \pi^*B) &\simeq \mathrm{Hom}_{\mathscr{X}}(F, \pi^*B) \\ &\simeq \mathrm{Hom}_S(\underline{H}^0(\pi_+(F)), B) \\ &\simeq \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_S)}(S^\bullet(\underline{H}^0(\pi_+(F))), B) \end{aligned}$$

□

Proposition 2.7. *Let π satisfy (L), and let $\mathcal{F}$ be an algebraic stack which is representable and affine over $\mathscr{X}$, so $\mathcal{F} \simeq \underline{\operatorname{Spec}}_{\mathscr{X}}(A)$ for some quasicoherent sheaf of $\mathcal{O}_{\mathscr{X}}$-algebras A. Then $\pi_*\mathcal{F} \simeq \underline{\operatorname{Spec}}_S(\pi_+^{alg}(A))$.*

Proof. First assume that S is affine. Consider $\pi_*\mathcal{F}(T)$ where $T = \underline{\operatorname{Spec}}(B)$ for some S-algebra B. We have

$$\pi_*\mathcal{F}(T) = \operatorname{Hom}_{\mathscr{X}}(\underline{\operatorname{Spec}}_{\mathscr{X}}(\pi^*B), \underline{\operatorname{Spec}}_{\mathscr{X}}(A)) \simeq \operatorname{Hom}_{\mathsf{Alg}(\mathcal{O}_{\mathscr{X}})}(A, \pi^*B)$$

It follows that $\pi_*\mathcal{F} \simeq \underline{\operatorname{Spec}}_S(\pi_+^{alg}(A))$ because both sheaves are determined by their values on schemes of the form $T = \underline{\operatorname{Spec}}_S(B)$. The isomorphism $\pi_*\mathcal{F} \simeq \underline{\operatorname{Spec}}_S(\pi_+^{alg}(A))$ is natural with respect to base change, so one can deduce the result for general S from the affine case. ☐

Remark 2.8. Lemma 2.6 and Proposition 2.7 are essentially formal consequences of the existence of a left adjoint π_+, and thus following the proof of [?DAG-XIV, Proposition 3.3.3], it is possible to show that $\pi^* : \operatorname{CAlg}(QC(S)) \to \operatorname{CAlg}(QC(\mathscr{X}))$ admits a left adjoint (in the derived setting) and that the derived analog of Proposition 2.7 holds as well.

Corollary 2.9. *If $\pi : \mathscr{X} \to S$ is flat and satisfies (L), and $\mathcal{F}/\mathscr{X}$ is geometric, then the diagonal $\pi_*\mathcal{F} \to \pi_*\mathcal{F} \times_S \pi_*\mathcal{F}$ is affine.*

Proof. Let T be an S-scheme and consider a morphism $T \to \pi_*\mathcal{F} \times_S \pi_*\mathcal{F}$ defined by a pair of morphisms $f_0, f_1 : \mathscr{X}_T \to \mathcal{F}$ over $\mathscr{X}$. We consider the sheaf of isomorphisms between f_0 and f_1, regarded as a sheaf over $\mathscr{X}_T$. It is defined as the pullback $\mathscr{X}_T \times_{(\mathcal{F} \times_{\mathscr{X}} \mathcal{F})} \mathcal{F}$, so it is representable and affine over $\mathscr{X}_T$ as long as $\mathcal{F}$ is geometric over $\mathscr{X}$. Explicitly, we have

$$\underline{\operatorname{Iso}}_{\mathscr{X}_T}(f_0, f_1)(U/\mathscr{X}_T) = \left\{ \begin{array}{c} \text{2-isomorphisms over } \mathscr{X} : \\ \begin{array}{ccccc} & \nearrow & \mathscr{X}_T & \overset{f_0}{\searrow} & \\ U & & \Downarrow & & \mathcal{F} \\ & \searrow & \mathscr{X}_T & \underset{f_1}{\nearrow} & \end{array} \end{array} \right\}$$

Let $\pi_T : \mathscr{X}_T \to T$ be the base change to T. Then we have a Cartesian diagram

$$\begin{array}{ccc} (\pi_T)_*\underline{\operatorname{Iso}}_{\mathscr{X}_T}(f_0, f_1) & \longrightarrow & \pi_*\mathcal{F} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \pi_*\mathcal{F} \times_S \pi_*\mathcal{F} \end{array}$$

$(\pi_T)_+$ admits a left adjoint by (L), so by Proposition 2.7, the Weil restriction to T is affine over T. This holds for any T, so the diagonal is representable. ☐

2.2. **Deformation theory of the mapping stack.** Here we verify the Rim-Schlessinger criterion and the existence of a deformation-obstruction theory for the mapping stack.

Schlessinger's criterion, in the form of $(S1')$ of [?Artin], is a special case of a property which holds for all algebraic stacks. Namely if $\mathscr{Z}$ is an algebraic stack over $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ and $\operatorname{Spec}(A')$ are affine schemes over $\operatorname{Spec}(A)$ with A' a nilpotent extension of A, then $\mathscr{Z}(A' \times_A B) \to \mathscr{Z}(A') \times_{\mathscr{Z}(A)} \mathscr{X}(B)$ is an equivalence of categories.[5].

Verifying this property for mapping stacks amounts to the following

Lemma 2.10. *Let $Y = Y_0 \coprod_{Y_{01}} Y_1$ be a push out of schemes, where $Y_{01} \hookrightarrow Y_0$, is an infinitesimal thickening. Let $\mathscr{X}$ be an algebraic stack, flat and l.f.p. over Y. Then*

$$\begin{array}{ccc} Y_{01} \times_Y \mathscr{X} & \longrightarrow & Y_1 \times_Y \mathscr{X} \\ \downarrow & & \downarrow \\ Y_1 \times_Y \mathscr{X} & \longrightarrow & \mathscr{X} \end{array}$$

is a pushout diagram of stacks.

[5]In fact, this still holds even when A' is not a nilpotent extension of A

Proof. Choose a simplicial scheme $X_\bullet$ which presents the stack $\mathscr{X}$. By hypothesis each X_i is flat over Y. $X_i \times_Y Y_0$ is a presentation for $\mathscr{X} \times_Y Y_0$ and we have analogous presentations for Y_0, Y_{01}. $\mathscr{X}$ is the homotopy colimit of the simplicial diagram $X_\bullet$, and homotopy colimits commute, so it suffices to prove the proposition for the schemes X_i, which is [?Ol06, Lemma 5.6]. ◻

Note that the fully faithful embedding of the ∞-category of classical stacks into the ∞-category of derived stacks preserves colimits, so this lemma applies to $\mathscr{X}/Y$ regarded as a derived stack. Thus we have

Corollary 2.11. *Let $\mathscr{X}$ be an algebraic stack, flat and locally of finite presentation over a classical affine scheme S, and let $\mathscr{Y}$ be a derived algebraic stack. Then the functor $\mathrm{Map}_S(\mathscr{X}, \mathscr{Y})^{cl}$ satisfies condition $(S1')$ of* [?Artin].

We have verified the Rim-Schlessinger condition for the classical mapping stack. We observe that the derived versions hold as well. For convenience we recall the following notions for a prestack:

- *nilcomplete*: The natural map $\mathscr{F}(R) \longrightarrow \mathscr{F}(\varprojlim_n \tau_{\leq n} R)$ is an equivalence.
- *cohesive*: Suppose that
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \oplus M[+1] \end{array}$$
is a pullback square in CAlg. Then the natural map
$$\mathscr{F}(A) \longrightarrow \mathscr{F}(A') \times_{\mathscr{F}(B')} \mathscr{F}(B)$$
is an equivalence.
- *infinitesimally cohesive*: $\mathscr{F}$ preserves those pullback squares in CAlg such that $\pi_0 A \to \pi_0 A'$ and $\pi_0 A \to \pi_0 B$ are surjective with nilpotent kernel.

Scholium 2.12. *With the hypotheses of Theorem 2.1, the functor $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})$ is nilcomplete and infinitesimally cohesive in the sense of* [?DAG-XIV].

Proof. As above, we can realize $\mathscr{X}$ as a colimit of a simplicial diagram $X_\bullet$ of flat S-schemes. Because $\mathscr{Y}$ satisfies fppf descent, $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})$ is a limit of $\underline{\mathrm{Map}}_S(X_i, \mathscr{Y})$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}/S, \mathcal{S})$. The subcategory of infinitesimally cohesive and nilcomplete functors is closed under small limits ([?DAG-XIV, Remark 2.1.11]), so it suffices to prove the claim for X_i which is [?DAG-XIV, Proposition 3.3.6]. ◻

Next we construct the cotangent complex for $\mathrm{Hom}_S(\mathscr{X}, \mathscr{Y})$. In derived algebraic geometry, a cotangent complex of a functor $\mathscr{X} : \mathrm{CAlg} \to \mathcal{S}$ is an object $L_{\mathscr{X}} \in QC(\mathscr{X})$ such that for any $a : \mathrm{Spec}(A) \to \mathscr{X}$ corresponding to a point in $\mathscr{X}(A)$,
$$\mathscr{X}(A \oplus M) \times_{\mathscr{X}(A)} \{a\} \simeq \Omega^\infty R\,\mathrm{Hom}_A(a^* L_{\mathscr{X}}, M)$$
The existence of such an object is not automatic, but when $\mathscr{X}$ admits a cotangent complex $L_{\mathscr{X}}$ is defined up to canonical isomorphism.

Formation of the split square zero extension $A \oplus M$ is compatible with pullback in the sense that for any map of rings $A \to B$, corresponding to a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, we have $B \otimes_A (A \oplus M) \simeq B \oplus (B \otimes_A M)$. This motivates the definition of the trivial square-zero extension $\mathscr{X}[F]$ for any functor $\mathscr{X}$ and any $F \in QC(\mathscr{X})^{cn}$ by the fiber square
$$\begin{array}{ccc} \mathrm{Spec}(A \oplus a^* F) & \longrightarrow & \mathscr{X}[F] \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{a} & \mathscr{X} \end{array} \quad (1)$$
In other words
$$\mathscr{X}[F](A) := \{a \in \mathscr{X}(A) \text{ and a section of } \mathrm{Spec}(A \oplus a^* F) \to \mathrm{Spec}(A)\},$$
where the space of sections of $\mathrm{Spec}(A \oplus a^* F) \to \mathrm{Spec}(A)$ can further be identified with $\Omega^\infty R\,\mathrm{Hom}_A(L_A, a^* F)$.

Lemma 2.13. *Let $S = \operatorname{Spec}(A)$, and let $\mathscr{X}, \mathscr{Y} : \mathrm{CAlg}_{A/} \to \mathcal{S}$ be functors such that $\mathscr{Y}$ admits a cotangent complex. Then for any $f : \mathscr{X} \to \mathscr{Y}$ over S there is a canonical isomorphism*

$$\operatorname{Map}_S(\mathscr{X}[F], \mathscr{Y}) \times_{\operatorname{Map}_S(\mathscr{X},\mathscr{Y})} \{f\} \simeq \Omega^\infty R\operatorname{Hom}_{\mathscr{X}}(f^* L_{\mathscr{Y}/S}, F)$$

as functors $QC(\mathscr{X})^{cn} \to \mathcal{S}$

Proof. Let $\operatorname{Aff}/\mathscr{X}[F]$ denote the ∞-category of affine schemes along with a morphism to $\mathscr{X}[F]$. By the ∞-categorical Yoneda lemma, we have

$$\mathscr{X}[F] = \operatorname*{colim}_{T \in \operatorname{Aff}/\mathscr{X}[F]} T$$

By the canonical fiber square (1), we have a functor $\operatorname{Aff}/\mathscr{X} \to \operatorname{Aff}/\mathscr{X}[F]$ mapping $T \mapsto T \times_{\mathscr{X}} \mathscr{X}[F] \simeq T[F|_T]$. This functor is cofinal because any morphism $T \to \mathscr{X}[F]$ factors canonically through $T \times_{\mathscr{X}} \mathscr{X}[F] \to \mathscr{X}[F]$, and this factorization is initial in the category of factorizations $T \to T'[F|_{T'}] \to \mathscr{X}[F]$ for varying T'. Thus we can write $\mathscr{X}[F]$ as a colimit over $\operatorname{Aff}/\mathscr{X}$

$$\mathscr{X}[F] = \operatorname*{colim}_{\eta : T \to \mathscr{X}} T[\eta^* F]$$

Hence on mapping spaces of presheaves we have

$$\operatorname{Map}_S(\mathscr{X}[F], \mathscr{Y}) = \lim_{\substack{(\operatorname{Aff}/\mathscr{X})^{op} \\ \eta : T \to \mathscr{X}}} \mathscr{Y}(T[\eta^* F]), \text{ whereas}$$

$$\operatorname{Map}_S(\mathscr{X}, \mathscr{Y}) = \lim_{\substack{(\operatorname{Aff}/\mathscr{X})^{op} \\ \eta : T \to \mathscr{X}}} \mathscr{Y}(T)$$

Taking fibers commutes with limits, so

$$\operatorname{Map}_S(\mathscr{X}[F], \mathscr{Y}) \times_{\operatorname{Map}_S(\mathscr{X},\mathscr{Y})} \{f\} \simeq \lim_{\substack{(\operatorname{Aff}/\mathscr{X})^{op} \\ \eta : T \to \mathscr{X}}} \Omega^\infty R\operatorname{Hom}_T(\eta^* f^* L_{\mathscr{Y}/S}, \eta^* F)$$

Where we have used the defining property of $L_{\mathscr{Y}/S}$ as controlling square-zero extensions of morphisms from affine schemes. This last expression is essentially the definition of $\Omega^\infty R\operatorname{Hom}_{QC(\mathscr{X})}(f^* L_{\mathscr{Y}/S}, F)$. □

From this lemma, we deduce that

Proposition 2.14. *Let $\mathscr{X}$ be a stack over S which satisfies (L) and let $\mathscr{Y}$ be a functor which admits a cotangent complex, then $\underline{\operatorname{Map}}_S(\mathscr{X}, \mathscr{Y})$ admits a cotangent complex. Furthermore if $\mathscr{S}$ is flat over S and $L_{\mathscr{Y}/S}$ is almost perfect then $L_{\underline{\operatorname{Map}}_S(\mathscr{X},\mathscr{Y})/S}$ is almost perfect as well.*

Proof. Let $\mathfrak{M} := \underline{\operatorname{Map}}_S(\mathscr{X}, \mathscr{Y})$. Let $f \in \mathfrak{M}(A)$ correspond to an affine scheme $\operatorname{Spec} A$ over S, together with a map $f : \mathscr{X}_A \to \mathscr{Y}$ over S. Let $M \in QC(A)^{cn}$. Then by definition $\mathfrak{M}(A \oplus M) = \operatorname{Map}_S(\mathscr{X}_{A \oplus M}, \mathscr{Y})$. If $\pi_A : \mathscr{X}_A \to \operatorname{Spec}(A)$ is the structure morphism, then $\mathscr{X}_{A\oplus M} \simeq \mathscr{X}_A[p^* M]$ over $\operatorname{Spec}(A)$, by the construction of the square zero extension of functors. Hence by Lemma 2.13 we have a canonical isomorphism

$$\mathfrak{M}(A \oplus M) \times_{\mathfrak{M}(A)} \{f\} \simeq \Omega^\infty R\operatorname{Hom}(f^* L_{\mathscr{Y}/S}, p^* M)$$

By hypothesis (L), the functor π_A^* has a left adjoint $(\pi_A)_+$, hence we can define $L_{\mathfrak{M}/S}|_{\operatorname{Spec}(A)} := (\pi_A)_+ f^* L_{\mathscr{Y}/S}$. For any map of rings $\phi : A \to B$ we have the pullback square

$$\begin{array}{ccccc} \mathscr{X}_B & \xrightarrow{\phi'} & \mathscr{X}_A & \xrightarrow{f} & \mathscr{Y} \\ \downarrow & & \downarrow & & \\ \operatorname{Spec}(B) & \xrightarrow{\phi} & \operatorname{Spec}(A) & & \end{array}$$

By Corollary B.17, we have a natural isomorphism $\phi^*(\pi_A)_+ f^* L_{\mathscr{Y}/S} \simeq (\pi_B)_+(\phi' \circ f)^* L_{\mathscr{Y}/S}$. Hence the assignment $\operatorname{Spec}(A)/\mathfrak{M} \mapsto (\pi_A)_+ f^* L_{\mathscr{Y}/S}$ determines an object of $L_{\mathfrak{M}/S} \in QC(\mathfrak{M})$.

In order to verify that $L_{\mathfrak{M}/S}$ is almost perfect if $L_{\mathscr{X}/S}$ is almost perfect and π is flat, we must show that $(\pi_A)_+$ preserves almost perfect objects, which follows from Lemma 6.13. □

Finally, in order to apply the classical Artin representability criterion to $\underline{\operatorname{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$, we relate derived deformation theory to classical deformation theory. Let $F : \mathrm{Ring} \to \mathrm{Gpd}$. Recall that an obstruction theory [?Artin] for the functor F assigns to each nilpotent extension $A \to A_0$ with A_0 reduced and each $a \in F(A)$:

- an A_0-linear functor, $\mathcal{O}_a : \mathrm{Coh}(A_0) \to \mathrm{Coh}(A_0)$, and
- to each surjection $A' \to A$ whose kernel is a finite A_0-module M (in particular, a square-zero extension) an element $o_a \in \mathcal{O}_a(M)$ which vanishes iff a extends to $F(A')$.

The module $\mathcal{O}_a(M)$ is functorial in the triple (A_0, a, M), and the element o_a is functorial in the "deformation situation" given by the pair $(A' \to A, a \in F(a))$.

Lemma 2.15. *Let $\mathscr{X} : \mathrm{CAlg} \to \mathcal{S}$ be a functor which admits an almost perfect cotangent complex, is infinitesimally cohesive in the sense of* [?DAG-XIV]*, and such that $\mathscr{X}$ takes 1-truncated values on discrete rings. Then $\mathscr{X}^{cl} := \Pi_1 \circ \mathscr{X}|_{\mathrm{CAlg}^0} : \mathrm{CAlg}^0 \to \mathrm{Gpd}$ admits an obstruction theory and satisfies conditions (S2) and (4) of* [?Artin, Corollary 5.2].

Proof. Given a sequence of nilpotent extensions of classical rings $A' \to A \to A_0$ with $M = \ker(A' \to A)$ a finite A_0 module, we can regard them as 0-truncated derived rings, and $A' \to A$ has a canonical structure of a derived square-zero extension classified by a homomorphism $\eta : L_A \to M[1]$.

We have [?HigherAlgebra, Remark 8.4.1.7] a pullback diagram of algebras, which remains a pullback square of spaces because $\mathscr{X}$ is infinitesimally cohesive

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle d_\eta} \\ A & \xrightarrow{d_0} & A \oplus M[1] \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathscr{X}(A') & \longrightarrow & \mathscr{X}(A) \\ \downarrow & & \downarrow{\scriptstyle d_\eta^*} \\ \mathscr{X}(A) & \xrightarrow{d_0^*} & \mathscr{X}(A \oplus M[1]) \end{array}$$

Thus $\mathrm{fib}_a(\mathscr{X}(A') \to \mathscr{X}(A))$ is canonically identified with the space of paths from $d_0^*(a)$ to $d_\eta^*(a)$ in the space

$$\mathrm{fib}_a\left(\mathscr{X}(A \oplus M[1]) \to \mathscr{X}(A)\right) \simeq \Omega^\infty R\,\mathrm{Hom}(a^* L_{\mathscr{X}}, M[1])$$

Consequently we can define the obstruction theory

$$\mathcal{O}_a(M) := \pi_0 \Omega^\infty R\,\mathrm{Hom}_A(a^* L_{\mathscr{X}}, M[1]) \simeq H_0 R\,\mathrm{Hom}(a^* L_{\mathscr{X}}, M[1])$$
$$o_a(A' \to A) := d_\eta^*(a)$$

The functor $\mathcal{O}_a$ is A_0-linear and functorial in (A_0, a, M) by its construction. From the above discussion, we have that a extends to $\mathscr{X}(A')$ if and only if $o_a(A' \to A)$ vanishes, if the obstruction vanishes then the set of lifts is canonically a torsor under $H_1 R\,\mathrm{Hom}(a^* L_{\mathscr{X}}, M[1])$, and the set of automorphisms of a lift which induce the identity on a is canonically isomorphic to $H_2 R\,\mathrm{Hom}(a^* L_{\mathscr{X}}, M[1])$. This and the fact that $L_{\mathscr{X}}$ is almost perfect imply the conditions (S2) and (4) of [?Artin, Corollary 5.2] ☐

2.3. **Integrability via the Tannakian formalism.** Recall that a functor $\mathcal{F} : \mathrm{CAlg}_S \to \mathcal{S}$ is said to be integrable if for any complete local Noetherian derived ring R over S, the canonical morphism $\mathcal{F}(\mathrm{Spec}\, R) \to \mathcal{F}(\mathrm{Spf}\, R)$ is an equivalence. If $\mathcal{F}$ is an integrable functor and R is a classical with maximal ideal $\mathfrak{m}$, then Proposition C.5, along with the fact that the Kan extension from classical prestacks to derived prestacks respects colimits, implies that

$$\mathcal{F}(R) \to \varprojlim_n \mathcal{F}(R/\mathfrak{m}^n)$$

is an equivalence. In classical terminology, this means that $\mathcal{F}^{cl}$ is compatible with completions, i.e. it satisfies condition (3) of [?Artin, Corollary 5.2]. In this section we will prove the integrability of the derived mapping stack $\underline{\mathrm{Map}}_S(\mathscr{X}, \mathscr{Y})$ under the hypotheses of Theorem 2.1.

Theorem 2.16 ([?DAG-VIII, Theorem 3.4.2][6])**.** *Let Y be a geometric stack and $X : \mathrm{CAlg} \to \mathcal{S}$ be an arbitrary functor. Then assigning $f : X \to Y$ to $f^* : QC(Y)^{cn} \to QC(X)^{cn}$ determines a fully faithful embedding*

$$\mathrm{Map}_{Fun(\mathrm{CAlg}, \mathcal{S})}(X, Y) \to Fun^{\otimes}(QC(Y)^{cn}, QC(X)^{cn})$$

whose essential image is the full subcategory spanned by those symmetric monoidal functors which preserve small colimits and flat objects.

[6] We have rephrased [?DAG-VIII, Theorem 3.4.2] in a manner which is more convenient to our application. By [?DAG-XII, Lemma 5.4.6], the ∞-category of symmetric monoidal functors $QC(\mathscr{Y}) \to QC(\mathscr{X})$ which are right t-exact and preserve small colimits is equivalent to the ∞-category of symmetric monoidal functors $QC(\mathscr{Y})^{cn} \to QC(\mathscr{X})^{cn}$ which preserve small colimits.

It is straightforward to extend Theorem 2.16 to a relative statement identifying maps over $\operatorname{Spec}(R)$ with symmetric monoidal functors of module categories over $QC(R)$.

Proposition 2.17. *Let $S = \operatorname{Spec}(R)$, where R is Noetherian derived ring, complete with respect to an ideal $I \subset \pi_0 R$, let $\mathscr{Y} \to S$ be a Noetherian geometric stack, and let $\pi : \mathscr{X} \to S$ be a locally Noetherian algebraic stack. Assume that either of the following hypotheses hold:*

(1) $\mathscr{X}$ satisfies $(GE)_R$; or
(2) $\mathscr{X}$ satisfies $(pGE)_R$ and the functor in $(GE)_R$ is fully-faithful, and $\mathscr{Y}$ is perfectly generated.

Let $\cdots \to R_1 \to R_0$ be tower of perfect R-algebras such that $\operatorname{Spf} R \simeq \varinjlim \operatorname{Spec} R_n$, as provided by Proposition C.3 for instance. If we let $\mathscr{X}_n := \mathscr{X} \times_S \operatorname{Spec} R_n$, then the canonical functor

$$\operatorname{Map}_S(\mathscr{X}, \mathscr{Y}) \to \varprojlim \operatorname{Map}_S(\mathscr{X}_n, \mathscr{Y})$$

is an equivalence of ∞-groupoids.

Remark 2.18. It is fairly easy to establish the fully faithful part of $(GE)_R$, which states that for any $F, G \in \operatorname{DCoh}(\mathscr{X})$, the canonical map $\operatorname{RHom}_{\mathscr{X}}(F,G) \to \operatorname{RHom}_{\widehat{\mathscr{X}}}(\hat{i}^* F, \hat{i}^* G)$ is a quasi-isomorphism. This holds when either

- $\mathscr{X}$ satisfies $(pGE)_R$, (CD), and is perfectly generated (Lemma 3.6), or
- $\mathscr{X}$ is geometric, Noetherian, and satisfies $(CP)_R$ (Proposition 6.8).

Before proving the proposition, we note that in Theorem 2.16, flat objects are those whose pullback to any affine scheme are flat. We observe that local flatness and global flatness agree (When $\mathscr{X}$ is a geometric Deligne-Mumford stack, this is [?DAG-XII, Lemma 5.4.8]):

Lemma 2.19. *Let $\mathscr{X}$ be an algebraic stack, and let $\phi : X \to \mathscr{X}$ be an fpqc morphism from an affine scheme, and let $M \in QC(\mathscr{X})^{cn}$. Then the following are equivalent:*

(1) M is flat on $\mathscr{X}$,
(2) $\phi^* M \in QC(X)$ is flat,
(3) For every $F \in QC(\mathscr{X})_{<0}$, $M \otimes F \in QC(\mathscr{X})_{<0}$, and
(4) For every $F \in QC(X)^{\heartsuit}$, $M \otimes F \in QC(X)^{\heartsuit}$.

If $\mathscr{X}$ is locally Noetherian and qc.qs., then these are also equivalent to

(5) For every $F \in \operatorname{Coh}(X)^{\heartsuit}$, $M \otimes F \in QC(X)_{<0}$

Proof. (1) and (2) are equivalent because flatness of modules on schemes can be checked fpqc locally. (3) and (4) are equivalent because $M \otimes (\bullet)$ is right t-exact. The implication (2) $\Rightarrow$ (3) follows from the fact that the t-structure on $\mathscr{X}$ is fpqc local.

For (3) $\Rightarrow$ (2):

Let $X_1 = X \times_{\mathscr{X}} X$ and let $p_1, p_2 : X_1 \to X$ be the respective projections, and let $e : X \to X_1$ be the diagonal. Note that for any $F \in QC(X)_{<0}$,

$$\phi^* \phi_*(F) \simeq (p_2)_* p_1^*(F)$$

and the canonical morphism $F \to (p_2)_* p_1^*(F)$ admits a retract

$$(p_2)_* p_1^*(F) \to (p_2)_* e_* e^* p_1^*(F) \simeq F$$

It follows that it is enough to prove that $\phi^* \phi_*(\phi^* M \otimes F) = \phi^*(M \otimes \phi_*(F))$ is co-connective, which follows from the flatness of ϕ and (3).

Finally, the fact that these are equivalent to (5) for a locally Noetherian qc.qs. stack follows from Theorem B.11 and the fact that the t-structure is compatible with filtered colimits. $\square$

Proof of Proposition 2.17. Theorem 2.16 reduces Proposition 2.17 to showing that

$$Fun^{\otimes}_{QC(R)^{cn}}(QC(\mathscr{Y})^{cn}, QC(\mathscr{X})^{cn}) \to \varprojlim Fun^{\otimes}_{QC(R)^{cn}}(QC(\mathscr{Y})^{cn}, QC(\mathscr{X}_n)^{cn})$$

is a homotopy equivalence on the components which correspond to functors preserving small colimits, connective objects, and flat objects. The proof is only slightly different with hypotheses (1) versus (2), and it essentially follows the proof of [?DAG-XII, Theorem 5.4.1].

Let $f_n^* : QC(\mathscr{Y})^{cn} \to QC(\mathscr{X}_n)^{cn}$, $f_{n+1}^*|_{\mathscr{X}_n} \simeq f_n^*$ be an inverse system of R-linear symmetric monoidal functors induced by an inverse system of maps in $f_n \in \mathrm{Map}_S(\mathscr{X}_n, \mathscr{Y})$. We must show that this system extends uniquely, up to a contractible space of choices, to an R-linear symmetric monoidal functor $f^* : QC(\mathscr{Y})^{cn} \to QC(\mathscr{X})^{cn}$ which preserves small colimits, connective objects, and flat objects. We prove this in three steps:

Claim 1: There is a unique extension (up to a contractible space of choices) of the functors f_n^* to a symmetric monoidal functor f^* preserving small colimits and almost-perfect objects.

Because the functors f_n^* are induced from maps of stacks $\mathscr{X}_n \to \mathscr{Y}$, they induce a symmetric monoidal functor

$$\hat{f}^* : \mathrm{APerf}(\mathscr{Y}) \to \mathrm{APerf}(\widehat{\mathscr{X}}) := \varprojlim \mathrm{APerf}(\mathscr{X}_n)$$

preserving perfect objects, connective objects, and small colimits which exist in $\mathrm{APerf}(\mathscr{Y})$.[7]

For any stable symmetric monoidal category $\mathcal{C}$ with a compatible t-structure, the subcategory $\mathcal{C}_{\leq p}$ is a (left) localization and thus inherits a symmetric monoidal structure (for $p > 0$). The functor $\hat{f}^*$ maps $\mathrm{APerf}(\mathscr{Y})_{\geq p}$ to $\mathrm{APerf}(\widehat{\mathscr{X}})_{\geq p}$ and thus induces a symmetric monoidal functor

$$\mathrm{DCoh}(\mathscr{Y})^{cn}_{\leq p} \to \mathrm{DCoh}(\widehat{\mathscr{X}})^{cn}_{\leq p}. \tag{2}$$

Under hypothesis (3), $(\mathrm{GE})_R$ implies that the restriction functor $\mathrm{DCoh}(\mathscr{X})^{cn}_{\leq p} \to \mathrm{DCoh}(\widehat{\mathscr{X}})^{cn}_{\leq p}$ is an equivalence, so there is a unique extension to a symmetric monoidal functor $\mathrm{DCoh}(\mathscr{Y})^{cn}_{\leq p} \to \mathrm{DCoh}(\mathscr{X})^{cn}_{\leq p}$.

By Theorem B.11 we have $QC(\mathscr{Y})^{cn}_{\leq p} \simeq \mathrm{Ind}(\mathrm{DCoh}(\mathscr{Y})^{cn}_{\leq p})$ as symmetric monoidal categories. It follows that $\hat{f}^*$ extends uniquely to a family of functors

$$QC(\mathscr{Y})^{cn} \to QC(\mathscr{Y})^{cn}_{\leq p} \to QC(\mathscr{X})^{cn}_{\leq p}$$

which preserve colimits and almost perfect objects by construction,[8] and are compatible with the restriction functors $QC(\mathscr{X})^{cn}_{\leq p+1} \to QC(\mathscr{X})^{cn}_{\leq p}$. Using the left-completeness of the t-structure on $QC(\mathscr{X})$, this family of functors extends uniquely to a symmetric monoidal functor $f^* : QC(\mathscr{Y})^{cn} \to QC(\mathscr{X})^{cn}$ which preserves colimits and almost perfect objects.

Under hypothesis (1) or (2), we proceed differently:

Let $\mathcal{C} \subset \mathrm{DCoh}(\mathscr{Y})_{\leq p}$ be the full symmetric monoidal subcategory which is the essential image of $\mathrm{Perf}(\mathscr{Y}) \to \mathrm{DCoh}(\mathscr{Y})_{\leq p}$ by truncation. The fully-faithfulness part of $(\mathrm{GE})_R$ identifies $\mathrm{DCoh}(\mathscr{X})_{\leq p}$ with a full subcategory of $\mathrm{DCoh}(\widehat{\mathscr{X}})_{\leq p}$, and $(\mathrm{pGE})_R$ implies that $\hat{f}^*$ maps $\mathcal{C}$ to this subcategory. Thus there is a unique lift of $\hat{f}^*|_{\mathcal{C}}$ to a functor $f^* : \mathcal{C} \to \mathrm{DCoh}(\mathscr{X})_{\leq p}$ preserving colimits which exist in $\mathcal{C}$. By Corollary B.12, the inclusion $\mathcal{C} \subset \mathrm{DCoh}(\mathscr{Y})_{\leq p}$ exhibits the latter as an idempotent completion of $\mathcal{C}$, thus because $\mathrm{DCoh}(\mathscr{X})_{\leq p}$ is idempotent complete f^* extends uniquely to a functor $f^* : \mathrm{DCoh}(\mathscr{Y})_{\leq p} \to \mathrm{DCoh}(\mathscr{X})_{\leq p}$ [?HigherTopos, Section 4.4.5].

By fully faithfulness and the fact that f^* extends $\hat{f}^*$ on the subcategory $\mathcal{C}$, it follows that f^* extends $\hat{f}^*$ on all of $\mathrm{DCoh}(\mathscr{Y})_{\leq p}$. Furthermore an object $F \in \mathrm{DCoh}(\mathscr{X})_{\leq p}$ is connective if and only $R\,\mathrm{Hom}(F, G) = 0$ for all $G \in \mathrm{DCoh}(\mathscr{X})_{<0}$, so fully-faithfullness and the fact that $\hat{f}^*$ preserves connective objects implies that f^* preserves connective objects. Thus we have our extension $\hat{f}^* : \mathrm{DCoh}(\mathscr{Y})^{cn}_{\leq p} \to \mathrm{DCoh}(\mathscr{X})^{cn}_{\leq p}$, and the rest of the argument proceeds as above.

[7] Colimit diagrams in $\mathrm{APerf}(\widehat{\mathscr{X}})$ are exactly those diagrams whose restriction to each $\mathscr{X}_n$ is a colimit, and for each n the functor f_n^* preserve colimits

[8] A slightly different approach, taken in the proof of [?DAG-XII, Theorem 5.4.1], would be to define a functor $\mathrm{DCoh}(\bullet)^{cn}_{\leq p}$, from stacks to symmetric monoidal categories, and then show that $\mathrm{DCoh}(\mathscr{X})^{cn}_{\leq n} \to \varprojlim \mathrm{DCoh}(\mathscr{X}_i)^{cn}_{\leq n}$ is an equivalence for each n. Although we have not used this version of Grothendieck existence, we remark that it still holds in our context by a slight modifications of Lemma 6.9

Claim 2: There are affine schemes U, V with smooth surjective morphisms $U \to \mathscr{X}$ and $V \to \mathscr{Y}$ which fit into a commutative diagram

$$\begin{array}{ccccc} \hat{U} & \longrightarrow & \hat{\mathscr{X}} & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow & \swarrow & \\ V & \longrightarrow & \mathscr{Y} & & \end{array} \tag{3}$$

Where $\hat{U}$ is the formal completion of U along the morphism $U \to \mathscr{X} \to \operatorname{Spec}(R)$, and the dotted arrow indicates the existence of the symmetric monoidal functor $f^* : QC(\mathscr{Y}) \to QC(\mathscr{X})$ which we do not yet know is induced by a morphism of stacks.

First note that if $U \to \mathscr{X}$ were an atlas such that the composition $U_0 := U \times_R R_0 \to \mathscr{X}_0 \to \mathscr{Y}$ lifted to V over $\mathscr{Y}$, then we could lift $\hat{U} \to \hat{\mathscr{X}} \to \mathscr{Y}$ to V over $\mathscr{Y}$. This is because $V \to \mathscr{Y}$ is formally smooth, and each U_{i+1} is a square-zero extension of the affine scheme $U_i := U \times_R R_i$. Thus it suffices to find some affine presentation of $\mathscr{X}$ such that U_0 lifts to V.

Start with an arbitrary affine presentation $U' \to \mathscr{X}$. Because $V \to \mathscr{Y}$ is a representable smooth surjective morphism, the map $U_0' \to \mathscr{Y}$ lifts to V after restricting to some étale cover $U_0'' \to U_0'$. From the structure theory of étale morphisms of affine schemes [?DAG-VII, Proposition 8.10], there is an étale $U'' \to U'$ whose restriction $U'' \times_{U'} U_0' \simeq U_0''$. Combining U'' with a cover of $U \setminus U_0$ by affine opens gives the desired presentation $U \to \mathscr{X}$.

Claim 3: If f^* is a symmetric monoidal functor which preserves connective objects and almost perfect objects and extends the f_n^*, then it must preserve flat objects.

Let $M \in QC(\mathscr{Y})$ be flat. We will use the characterization, which follows immediately from the lemma above, that $f^*(M) \in QC(\mathscr{X})$ is flat if and only if $\forall N \in \operatorname{Coh}(\mathscr{X})^\heartsuit$ and $\forall m > 0$,

$$H_m(f^*(M) \otimes N) = H_m(f^*(\tau_{\leq m} M) \otimes N) = 0.$$

Choose a U and V fitting into the diagram (3) above. By Theorem B.11, we can write $\tau_{\leq m} M$ as a filtered colimit $\tau_{\leq m} M = \varinjlim M_\alpha$ with $M_\alpha \in \operatorname{Coh}(\mathscr{Y})^{cn}_{\leq m}$. In addition $M|_V$ can be written as a a filtered colimit of free modules P_i of finite rank because it is flat [?HigherAlgebra, 7.2.2.15], so

$$\tau_{\leq m} M|_V = \varinjlim M_\alpha|_V \simeq \varinjlim \tau_{\leq m} P_i.$$

So for each α, there is a $\beta \geq \alpha$ such that $M_\alpha|_V \to M_\beta|_V$ factors though a $\tau_{\leq m} P_i$ for some i. It follows that $\tau_{\leq m}(f^*(M_\alpha) \otimes N)|_{\hat{U}} \to \tau_{\leq m}(f^*(M_\beta) \otimes N)|_{\hat{U}}$ factors through $N^k|_{\hat{U}}$ for some k.

Because $\hat{U}$ is a completion of a scheme U over $\operatorname{Spec}(R)$, the restriction $\operatorname{APerf}(U) \to \operatorname{APerf}(\hat{U})$ is t-exact Proposition C.8. The restriction $\operatorname{APerf}(\mathscr{X}) \to \operatorname{APerf}(U)$ is also t-exact, and hence so is the restriction $\operatorname{APerf}(\mathscr{X}) \to \operatorname{APerf}(\hat{U})$. In other words we have a natural isomorphism $H_m(Q|_{\hat{U}}) \simeq H_m(Q)|_{\hat{U}}$ for $Q \in \operatorname{APerf}(\mathscr{X})$. It follows that the morphism

$$H_m(f^*(M_\alpha) \otimes N)|_{\hat{U}} \to H_m(f^*(M_\beta) \otimes N)|_{\hat{U}}$$

factors through $H_m(N^k|_{\hat{U}})$ and thus is the zero homomorphism for $m > 0$.

The morphism $\hat{U} \to \hat{\mathscr{X}}$ is representable and fppf, so it follows from faithfully flat decent that if the homomorphism between objects in the heart of the t-structure on $\operatorname{APerf}(\hat{\mathscr{X}})$, $H_m(f^*(M_\alpha) \otimes N) \to H_m(f^*(M_\beta) \otimes N)$, is zero when restricted to $\hat{U}$, it is already zero on $\hat{\mathscr{X}}$. Finally, from the the fully-faithful part of (GE) (which holds under hypotheses (1) and (2) as well) we know that if a homomorphism between coherent sheaves on $\mathscr{X}$ is zero after restricting to $\hat{\mathscr{X}}$, then it is already zero on $\mathscr{X}$.

We have shown that for any α, there is a $\beta \geq \alpha$ such that $H_m(f^*(M_\alpha) \otimes N) \to H_m(f^*(M_\beta) \otimes N)$ vanishes for $m > 0$. This implies finally that $H_m(f^*(M) \otimes N) = \varinjlim H_m(f^*(M_\alpha) \otimes N)$ vanishes for $m > 0$. Hence $f^*(M)$ is flat. □

Remark 2.20. The proof of Proposition 2.17 is essentially "categorical," but we are only working in the derived setting because some of our potential applications are for derived stacks. There one can prove integrability for classical stacks along the same lines as above, but using the Tannakian formalism for the abelian tensor category $QC(\mathscr{X})^\heartsuit$, as discussed in [?Lu1].

Corollary 2.21. *Let $\mathscr{X}, \mathscr{Y}$, and $\mathscr{S}$ be derived locally Noetherian algebraic 1-stacks. Let $f : \mathscr{X} \to \mathscr{S}$ be a morphism satisfying (GE), and let $\mathscr{Y} \to \mathscr{S}$ be a morphism locally almost of finite presentation with affine diagonal. Then for any complete local Noetherian ring R over $\mathscr{S}$, the canonical map*

$$\operatorname{Map}_{\mathscr{S}}(\mathscr{X} \times_{\mathscr{S}} \operatorname{Spec} R, \mathscr{Y}) \to \varprojlim \operatorname{Map}_{\mathscr{S}}(\mathscr{X} \times_{\mathscr{S}} \operatorname{Spec} R_n, \mathscr{Y})$$

is a weak equivalence. In other words the functor $\underline{\operatorname{Map}}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y})$ is integrable.

Proof. We can assume without loss of generality that $\mathscr{S}$ is Noetherian and then affine by passing to an affine atlas. Let $\coprod_{\alpha \in I} T_\alpha \to \mathscr{Y}$ be a smooth surjection, where each of the T_α are affine. Let S be an affine scheme over $\mathscr{S}$. Let $\psi : \mathscr{X}_S := \mathscr{X} \times_{\mathscr{S}} S \to \mathscr{Y}$ be a morphism. Consider the fiber product

$$\begin{array}{ccc} \coprod_\alpha T'_\alpha & \longrightarrow & \coprod_\alpha T_\alpha \\ \downarrow & & \downarrow \\ \mathscr{X}_S & \longrightarrow & \mathscr{Y} \end{array}$$

The morphism $\coprod_{\alpha \in I} T' \to \mathscr{X}_S$ is smooth and surjective, thus because $\mathscr{X}_S$ is quasicompact there is a finite subset $I' \subset I$ such that $\coprod_{\alpha \in I'} T'_\alpha \to \mathscr{X}_S$ is still surjective. Furthermore, if $S' \subset S$ is a nilpotent thickening of schemes, and we consider

$$\begin{array}{ccccc} \coprod_\alpha T''_\alpha & \longrightarrow & \coprod_\alpha T'_\alpha & \longrightarrow & \coprod_\alpha T_\alpha \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{X}_{S'} & \longrightarrow & \mathscr{X}_S & \longrightarrow & \mathscr{Y} \end{array}$$

then because the open substacks of $\mathscr{X}_S$ correspond bijectively with open substacks of $\mathscr{X}_{S'}$, a finite subset $I' \subset I$ suffices to surject onto $\mathscr{X}_S$ iff it suffices to surject onto $\mathscr{X}_{S'}$.

For each finite subset $I' \subset I$ we introduce the notation $\mathscr{Y}_{I'} := \operatorname{im}(\coprod_{\alpha \in I'} T_\alpha \to \mathscr{Y})$. Applying the above discussion to a complete local ring R over $\mathscr{S}$ we conclude that

$$\begin{aligned} \operatorname{Map}_{\mathscr{S}}(\mathscr{X}_{\operatorname{Spec} R}, \mathscr{Y}) &= \varinjlim_{I' \subset I \text{ finite}} \operatorname{Map}_{\mathscr{S}}(\mathscr{X}_{\operatorname{Spec} R}, \mathscr{Y}_{I'}) \\ \varprojlim \operatorname{Map}_{\mathscr{S}}(\mathscr{X}_{\operatorname{Spec} R_n}, \mathscr{Y}) &\simeq \varinjlim_{I' \subset I \text{ finite}} \varprojlim \operatorname{Map}_{\mathscr{S}}(\mathscr{X}_{\operatorname{Spec} R_n}, \mathscr{Y}_{I'}) \end{aligned}$$

Each $\mathscr{Y}_{I'}$ is geometric, hence we may apply Proposition 2.17 to the terms in the colimit on the right hand side and thus conclude the integrability of the functor $\underline{\operatorname{Map}}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y})$. ☐

2.4. **Derived representability from classical representability.** We have verified Artin's criterion for $\underline{\operatorname{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$ under the hypotheses of Theorem 2.1, and thus we have shown that $\underline{\operatorname{Map}}_S(\mathscr{X}, \mathscr{Y})^{cl}$ is algebraic and (classically) locally of finite presentation over S, with affine diagonal. At the same time we have verified that the derived stack $\underline{\operatorname{Map}}_S(\mathscr{X}, \mathscr{Y})$ is nilcomplete and infinitesimally cohesive (Lemma 2.10) and admits a cotangent complex (Proposition 2.14). Thus we complete the proof of Theorem 2.1 with the following:

Theorem 2.22. *Suppose that $\mathscr{X} : \mathrm{CAlg} \to \mathcal{S}$ is a pre-stack satisfying the following conditions:*

1. *(Classical Representability) The restriction $\mathscr{X}|_{\mathrm{CAlg}_0}$ of $\mathscr{X}$ to classical rings is representable by an Artin stack;*
2. *$\mathscr{X}$ admits a cotangent complex;*
3. *$\mathscr{X}$ is nilcomplete;*
4. *$\mathscr{X}$ is infinitesimally cohesive.*

Then, $\mathscr{X}$ is representable by a (derived) Artin stack.

Proof. This is essentially the argument of [?DAG-XIV, Theorem 3.1.2] and [?HAG-II, Theorem C.0.9]. For completeness we provide references to these sources, though for spaces reasons we do not aim to give a particularly readable account – for that, the reader is directed to the source material!

Note that if $\mathscr{X}|_{\mathrm{CAlg}_0}$ is representable by an algebraic space, the assertion that $\mathscr{X}$ is representable by a derived algebraic space is contained in [?DAG-XIV, Theorem 3.1.2]. We will first use this case to prove that the diagonal $\Delta_{\mathscr{X}}: \mathscr{X} \to \mathscr{X}^2$ is representable by a relative algebraic space: That is, we must show that for all

$$\eta\colon \operatorname{Spec} R \longrightarrow \mathscr{X}^2$$

the fiber product

$$\operatorname{Spec} R \times_{\mathscr{X}^2} \mathscr{X}$$

is representable by a derived algebraic space. We will do this by verifying the conditions of [?DAG-XIV, Theorem 3.1.2]: The underlying classical functor is representable by an algebraic space, since the restriction to classical rings preserves fiber products and the diagonal of $\mathscr{X}|_{\mathrm{CAlg}_0}$ is a relative algebraic space by assumption (1). It admits a cotangent complex because any morphism between spaces admitting cotangent complexes, and any fiber product of such, admits a cotangent complex by [?DAG-XIV, Prop. 1.3.18, Remark 1.3.21]. It is nilcomplete and infinitesimally cohesive for similar reasons [?DAG-XIV, Remark 2.2.3, Prop. 2.2.7]. Thus, $\Delta_{\mathscr{X}}$ is representable by a relative algebraic space.

Note that by [?DAG-XIV, Remark 3.1.6], $\mathscr{X}$ is a sheaf for the flat topology, and we have seen that its diagonal is representable by a derived algebraic space. It remains to show that $\mathscr{X}$ admits a smooth atlas from a derived scheme. Begin with a smooth atlas

$$f_0\colon U_0 = \coprod_{\alpha} \operatorname{Spec} R_\alpha^0 \longrightarrow \mathscr{X}|_{\mathrm{CAlg}_0}$$

We will show that one can lift it to a smooth atlas of $\mathscr{X}$. More precisely, we will construct a commutative diagram

$$\begin{array}{ccc} U_0 = \coprod_\alpha \operatorname{Spec} R_\alpha^0 & \longrightarrow & U = \coprod_\alpha \operatorname{Spec} R_\alpha \\ \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f} \\ \mathscr{X}_{cl} & \longrightarrow & \mathscr{X} \end{array}$$

with f smooth and faithfully flat. Here, $\mathscr{X}_{cl}$ denotes (the smooth sheafification of) the Kan extension of $\mathscr{X}|_{\mathrm{CAlg}_0}$ along $\mathrm{CAlg}_0 \hookrightarrow \mathrm{CAlg}$.

We first describe the construction of U and f. One uses obstruction theory, as in [?HAG-II, Lemma C.0.11] to iteratively construct directed system of maps $f_k\colon U_k = \operatorname{Spec} R_\alpha^k \to \mathscr{X}$, such that $\pi_i R_\alpha^k = 0$ for $i > k$, $\tau_{\leq i} R_\alpha^k = R_\alpha^i$ for $i \leq k$, and

$$\operatorname{Map}_{R_\alpha^k\text{-mod}}(\mathbb{L}_{U_k/\mathscr{X}}, M) = *$$

for $M \in R_\alpha^k$-mod satisfying $\pi_i M = 0$ for $i > k+1$ and $\pi_0 M = 0$ – this uses the cotangent complex formalism for $\mathscr{X}$, together with the fact that $\mathscr{X}$ is infinitesimally cohesive. Next, one sets $U = \operatorname{Spec} R_\alpha$ where $R_\alpha = \varprojlim_k R_\alpha^k$ – since $\mathscr{X}$ is nilcomplete, the f_k assempt to a map $f\colon U \to \mathscr{X}$. The deformation theory argument shows that $U \to \mathscr{X}$ is formally smooth. Furthermore, the underlying classical map of f is f_0, which is locally of finite presentation; thus, we conclude that f is itself smooth. Finally, checking that a flat map is faithfully flat may be done on classical (or even classical, reduced) schemes: so that f is faithfully flat since f_0 is. □

2.5. **Application: (pGE) and the moduli of perfect complexes.** Let $\underline{\mathrm{Perf}}_S$ denote the stack which assigns to a derived ring R over S the maximal sub-∞-groupoid $\mathrm{Perf}(R)^{\simeq}$ of the ∞-category of perfect R-modules. In the language of [?HAG-II], it is a locally geometric D^--stack (see [?dgModuli]). This means that – locally – it is a geometric n-stack for *some* n that is generally not equal to 1. In particular, even locally, it is not a derived 1-stack: For instance, it does not assign classical rings to 1-truncated spaces (i.e., spaces equivalent to the nerve of a groupoid).

Nevertheless, it satisfies the derived form of Artin's criteria appearing in op.cit. and, in principle, this could be used to prove that it is a locally geometric stack. Now let $\pi : \mathscr{X} \to S$ be a flat morphism. We define the moduli stack of perfect complexes on $\mathscr{X}$ to be the functor assigning to $T = \operatorname{Spec}(R) \to S$ the space

$$\underline{\mathrm{Perf}}(\mathscr{X}/S)(T) := \mathrm{Perf}(\mathscr{X} \times_S T)$$

Because perfect complexes satisfy descent we have

$$\underline{\mathrm{Perf}}(\mathscr{X}/S)(T) \simeq \mathrm{Map}(\mathscr{X} \times_S T, \underline{\mathrm{Perf}}) \simeq \mathrm{Map}_S(\mathscr{X} \times_S T, \underline{\mathrm{Perf}}_S)$$

Thus the $\underline{\mathrm{Perf}}(\mathscr{X}/S)$ is a mapping object into $\underline{\mathrm{Perf}}_S$.

In the preceding discussion, we have chosen to rely on the classical form of Artin representability – rather than a derived analog as in [?DAG-XIV] or [?HAG-II] – because a suitable form for our purposes does not yet exist (though it is suggested [?DAG-XIV, Remark 3.2.3]). Such a Theorem should state:

Theorem 2.23 (Pre-Theorem "Derived Artin's Criterion")**.** *Let $\mathscr{X}$ be a pre-stack over $S = \operatorname{Spec} R$, where R is a derived Grothendieck ring. Let $n \geq 0$. Then $\mathscr{X}$ is representable by a derived Artin n-stack which is locally almost of finite presentation over S if and only if it satisfies the following conditions:*

- *(1) For every discrete commutative ring A, the space $\mathscr{X}(A)$ is n-truncated;*
- *(2) The functor $\mathscr{X}$ is a sheaf for the smooth topology;*
- *(3) The functor $\mathscr{X}$ is nilcomplete, infinitesimally cohesive, and integrable;*
- *(4) The map $\mathscr{X} \to S$ admits a $(-n)$-connective cotangent complex;*
- *(5) The map $\mathscr{X} \to S$ is locally almost of finite presentation.*

Remark 2.24. Note that the case $n = 0$ of the above theorem is the same as the case $n = 0$ of [?DAG-XIV, Theorem 3.2.1], since a derived Artin 0-stack is defined to be a derived algebraic space, which in turn is defined to be a derived Deligne-Mumford 0-stack.

The proof is by induction on n, noting that the properties for $\mathscr{X}$ implies that the diagonal morphism $\mathscr{X} \to \mathscr{X}^2$ satisfies the analogous conditions for $(n-1)$ after base-change to any affine over $\mathscr{X}^2$. This is what makes this "derived Artin criterion" more pleasant to work with than the ordinary Artin's criterion: Since the obstruction theory is now a condition, rather than extra data, it can automatically pass to the diagonal without the user having to explicitly guarantee representability of the diagonal!

As in op.cit. it is thus enough to show that there are enough formally smooth morphisms $\operatorname{Spec} B \to \mathscr{X}$ where B is an R-algebra locally almost of finite presentation, to ensure the existence of a smooth surjection from a disjoint union of such. To make the Pre-Theorem a Theorem, it thus remains to generalize [?DAG-XIV, Prop. 3.2.4] to the case where $L_{X/Y}$ is only assumed $(-n)$-connective, and the relative cotangent complex

Definition 2.25. The pre-Theorem requires a fixed n, while $\underline{\mathrm{Perf}}$ and friends are not, globally, n-stacks for any fixed n. To remedy this, we define the open sub-pre-stack

$$\underline{\mathrm{Perf}}(\mathscr{X}/S)^{\leq n} \subset \underline{\mathrm{Perf}}(\mathscr{X}/S)$$

of *universally n-gluable* objects as follows. For each affine test S-scheme $T = \operatorname{Spec} A$, a perfect complex

$$E \in \underline{\mathrm{Perf}}(\mathscr{X}/S)(A) = \mathrm{Perf}(X \times_S T)$$

belongs to $\underline{\mathrm{Perf}}(\mathscr{X}/S)^{\leq n}(A)$ if and only if

$$R\Gamma(\mathscr{X} \times_S T, E \otimes E^*)[+1] \in QC(T)$$

is of formation compatible with base-change on T and is of Tor amplitude at most n.

In case $n = 1$, these are the *universally gluable* objects of [?Lieblich].

Remark 2.26. Note that when these complexes are all perfect – as they will be in our examples below – this condition is simply requiring that the putative cotangent complex

$$L_{\underline{\mathrm{Perf}}(\mathscr{X}/\operatorname{Spec} T)} = R\Gamma(\mathscr{X} \times_S T, E \otimes E^*)[+1]^*$$

be $(-n)$-connective since for any $M \in QC(T)_{<-n}$ we have that

$$\mathrm{RHom}_T(L_{\underline{\mathrm{Perf}}(\mathscr{X}/T)/T}, M) = R\Gamma(\mathscr{X} \times_S T, E \otimes E^*)[+1] \otimes_{\mathcal{O}_T} M.$$

Lemma 2.27. *Suppose that S is an n-truncated base; that $\mathscr{X}/S$ is a flat, derived ∞-quasi-compact higher stack; and that Y/S is Zarski locally a derived higher stack locally almost of finite presentation. Then, $\underline{\mathrm{Map}}_S(X, Y)/S$ is locally almost of finite presentation.*

Proof. The claim is local on $S = \operatorname{Spec} R$, so we may suppose it affine for an n-truncated ring R. If $\mathscr{X} = \operatorname{Spec} A$ is affine, then this is clear because $A \otimes_R -$ preserves filtered colimits of n-truncated algebras for all n.

Suppose first that Y/S is a derived k-stack. Let X' denote the colimit of the first $(n+k)$-pieces of some Cech-type resolution of X by affine schemes along smooth morphisms. Note that $\underline{\operatorname{Map}}_S(X', Y)$ and $\underline{\operatorname{Map}}_S(X, Y)$ coincide on n-truncated algebras A, since we may compute $Y(X \times_S \operatorname{Spec} A)$ using the Cech nerve of X and each of the spaces involved will be $(n+k)$-truncated.

Next, notice that the natural map

$$\varinjlim_{\alpha} \underline{\operatorname{Map}}_S(X, U_\alpha)(R) \longrightarrow \underline{\operatorname{Map}}_S(X, Y)(R)$$

is an equivalence whenever Y is the increasing union of a filtered system of open sub-pre-stacks, since $X \times_S \operatorname{Spec} R$ is quasi-compact. Applying this observation to the filtered system consisting of all $U_\beta \subset Y$ Zariski opens which are n-stacks for some n, noting that this is filtered since n-stacks are closed under Zariski pushouts, we conclude the proof of the Lemma. ☐

As a result one can deduce:

Corollary 2.28 (Pre-Corollary)**.** *Let $\pi : \mathscr{X} \to S$ be a flat, perfect morphism satisfying (L) and (pGE). Then, $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n} \subset \underline{\operatorname{Perf}}(\mathscr{X}/S)$ is an open sub-pre-stack and satisfies the condition of the above Pre-Theorem. Thus, it is representable by a derived Artin n-stack. Furthermore, $\underline{\operatorname{Perf}}(\mathscr{X}/S)$ is the union of these open sub-stacks, so it is Zariski locally a derived Artin stack.*

Proof. Let us verify that $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n}$ satisfies each of the indicated conditions. We begin by verifying that $\underline{\operatorname{Perf}}(\mathscr{X}/S)$ satisfies those not involving an "n," and check that these are in fact open sub-functors so that the relevant conditions pass to them. Note that (ii) follows from smooth descent for perfect complexes, and that the Tor-dimension conditions are also local.

To prove (iii) we will use mapping stack tricks: The functor $\underline{\operatorname{Perf}}_S : \operatorname{CAlg}_S \to \mathcal{S}$ is nilcomplete, cohesive, and admits a perfect cotangent complex by [?DAG-XIV, Proposition 3.4.10] and is integrable by 6.6 and the fact that perfect complexes are the dualizable objects of $\operatorname{APerf}(R)$. By Scholium 2.12 and Proposition 2.14, the mapping object

$$\underline{\operatorname{Perf}}(\mathscr{X}/S) \simeq \underline{\operatorname{Map}}_S(\mathscr{X}, \underline{\operatorname{Perf}}_S)$$

is also nilcomplete, infinitesimally cohesive and admits a cotangent complex. Notice that it is also integrable, though now this is not a formal deduction – instead, we notice that integrability here is precisely (pGE)!

Each of the $\underline{\operatorname{Perf}}_S^{\leq n}$ are open subfunctors – because Tor amplitude of a perfect complex is upper semi-continuous – and thus also satisfy the conditions of (iii).

Furthermore, since S is quasi-compact every perfect complex has finite Tor amplitude so that these open subfunctors exhaust $\underline{\operatorname{Perf}}_S^{\leq n}$. By [?DAG-XIV, Proposition 3.4.10] it is locally almost of finite presentation, so that the conditions of the previous Lemma (on "Y") apply to it – we thus conclude that

$$\underline{\operatorname{Perf}}(\mathscr{X}/S) = \underline{\operatorname{Map}_S}(\mathscr{X}, \underline{\operatorname{Perf}}_S)$$

is locally almost of finite presentation, i.e. satisfies (v).

Since π is a flat perfect morphism, and it satisfies (L), we see that π_* preserves perfect objects so that $\operatorname{Perf}(\mathscr{X}/S)$ has a perfect cotangent complex, given by

$$L_{\underline{\operatorname{Perf}}(\mathscr{X}/S)/S}\big|_T = f_+(E \otimes E^*[-1]) = R\Gamma(\mathscr{X} \times_S T, E \otimes E^*)[+1]^*$$

as indicated above. The formation of this is compatible with base-change by Corollary B.17, so that each $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n}$ is an open sub-functor by the upper semi-continuity of Tor-dimension of a perfect complex. Furthermore, since S is quasi-compact each perfect complex has a bounded Tor amplitude – this proves that $\underline{\operatorname{Perf}}(\mathscr{X}/S)$ is the union of these open sub-functors.

It follows that each $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n}$ satifies (ii), (iii), and (v). It remains to verify the properties that involve n. Remark 2.26 shows that the cotangent complex of $\underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n}$ is $(-n)$-connective, verifying (iv). Finally, if A is a discrete ring and $E \in \underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n}(A)$ then

$$\Omega_E \underline{\operatorname{Perf}}(\mathscr{X}/S)^{\leq n}(A) \subset \operatorname{Map}_{\mathscr{X} \times_S A}(E, E) = \Omega^\infty R\Gamma(X \times_S A, E \otimes E^*)$$

is $(n-1)$-truncated. This verifies (i).

This completes the proof. ☐

Remark 2.29. Note that a special case of this, taking $n = 1$, is again Lieblich's moduli of universally gluable objects. In this case, we do not have to make use of the pre-Theorem – instead we apply Theorem 2.22 and the classical Artin's Criterion, with Lemma 2.15 giving the obstruction theory, and a form of [?DAG-XIV, Theorem 3.2.1] with $n = 0$ applied to the diagonal to ensure that it is a relative algebraic space.

3. Perfect Grothendieck existence

Before developing general methods for establishing (GE), we discuss the slightly simpler property (pGE).[9] We regard (pGE) as simpler because when $\mathscr{X}$ is a perfect stack, (pGE)_R can be re-phrased as purely a property of the R-linear ∞-category $\mathcal{C} = \operatorname{Perf}(\mathscr{X})$, and without reference to the geometry of $\mathscr{X}$.

Let $\{R_n\}$ be as in Proposition C.3, and set $\mathscr{X}_n := \mathscr{X} \times_S \operatorname{Spec} R_n$ so that $\operatorname{Perf}(\widehat{X}) = \varprojlim_n \operatorname{Perf}(\mathscr{X}_n)$. For each n, tensor product over R determines a functor

$$\operatorname{Perf}(\mathscr{X}) \otimes_{\operatorname{Perf} R} \operatorname{Perf} R_n \longrightarrow \operatorname{Perf}(\mathscr{X}_n) \tag{4}$$

which we will see below is an equivalence. Thus when $\mathcal{C}$ is a small, stable, idempotent complete R-linear ∞-category, we will say that $\mathcal{C}$ satisfies (pGE)_R when the natural map

$$\mathcal{C} \longrightarrow \varprojlim_n (\mathcal{C} \otimes_{\operatorname{Perf} R} \operatorname{Perf} R_n)$$

is an equivalence for any (and thus all) $\{R_n\}$ as in Proposition C.3.

Lemma 3.1. *The functor of Equation 4 is an equivalence provided that $\mathscr{X}$ is a perfect stack.*

Proof. Note first that the analogous functor on presentable categories

$$QC(\mathscr{X}) \otimes_{R\text{-mod}} R_n\text{-mod} \longrightarrow QC(\mathscr{X}_n)$$

is an equivalence: Both sides identify with R_n-module objects of $QC(\mathscr{X})$ (c.f., [?BFN]). Note that the tensor product in the previous displayed equation was in the sense of *presentable* ∞-categories, which is compatible with the tensor product of small, finitely co-complete, idempotent complete ∞-categories via

$$\operatorname{Ind}(\mathcal{C}) \otimes_{\operatorname{Ind}(\mathcal{A})} \operatorname{Ind}(\mathcal{D}) \simeq \operatorname{Ind}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}) \tag{5}$$

by the construction in [?HigherAlgebra]. Thus,

$$\operatorname{Perf}(\mathscr{X}) \otimes_{\operatorname{Perf} R} \operatorname{Perf} R_n \simeq QC(\mathscr{X}_n)^c$$

Since $\mathscr{X}_n \to \mathscr{X}$ is an affine morphism, it is perfect and so $\mathscr{X}_n$ is a perfect stack by [?BFN]. Thus $QC(\mathscr{X}_n)^c$ identifies with $\operatorname{Perf}(\mathscr{X}_n)$. □

By the same logic a morphism $f : \mathscr{X} \to \mathscr{Y}$ between perfect stacks satisfies (pGE) if and only if

$$\operatorname{Perf}(\mathscr{X}) \otimes_{\operatorname{Perf} \mathscr{Y}} \operatorname{Perf}(R) \to \varprojlim_n \operatorname{Perf}(\mathscr{X}) \otimes_{\operatorname{Perf} \mathscr{Y}} \operatorname{Perf}(R_n)$$

is an isomorphism for all morphisms $\operatorname{Spec} R \to \mathscr{Y}$ and any one (equivalently, all) pro-R-algebras $\{R_n\}$ as in Proposition C.3. Note, however, that there are potentially more tensor functors $\operatorname{Perf}(\mathscr{Y}) \to \operatorname{Perf} R$ than morphisms of stacks $\operatorname{Spec} R \to \mathscr{Y}$.

We will see that for some stacks, one can prove (pGE)_R using very different methods than those used to prove the Grothendieck existence theorem in Section 6. The key fact is the following

Lemma 3.2. *Suppose that R is a complete local derived ring, and $(\mathcal{A}, \otimes)$ is an R-linear (small, stable) symmetric monoidal ∞-category, and suppose that $\mathcal{A}$ satisfies (pGE)_R. Suppose that $\mathcal{C}$ is a (small, stable) $\mathcal{A}$-module category, which is fully dualizable as an $\mathcal{A}$-module category (i.e., "smooth and proper" over $\mathcal{A}$). Then, $\mathcal{C}$ also satisfies (pGE)_R*

[9] In fact, we will see below that (GE) implies (pGE)

Proof. Consider the commutative diagram of natural functors

$$\begin{array}{ccccc} \mathcal{C} & & \longrightarrow & & \varprojlim_n (\mathcal{C} \otimes_{\operatorname{Perf} R} \operatorname{Perf} R_n) \\ \uparrow \sim & & & & \uparrow \sim \\ \mathcal{C} \otimes_{\mathcal{A}} \mathcal{A} & \xrightarrow{\sim} & \mathcal{C} \otimes_{\mathcal{A}} \left(\varprojlim_n \mathcal{A} \otimes_{\operatorname{Perf} R} \operatorname{Perf} R_n \right) & \longrightarrow & \varprojlim_n \left(\mathcal{C} \otimes_{\mathcal{A}} \left(\mathcal{A} \otimes_{\operatorname{Perf} R} \operatorname{Perf} R_n \right) \right) \end{array}$$

The vertical arrows are equivalenced by general non-sense on tensor products. The horizontal arrow in the bottom left is an equivalence by hypothesis on $\mathcal{A}$. Thus, it is enough to show that horizontal arrow in the bottom right is an equivalence.

But notice that since $\mathcal{C}$ is fully dualizable as an $\mathcal{A}$-module category, the functor on $\mathcal{A}$-linear categories

$$\mathcal{C} \otimes_{\mathcal{A}} -$$

identifies with the inner-Hom $\operatorname{Fun}^{ex}_{\mathcal{A}}(\mathcal{C}^\vee, -)$. Formal non-sense ensures that this inner-Hom preserves inverse limits, and thus $\mathcal{C} \otimes_{\mathcal{A}} -$ preserves inverse limits as well. In particular, the natural map

$$\mathcal{C} \otimes_{\mathcal{A}} \left(\varprojlim_n \mathcal{A} \otimes_{\operatorname{Perf} R} \operatorname{Perf} R_n \right) \longrightarrow \varprojlim_n \left(\mathcal{C} \otimes_{\mathcal{A}} \left(\mathcal{A} \otimes_{\operatorname{Perf} R} \operatorname{Perf} R_n \right) \right)$$

is an equivalence. □

Example 3.3. Setting $\mathcal{A} = \operatorname{Perf}(R)$ in the previous lemma, we have that a fully dualizable R-linear category satisfies (pGE).

Example 3.4. When $\mathscr{X} \to \mathscr{Y}$ is a morphism of perfect R-stacks which is a relative smooth and proper Deligne-Mumford stack, then $\operatorname{Perf}(\mathscr{X})$ is a fully dualizable $\operatorname{Perf}(\mathscr{Y})$-module category. Thus if $\mathscr{Y}$ satisfies (pGE), so does $\mathscr{X}$. Letting $\mathscr{Y} = \operatorname{Spec} R$ we see that any smooth and proper Deligne-Mumford stack over $\operatorname{Spec} R$ satisfies (pGE).

Finally, we compare the property (pGE) with (GE).

Lemma 3.5. *Let R be a complete local Noetherian ring and let $\mathscr{X}$ be an R-stack. Then (GE)$_R$ implies (pGE)$_R$.*

Proof. The restriction functor $\hat{i}^* : \operatorname{APerf}(\mathscr{X}) \to \operatorname{APerf}(\widehat{\mathscr{X}})$ is an equivalence of tensor categories. It clearly maps perfect objects to perfect objects. Conversely if $\{F_n\} \in \operatorname{APerf}(\widehat{\mathscr{X}})$ is an inverse system of perfect complexes, then $\{F_n^\vee\}$ is an inverse system which is dual to $\{F_n\}$. It follows that the object $F \in \operatorname{APerf}(\mathscr{X})$ with $\hat{i}^* F = \{F_n\}$, is dualizable, hence perfect. □

When $\mathfrak{X}$ is a perfectly generated stack, we have a partial converse.

Lemma 3.6. *Let $\mathscr{X}$ be a perfectly generated algebraic stack satisfying (pGE)$_R$ and (CD) over a complete local Noetherian ring R. Then for any $F, G \in QC(\mathscr{X})$ with $G \in \operatorname{APerf}(\mathscr{X})$,*

$$R\operatorname{Hom}_{\mathscr{X}}(F, G) \to R\operatorname{Hom}_{\hat{\mathscr{X}}}(\hat{F}, \hat{G})$$

is an equivalence.

Proof. Because $\mathscr{X}$ is perfectly generated, any F can be written as a colimit of perfect objects, so it suffices to consider the case when F is perfect. A perfect F is dualizable, so one can replace F with $\mathcal{O}_{\mathscr{X}}$ and G with $G \otimes F^\vee$. Using Lemma 6.1, it thus suffices to prove that

$$R\Gamma(\mathscr{X}, G) \to \varprojlim R\Gamma(\mathscr{X}_n, G|_{\mathscr{X}_n}) \simeq \varprojlim R\Gamma(\mathscr{X}, G) \otimes_R R_n \tag{6}$$

is an equivalence for any almost perfect G.

Using the fact that $R^i \varprojlim$ and $R^i\Gamma$ vanish for sufficiently large i, we have that for $m \gg m' \gg p$

$$\begin{aligned} \tau_{\leq p} \varprojlim (R\Gamma(\mathscr{X}, G) \otimes R_n) &\simeq \tau_{\leq p} \varprojlim (\tau_{\leq m'} R\Gamma(\mathscr{X}, G)) \otimes R_n \\ &\simeq \tau_{\leq p} \varprojlim (\tau_{\leq m'} R\Gamma(\mathscr{X}, \tau_{\leq m} G)) \otimes R_n \end{aligned}$$

The canonical morphism from $\tau_{\leq p} R\Gamma(\mathscr{X}, G)$ to this last expression factors through the isomorphism $\tau_{\leq p} R\Gamma(\mathscr{X}, G) \simeq \tau_{\leq p} R\Gamma(\mathscr{X}, \tau_{\leq m} G)$. Thus (6) is an equivalence if and only if for every p,

$$\tau_{\leq p} R\Gamma(\mathscr{X}, \tau_{\leq m} G) \to \tau_{\leq p} \varprojlim(\tau_{\leq m'} R\Gamma(\mathscr{X}, \tau_{\leq m} G)) \otimes R_n \tag{7}$$

is an equivalence for $m \gg m' \gg p$.

Finally by Corollary B.12, $\tau_{\leq m} G$ is a retract of a $\tau_{\leq m} G'$ for some perfect complex G'. $(\text{pGE})_R$ implies that (6) and as a result (7) is an equivalence for G', hence (7) is an equivalence for G as well.

□

3.1. **Proving (pGE) via semiorthogonal decompositions.** The perfect Grothendieck existence theorem for a stack $\mathscr{X}$ over a complete local Noetherian ring R can be phrased entirely in terms of the R-linear ∞-category $\operatorname{Perf}(\mathscr{X})$. This allows one to take a different approach to establishing $(\text{pGE})_R$ for $\mathscr{X}$.

Notation 3.7. Let $\mathcal{C}$ be a stable ∞-category, let I be an index set, and let $\{\mathcal{A}_i \subset \mathcal{C}\}_{i \in I}$ be a collection of either ∞-subcategories, or simply sets of objects of $\mathcal{C}$. We let $\langle \mathcal{A}_i; i \in I \rangle$ denote the smallest full, saturated, stable ∞-subcategory of $\mathcal{C}$ containing all of the objects in $\mathcal{A}_i$, for all $i \in I$. Furthermore, we let $\overline{\langle \mathcal{A}_i; i \in I \rangle}$ denote the smallest full, saturated, stable ∞-subcategory of $\mathcal{C}$ which is *closed under retracts* and contains the objects of $\mathcal{A}_i$, $\forall i$.

Let $\mathcal{C}$ be an ∞-category and let $\mathcal{A} \subset \mathcal{C}$ be either a subcategory or a set of objects. We define the full, saturated ∞-subcategories

$$\mathcal{A}^{\perp} := \{F \in \mathcal{C} | \operatorname{Map}(E, F) \text{ contractible for all } E \in \mathcal{A}\}$$
$$^{\perp}\mathcal{A} := \{F \in \mathcal{C} | \operatorname{Map}(F, E) \text{ contractible for all } E \in \mathcal{A}\}$$

Note that $\mathcal{A}^{\perp}$ and $^{\perp}\mathcal{A}$ are closed under retracts, and if $\mathcal{C}$ is stable and closed under suspension and desuspension, then these categories are stable as well.

Recall that a *semiorthogonal decomposition* of an R-linear stable ∞-category $\mathcal{C}$ is a (possibly infinite) collection of full stable R-linear subcategories $\mathcal{A}_i \subset \mathcal{C}$ indexed by a totally ordered set such that

$$\mathcal{A}_i \subset \mathcal{A}_j^{\perp} \text{ for all } j > i, \text{ and } \mathcal{C} = \langle \mathcal{A}_i; i \in I \rangle.$$

In this case each subcategory $\mathcal{A}_i$ is characterized by properties $\mathcal{A}_i \subset \mathcal{A}_j^{\perp}$ for $j > i$ and $\mathcal{A}_j \subset \mathcal{A}_i^{\perp}$ for $j < i$, and thus $\mathcal{A}_i$ is automatically a thick subcategory (closed under retracts). In particular, if $\mathcal{C}$ is idempotent complete then so is $\mathcal{A}_i$, for each i.

Proposition 3.8. *For any nonvanishing $F \in \mathcal{C}$ there is a diagram*

$$\begin{array}{ccccccccccc} A_{i_K} = F_{i_K} & \longrightarrow & \cdots & \longrightarrow & F_{i_2} & \longrightarrow & F_{i_1} & \longrightarrow & F_{i_0} & = & F \\ & \nwarrow & & \overset{+}{\nwarrow} & \downarrow & \overset{+}{\nwarrow} & \downarrow & \overset{+}{\nwarrow} & \downarrow & & \\ & & \cdots & & A_{i_2} & & A_{i_1} & & A_{i_0} & & \end{array} \tag{8}$$

of exact triangles, where $A_{i_p} \in \mathcal{A}_{i_p}$, $i_0 < i_1 < \cdots < i_N$, and by convention $\mathcal{A}_{i_p} \neq 0, \forall p$. The indices i_p and the diagram is determined uniquely up to a contractible space of isomorphisms. Furthermore

1. *the assignment $F \mapsto A_i$ extends to a functor $\mathcal{C} \to \mathcal{A}_i$ which is a retract onto this full subcategory, and*
2. *this functor provides a left adjoint for the inclusion of the full subcategory $\mathcal{A}_i \subset \langle \mathcal{A}_j; j \geq i \rangle$ and a right adjoint for the inclusion $\mathcal{A}_i \subset \langle \mathcal{A}_j; j \leq i \rangle$.*

The proof of this Proposition proceeds just as in the case of triangulated categories, so we have omitted it. If $\mathcal{C}$ is an R-linear category and $\mathcal{A}_i$ are R-linear subcategories, then the projection functors $\mathcal{C} \to \mathcal{A}_i$ are R-linear as well.

The main result of this section reduces $(\text{pGE})_R$ for a category $\mathcal{A}$ to $(\text{pGE})_R$ for its semiorthogonal factors.

Proposition 3.9. *Let R be a complete local Noetherian ring, and let $\mathcal{A}$ be a small, stable, idempotent complete R-linear ∞-category with a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{A}_i; i \in I \rangle$. Assume that for any $F, G \in \mathcal{A}$, the R-module $\operatorname{RHom}(F, G)$ is almost perfect. Then $\mathcal{A}$ satisfies $(\text{pGE})_R$ if and only if $\mathcal{A}_i$ satisfies $(\text{pGE})_R$ for all i.*

We will collect a few preliminary lemmas about semiorthogonal decompositions.

Lemma 3.10. *Let $\mathcal{A}$ be a small, stable R-linear ∞-category and let $\mathcal{A} = \langle \mathcal{A}_i; i \in I\rangle$ be a semiorthogonal decomposition. Then $\mathcal{A}$ is idempotent complete if and only if each $\mathcal{A}_i$ is.*

Proof. The categories $\mathcal{A}_i$ are retracts of $\mathcal{A}$, so they are idempotent complete if $\mathcal{A}$ is. For the converse, a simple inductive argument shows that it suffices to consider the case where $I = \{0, 1\}$. We choose an idempotent completion $\mathcal{A} \subset \bar{\mathcal{A}}$ and assume that $\mathcal{A}_i$ are idempotent complete and thus closed under retracts in $\bar{\mathcal{A}}$. Let $F \in \mathcal{A}$, and let $F' \in \bar{\mathcal{A}}$ be a retract of F. In the ∞-categorical context (See [?HigherTopos, Section 4.4.5], this means that F' is the colimit in $\bar{\mathcal{A}}$ of a certain functor $f : \mathrm{Idem} \to \mathcal{A}$. The semiorthogonal decomposition of $\mathcal{A}$ induces a semiorthogonal decomposition

$$\mathrm{Fun}(\mathrm{Idem}, \mathcal{A}) = \langle \mathrm{Fun}(\mathrm{Idem}, \mathcal{A}_0), \mathrm{Fun}(\mathrm{Idem}, \mathcal{A}_1)\rangle$$

hence we have an extension of functors $f' \to f \to f'' \to$ with the image of f' in $\mathcal{A}_1$ and the image of f'' in $\mathcal{A}_0$. This gives an exact triangle of colimits in $\bar{\mathcal{A}}$. Because $\mathcal{A}_0$, and $\mathcal{A}_1$ are idempotent complete, we have that $F' \in \langle \mathcal{A}_0, \mathcal{A}_1\rangle$. □

Note that if $\phi : \mathcal{A} \to \mathcal{B}$ is an R-linear morphism of small stable ∞-categories which is a retract in the sense that there is a morphism $\pi : \mathcal{B} \to \mathcal{A}$ with $\pi \circ \phi \simeq \mathrm{id}_{\mathcal{A}}$, then ϕ is fully faithful and for any R-algebra, R', the morphism $\mathcal{A} \otimes_{\mathrm{Perf}\, R} \mathrm{Perf}\, R' \to \mathcal{B} \otimes_{\mathrm{Perf}\, R} \mathrm{Perf}\, R'$ is a retract as well.

Lemma 3.11. *Let R be a (derived) ring and let $\mathcal{A}$ and be a small, stable, idempotent complete R-linear ∞-category. Assume we have a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{A}_i; i \in I\rangle$. Then for any R-algebra R',*

$$\mathcal{A}' := \mathcal{A} \otimes_{\mathrm{Perf}\, R} \mathrm{Perf}\, R' = \langle \mathcal{A}'_i; i \in I\rangle$$

is a semiorthogonal decomposition as well, where

$$\mathcal{A}'_i = \overline{\langle A \otimes R'; A \in \mathcal{A}_i\rangle} \simeq \mathcal{A}_i \otimes_{\mathrm{Perf}\, R} \mathrm{Perf}\, R'.$$

Proof. Note first that because $\mathcal{A}_i \otimes_R R' \to \mathcal{A} \otimes_R R'$ admits a retract it is fully faithful, and thus $\mathcal{A}_i \otimes_R R' \simeq \overline{\langle A \otimes R'; A \in \mathcal{A}_i\rangle}$ because the former is idempotent complete.

It follows that to show $\mathcal{A}'_i \subset (\mathcal{A}'_j)^\perp$ for $j > i$, it suffices to verify the orthogonality of $A_j \otimes R'$ and $A_i \otimes R'$ for any $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$. We compute

$$R\,\mathrm{Hom}_{\mathcal{A}'}(A_j \otimes R', A_i \otimes R') \simeq R\,\mathrm{Hom}_{\mathcal{A}}(A_j, A_i) \otimes_R R' = 0$$

where "$R\,\mathrm{Hom}_{\mathcal{A}}$" denotes R-module valued homomorphism spectra induced by the R-linear structure on $\mathcal{A}$.

The category $\mathcal{C} := \langle \mathcal{A}'_i; i \in I\rangle$ admits a semiorthogonal decomposition by construction. Objects of the form $A \otimes R'$ with $A \in \mathcal{A}$ thickly generate $\mathcal{A}'$, so $\bar{\mathcal{C}} = \mathcal{A}'$. But the $\mathcal{A}'_i$ are idempotent complete, as retracts of $\mathcal{A}'$, so Lemma 3.10 implies that $\mathcal{C}$ is idempotent complete and hence $\mathcal{C} = \bar{\mathcal{C}}$. □

We apply Lemma 3.11 to the situation where R is a complete local Noetherian ring and $R' = R_n$ to get a system of semiorthogonal decompositions $\mathcal{A} \otimes_R R_n = \langle \mathcal{A}_{i;n}; i \in I\rangle$ which is compatible with the natural restriction morphisms $i_n^* : \mathcal{A} \otimes_R R_{n+1} \to \mathcal{A} \otimes_R R_n$. For any R-linear category $\mathcal{A}$ we denote the completion

$$\hat{\mathcal{A}} := \varprojlim_n \mathcal{A} \otimes_R R_n$$

Lemma 3.12. *Let $\mathcal{A}$ be a small, stable, idempotent complete R-linear ∞-category with a semiorthogonal decomposition $\langle \mathcal{A}_i; i \in I\rangle$. Assume for any $F, G \in \mathcal{A}$,*

$$\mathrm{RHom}(F, G) \in QC(R)_{\geq -n}, \text{ for some } n \gg 0.$$

Then the category $\hat{\mathcal{A}}$ has a semiorthogonal decomposition $\langle \hat{\mathcal{A}}_i; i \in I\rangle$.

Proof. Because each $\mathcal{A}_{i;n}$ is a full subcategory of $\mathcal{A} \otimes_R R_n$, we can regard $\hat{\mathcal{A}}_i$ as a full subcategory of $\hat{\mathcal{A}}$. Thus these categories have the appropriate semiorthogonality properties.

Let $\{F_n\} \in \hat{\mathcal{A}}$ be an inverse system with isomorphisms $i_n^* F_{n+1} \simeq F_n$. Applying i_n^* to the decomposition (8) for $F_{n+1} \in \mathcal{A} \otimes_R R_{n+1}$ is canonically isomorphic to the decomposition of $F_n \in \mathcal{A} \otimes_R R_n$, with the exception that the diagram might become smaller if some of the objects A_{i_p} vanish. We claim no object can vanish under the restriction functor i_n^* and thus the same indices appear in the decomposition of each F_n. It follows from the uniqueness of (8) that one has an inverse system of diagrams and hence a diagram of inverse systems, which shows that $\{F_n\} \in \langle \hat{\mathcal{A}}_i; i \in I\rangle$.

To verify the claim that i_n^* has trivial kernel, we observe the following more general fact: if $\mathcal{B}$ is an R-linear category satisfying the hypotheses above and R' is a nilpotent thickening of R, then the kernel of $\mathcal{B} \mapsto \mathcal{B} \otimes_R R'$ is trivial. This follows from the fact that an object is 0 if and only if its endomorphism algebra is 0, and

$$\mathrm{RHom}_{\mathcal{B}'}(M \otimes R', M \otimes R') \simeq \mathrm{RHom}_{\mathcal{B}}(M, M) \otimes_R R'$$

By hypothesis the derived endomorphism algebra lies in $(R\text{-mod})_{\geq -n}$ for some $n \gg 0$. For each n the restriction functor $(R\text{-mod})_{\geq -n} \to (R'\text{-mod})_{\geq -n}$ has a trivial kernel for nilpotent thickenings, which can be proved by reduction to the case of objects in the heart and then to the case of classical rings. □

Proof of Proposition 3.9. Note that $\mathrm{RHom}_{\mathcal{A}_n}(F \otimes R_n, G \otimes R_n) \simeq \mathrm{RHom}_{\mathcal{A}}(F, G) \otimes R_n$. Thus by Lemma 6.6, the fact that $\mathrm{RHom}(F, G) \in \mathrm{APerf}(R)$ implies that

$$\mathrm{RHom}_{\mathcal{A}}(F, G) \to \mathrm{RHom}_{\hat{\mathcal{A}}}(\hat{F}, \hat{G})$$

is an equivalence, i.e. that the functor $\hat{i} : \mathcal{A} \to \hat{\mathcal{A}}$ is fully faithful.[10]

Because $\mathrm{RHom}(F, G) \in \mathrm{APerf}(R)$ for all $F, G \in \mathcal{A}$, Lemma 3.12 applies, so $\hat{\mathcal{A}}$ has a semiorthogonal decomposition which is compatible with the semiorthogonal decomposition of $\mathcal{A}$ under $\hat{i}$. If $(\mathrm{pGE})_R$ holds for $\mathcal{A}$, then every object in $\hat{\mathcal{A}}_i$ extends to a unique object $F \in \mathcal{A}$. By fully-faithfulness of $\hat{i}$, $F \in \mathcal{A}_i$ because $\mathcal{A}_i$ is characterized by semiorthogonality with the other $\mathcal{A}_j$. Hence $(\mathrm{pGE})_R$ holds for $\mathcal{A}_i$.

Conversely, if $(\mathrm{pGE})_R$ holds for each $\mathcal{A}_i$ and $\{F_n\} \in \hat{\mathcal{A}}$, then we can consider the diagram (8) induced by the semiorthogonal decomposition of $\hat{\mathcal{A}}$. By $(\mathrm{pGE})_R$ for the categories $\mathcal{A}_i$ and fully-faithfulness of $\hat{i}$, this diagram extends to a diagram in $\mathcal{A}$, hence there is an object $F \in \mathcal{A}$ such that $\hat{i}F \simeq \{F_n\}$. □

We are mostly interested in applying Proposition 3.9 in the geometric context:

Scholium 3.13. *Let R be a complete local Noetherian ring, let $\mathscr{X}$ be a perfect stack over R satisfying $(\mathrm{CP})_R$, and assume we have a semiorthogonal decomposition of R-linear categories $\mathrm{Perf}(\mathscr{X}) = \langle \mathcal{A}_i; i \in I\rangle$. Then $\mathscr{X}$ satisfies $(\mathrm{pGE})_R$ if and only if each of the $\mathcal{A}_i$ do. In particular if each $\mathcal{A}_i$ is smooth and proper over R then $\mathcal{A}$ satisfies $(\mathrm{pGE})_R$.*

Proof. By Lemma 3.1 this is equivalent to showing that the R-linear category $\mathrm{Perf}(\mathscr{X})$ satisfies (pGE). If $\mathscr{X}$ satisfies (CD), then $(\mathrm{CP})_R$ implies that $\mathrm{RHom}(F, G) \in \mathrm{APerf}(R)$ for $F, G \in \mathrm{Perf}(\mathscr{X})$, so the claim follows from Proposition 3.9. The statement that (pGE) holds when all of the $\mathcal{A}_i$ are fully dualizable is Example 3.3.

When (CD) does not hold:

We must revisit the two points in the proof of Proposition 3.9 which used (CD). The first was to show that $\hat{i} : \mathrm{Perf}(\mathscr{X}) \to \mathrm{Perf}(\hat{\mathscr{X}})$ was fully faithful, but this follows from $(\mathrm{CP})_R$ by Proposition 6.8.

The second was in the proof of Lemma 3.12, where we argued that if $M \in \mathcal{A} \otimes_R R_{n+1}$ and $M \otimes_{R_{n+1}} R_n = 0 \in \mathcal{A} \otimes_R R_n$, then $M = 0$. When $\mathcal{A} = \mathrm{Perf}(\mathscr{X})$, this fact is true without the hypothesis (CD). Choose an affine presentation $U \to \mathfrak{X}$ and consider the presentations $U_n \to \mathscr{X}_n$ for each n. An object $M \in \mathrm{Perf}(\mathscr{X}_{n+1})$ vanishes if and only if its restriction to $\mathrm{Perf}(U_{n+1})$ vanishes. However, U_{n+1} is affine and thus satisfies (CD), so we can apply the argument in the proof of Lemma 3.12.

The rest of the proof of Proposition 3.9 applies as written. □

Corollary 3.14. *Let A be a Noetherian ring and let $\mathscr{X}$ be a perfect stack over $\mathrm{Spec}\, A$ satisfying (CP). Assume that we have a semiorthogonal decomposition $\mathrm{Perf}(\mathscr{X}) = \langle \mathcal{A}_i; i \in I\rangle$. If $\mathcal{A}_i$ are smooth and proper A-linear categories, then $\mathscr{X}$ satisfies (pGE).*

Proof. Using the characterization of smooth and proper categories as dualizable categories, one can show that this property is preserved by base change. Lemma 3.11 implies that semiorthogonal decompositions are preserved by base change as well. By definition, (CP) implies that the base change to any complete local A-algebra R satisfies $(\mathrm{CP})_R$, hence we can Scholium 3.13 to this base change. □

The simplest application is to stacks such that $\mathrm{Perf}(\mathscr{X})$ admits a possibly infinite full exceptional collection.

[10]This is a purely formal version of Proposition 6.7 for R-linear categories.

Example 3.15. Let X be a smooth projective variety over k with a full exceptional collection of line bundles $\mathrm{Perf}(X) = \langle E_1, \ldots, E_n \rangle$, and let G be a linearly reductive group acting on X, and let $\mathscr{X} = X/G$. It is possible to choose G-linearizations of the E_i, and for any representation W of G we construct the twist $E_i(W) := E_i \otimes_k W$. We have

$$\mathrm{Hom}_{X/G}(E_i(W_1), E_j(W_2)) = (\mathrm{Hom}_X(E_i, E_j)(W_2 \otimes W_1^\vee))^G = 0 \text{ if } i > j$$

Furthermore if $i = j$ then $R\,\mathrm{Hom}(E_i, E_i)$ is a self-dual 1-dimensional representation of G, hence trivial. Objects of the form $E_i(W)$ generate, so we have

$$\mathrm{Perf}(\mathscr{X}) = \langle E_1 \otimes \mathrm{Rep}(G), \ldots, E_n \otimes \mathrm{Rep}(G) \rangle$$

where $E_i \otimes \mathrm{Rep}(G)$ denotes the stable ∞-subcategory generated by $E_i(W)$ for all representations W. Each subcategory $E_i \otimes \mathrm{Rep}(G)$ is equivalent to $\mathrm{Rep}(G)$, and thus admits a full exceptional collection consisting of irreducible representations of G. It follows that $\mathrm{Perf}(\mathscr{X})$ admits a full exceptional collection of vector bundles. By Corollary 3.14, $\mathscr{X}$ satisfies (pGE).

Example 3.16. Let V be a linear representation of a linearly reductive group G. Assume that there is central one parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that V has positive weights with respect to $\mathbb{G}_m$. Then for two irreducible representations of G, W_1 and W_2, we have

$$\mathrm{Hom}_{V/G}(\mathcal{O}_V(W_1), \mathcal{O}_V(W_2)) = (k[V^\vee] \otimes W_1^\vee \otimes W_2)^G \tag{9}$$

where there are no higher Ext's because V is affine and G is reductive. Furthermore, being irreducible, the representations W_1 and W_2 are concentrated in single weights with respect to λ, say w_1 and w_2 respectively. If $w_2 < w_1$, then the λ weight of $W_1^\vee \otimes W_2$ is negative so (9) vanishes. Furthermore, if $w_1 = w_2$, then (9) vanishes unless $W_1 \simeq W_2$, in which case it is one dimensional. This shows that after choosing an ordering which refines the ordering by λ weights, the vector bundles $\mathcal{O}_V \otimes W$ form a full exceptional collection for $\mathrm{Perf}(V/G)$.

We generalize the previous example with the following

Corollary 3.17. *Let k be field of characteristic 0 and let G be a smooth linear algebraic group over k, and let X be a quasiprojective (classical) k-scheme with a linearizable G-action. Assume that there is a one parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that*[11]

(1) the morphism $\mathbb{G}_m \times X \to X$ extends to a morphism $\mathbf{A}^1 \times X \to X$,
(2) the conjugation action $\mathbb{G}_m \times G \to G$ extends to a morphism $\mathbf{A}^1 \times G \to G$

If we let $Z := X^{\mathbb{G}_m}$ be the scheme theoretic fixed locus and $L := G^{\mathbb{G}_m}$ be the centralizer of λ, then $\mathscr{X} := X/G$ satisfies (CP) if Z/L does. If Z/L satisfies (CP), and (pGE), then so does $\mathscr{X}$.

Condition (2) is equivalent to the weights of λ in the adjoint representation of G being nonnegative. It follows that we have a semidirect product decomposition $G = U \rtimes L$ where U is the (unipotent) subgroup of elements which limit to $1 \in G$ under conjugation by $\lambda(t)$ as $t \to 0$.

One example in which all of these conditions hold is when L is linearly reductive and $X = Z = \mathrm{Spec}(k)$, in which case we have (pGE) for BG. Alternatively, if we let $L' := L/\lambda(\mathbb{G}_m)$, then Z/L satisfies (CP) and (pGE) if Z/L' is a smooth proper Deligne-Mumford stack.

Proof of Corollary 3.17. Let $\pi : X = \{0\} \times X \subset \mathbf{A}^1 \times X \to X$ be the morphism induced by assumption (1). π identifies $X = \underline{\mathrm{Spec}}_Z \mathcal{A}$, where $\mathcal{A} = p_* \mathcal{O}_X$ is a G-equivariant algebra over Z which is graded in non-positive degree with respect to λ. Note that we are letting G act on Z via the projection $G \to L$.

For any $F \in \mathrm{DCoh}(\mathscr{X})^\heartsuit$, we regard $\pi_* F$ as a G-equivariant $\mathcal{O}_Z$-module, graded with respect to λ. The subsheaf $(\pi_* F)_{\geq 0} := \bigoplus_{w \geq 0} (\pi_* F)_w \subset \pi_* F$ is G equivariant and coherent as an $\mathcal{O}_Z$-module. Furthermore we have

$$R\Gamma(\mathscr{X}, F) = R\Gamma(Z/G, \pi_* F) \simeq R\Gamma(Z/G, (\pi_* F)_{\geq 0}).$$

[11](1) and (2) together imply that the quotient stack X/G consists of a single Kempf-Ness stratum, as studied in [?HL12], i.e. each connected component of X is contracted onto a connected component of Z by the action of $\lambda(t)$ as $t \to 0$.

The unipotent group U acts trivially on Z, hence $p : Z/G \to Z/L$ is a U-gerbe, and in particular $Rp_*((\pi_* F)_{\geq 0}) \in \mathrm{DCoh}(Z/L)$. Thus we have

$$R\Gamma(X/G, F) \simeq R\Gamma(Z/L, Rp_*((\pi_* F)_{\geq 0})) \in \mathrm{DCoh}(\operatorname{Spec} k).$$

By [?HL12, Proposition 3.17 and Amplification 3.19][12], one has an infinite semiorthogonal decomposition

$$\mathrm{Perf}(\mathscr{X}) = \langle \ldots, \mathrm{Perf}(\mathscr{X})_w, \mathrm{Perf}(\mathscr{X})_{w+1}, \ldots \rangle$$

Where $\mathrm{Perf}(\mathscr{X})_w$ is the essential image of the fully faithful functor $\pi^* : \mathrm{Perf}(Z/L)_w \to \mathrm{Perf}(\mathscr{X})_w$ is an equivalence, where

$$\mathrm{Perf}(Z/L)_w = \{F \in \mathrm{Perf}(Z/L) | \mathcal{H}^i(F) \text{ has } \lambda\text{-weight } w, \forall i\}.$$

□

Example 3.18. Let Z be a smooth projective variety, and let $X = \underline{\operatorname{Spec}}_Z(\operatorname{Sym}(\mathcal{E}^\vee))$ be the total space of a locally free sheaf $\mathcal{E}$ on X. We equip X with the $\mathbb{G}_m$ action of scaling in the fibers of $X \to Z$. If $\mathcal{E}^\vee$ is sufficiently ample then X is projective-over-affine, hence $X/\mathbb{G}_m$ is cohomologically projective, and (pGE) follows from Theorem 6.10 and Lemma 3.5. On the other hand, Corollary 3.17 implies that $X/\mathbb{G}_m$ satisfies (pGE) for any $\mathcal{E}$.

Remark 3.19. The truncation functors in the infinite semiorthogonal decomposition used in the proof of Corollary 3.17 are t-exact. It seems that the methods of Appendix A can be extended to prove (GE) in the situation of Corollary 3.17.

Example 3.20 (Geometric invariant theory)**.** If X is a smooth projective-over-affine variety over a field of characteristic 0, G is a linearly reductive group acting on X, and L is a G-equivariant ample invertible sheaf, then the GIT quotient is an open substack $X^{ss}/G \subset X/G$. The main theorem of [?HL12] provides an infinite semiorthogonal decomposition of $\mathrm{Perf}(X/G)$ with one factor isomorphic to $\mathrm{Perf}(X^{ss}/G)$, and the rest isomorphic to the derived categories of perfect complexes (twisted by a $\mathbb{G}_m$-gerbe) over various GIT quotients Z_i^{ss}/K_i where $Z_i \subset X$ are smooth closed subvarieties and $K_i \subset G$ are subgroups.

Using Scholium 3.13 one can prove that, provided $\Gamma(X, \mathcal{O}_X)^G$ is finite dimensional, (pGE) holds for X/G if and only if it long as it holds for X^{ss}/G and each of the Z_i^{ss}/K_i. In good cases, X^{ss}/G and Z_i^{ss}/K_i are proper Deligne-Mumford stacks, hence (pGE) holds for both by Lemma 3.2. It is possible to expand these techniques to prove that (pGE) holds under the hypotheses of Proposition 5.14, provided k is a field of characteristic 0 and X is smooth, which is an alternative to Theorem 6.10 and Lemma 3.5.

4. h-descent Theorems

In the context of this paper, the goal of this section will be to prove that the properties (GE) and (pGE) can be checked "proper locally."

The main tool for this will be a general "h-descent" Theorem for APerf and Perf that we believe is of independent interest. It is similar to – and can be deduced, in the presence of a dualizing complex, from – the h-descent theorem for Ind DCoh in [?tsd-mf]. Nevertheless, the proof is more elementary since it avoids use of the shriek pullback functors.

As a funny consequence of this, we will also note that geometric stacks satisfy derived h-descent.

4.1. Descent pattern for closed immersions.

The goal for this subsection is to investigate a general descent pattern for what one might call the *(derived) nil-immersion topology.* This will be the Grothendieck topology on CAlg generated by surjective finitely presented closed immersions.

Lemma 4.1. *Suppose that $\mathscr{F}$ is a pre-sheaf on* CAlg *which satisfies*

(i) ("nilcomplete") The natural map $\mathscr{F}(R) \longrightarrow \varprojlim_n \mathscr{F}(\tau_{\leq n} R)$ is an equivalence for all $R \in \mathrm{CAlg}$;

[12]The paper [?HL12] considered stacks over an algebraically closed field of characteristic 0, but the proofs in Section 3.1 apply as stated over an arbitrary Noetherian base ring

(ii) *("infinitesimally cohesive") For every pullback diagram*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

in CAlg^{cn} *for which the maps* $\pi_0 A \to \pi_0 B$ *and* $\pi_0 B' \to \pi_0 B$ *are surjections whose kernels are nilpotent ideals, the induced map*

$$\mathscr{F}(A') \longrightarrow \mathscr{F}(A) \times_{\mathscr{F}(B)} \mathscr{F}(B')$$

is an equivalence. (In fact, it is enough to require this only when the pullback diagram is a square-zero extension in the sense of [?HigherAlgebra, Section 8.4]*.*

Let $\pi\colon Z = \operatorname{Spec} R' \to X = \operatorname{Spec} R$ *be a finitely presented, surjective, closed immersion, and* $\mathrm{Cech}(\pi)$ *its Cech nerve. Then, the natural pullback functor*

$$\mathscr{F}(R) \longrightarrow \mathscr{F}(\mathrm{Cech}(\pi)) = \mathrm{Tot}\left\{\mathscr{F}(R'^{\otimes_R \bullet+1})\right\}$$

is an equivalence.

Before giving the proof of this lemma, let us make some remarks and state some corollaries:

Remark 4.2. There is a slight variant of the above where, rather than considering a single surjective closed immersion, we consider a finite family of closed immersions which are jointly surjective. To handle this one can, for instance, replace (ii) by an arbitrary pushout diagram along closed immersions. This is, in principle, convenient – but will be subsumed in our application by h-descent.

Remark 4.3. Note that in the derived context there is a difference between *descent*, i.e., for Cech covers, and *hyper-descent*, i.e., for hyper-covers that are not n-coskeletal for any n. This difference is *severe* for surjective closed immersions:

Lemma 4.4. *Suppose that* $\mathscr{F}$ *is a pre-sheaf on* CAlg*. Then, the following are equivalent:*

(1) $\mathscr{F}$ *satisfies hyper-descent for the derived nil-immersion topology;*
(2) The natural map $\mathscr{F}(R) \to \mathscr{F}(\pi_0 R)$ *is an equivalence for all* $R \in \mathrm{CAlg}$*, and the restriction* $\mathscr{F}|_{\mathrm{CAlg}_0}$ *of* $\mathscr{F}$ *to discrete commutative algebras has descent for the ordinary nil-immersion topology.*

Proof. Note that the map from the constant simplicial diagram

$$\operatorname{Spec} R \leftarrow \{\operatorname{Spec} \pi_0 R\}$$

is a hyper-cover for the closed topology. □

Corollary 4.5. *Let* $\mathscr{F}$ *satisfy the conditions of Lemma 4.1, and suppose now only that* $\pi\colon Z \to X$ *is a finitely presented closed immersion, but not necessarily surjective. Then the natural pullback functor*

$$\mathscr{F}(\operatorname{Spf} R) \longrightarrow \mathscr{F}(\mathrm{Cech}(\pi))$$

is an equivalence.

Proof. Recall that $\operatorname{Spf} R \simeq \varinjlim_n \operatorname{Spec} R_n$ for $\{R_n\}$ as in Proposition C.3, so that

$$\mathscr{F}(\operatorname{Spf} R) \simeq \varprojlim_n \mathscr{F}(R_n)$$

and applying Lemma 4.1 to the base change of π to each $\operatorname{Spec} R_n$ we obtain that

$$\mathscr{F}(R_n) \simeq \mathscr{F}(\mathrm{Cech}(\pi \times_{\operatorname{Spec} R} \operatorname{Spec} R_n)).$$

Taking the inverse limit of these equivalences we complete the proof, since

$$\varprojlim_n \mathscr{F}(R'^{\otimes_R \bullet+1} \otimes_R R_n) \simeq \mathscr{F}(\operatorname{Spec}(R'^{\otimes_R \bullet+1}) \times_{\operatorname{Spec} R} \operatorname{Spf} R) \simeq \mathscr{F}(R'^{\otimes_R \bullet+1})$$

as $\operatorname{Spec}(R'^{\otimes_R \bullet+1}) \to \operatorname{Spec} R$ factors through the monomorphism $\operatorname{Spf} R \to \operatorname{Spec} R$. □

We now give a proof of the lemma:

Proof of Lemma 4.1. Our plan of proof is as follows: Let $\mathcal{C}$ denote the class of all finitely presented nil-thickenings i of affine schemes, such that every base change of i satisfies the conclusion of the lemma. We wish to show that $\mathcal{C}$ consists of all finitely presented nilthickenings. Our plan will be to prove that $\mathcal{C}$ contains increasingly large classes of morphisms: the basic building blocks will be closed gluings and thus square-zero extensions, as suggested in the discussion above the lemma.

Step 0: Case of π with a section. Note that if π has a section, then the result is immediate: In this case, the augmented cosimplicial diagram

$$\mathscr{F}(\widehat{X}) \longrightarrow \left\{\mathscr{F}(Z^{\times_X \bullet+1})\right\}$$

extends to a *split* augmented cosimplicial diagram. And any split augmented cosimplicial diagram is a limit diagram.

Step 1: Case of square-zero extensions.
Suppose that π fits into a pushout square

$$\begin{array}{ccc} \operatorname{Spec} R & \xleftarrow{\ \pi\ } & \operatorname{Spec} R' \\ \uparrow & & \uparrow \\ \operatorname{Spec} R' & \longleftarrow & \operatorname{Spec} R' \oplus M[+1] \end{array}$$

for some connective almost perfect R'-module M – i.e., it is a square-zero extension. Then, by hypothesis

$$\mathscr{F}(R) \xrightarrow{\sim} \mathscr{F}(R') \times_{\mathscr{F}(R' \oplus M[+1])} \mathscr{F}(R')$$

Furthermore, square-zero extensions are stable under base change so that there are also equivalences

$$\mathscr{F}(R'^{\otimes_R \bullet}) \xrightarrow{\sim} \mathscr{F}(R'^{\otimes_R \bullet} \otimes_R R') \times_{\mathscr{F}(R'^{\otimes_R \bullet} \otimes_R (R' \oplus M[+1]))} \mathscr{F}(R'^{\otimes_R \bullet} \otimes_R R')$$

Taking totalizations, and commuting totalizations and inverse limits, we conclude that to show the conclusion for π_R it suffices to know it for each of $\pi_{R'}$ and $\pi_{R' \oplus M[+1]}$. But each of these belong to $\mathcal{C}$ by Step 0. Thus, the conclusion holds for any square-zero extension π_R. Thus, square-zero extensions lie in $\mathcal{C}$.

Step 2: Composites, refinements, and locality. Note that $\mathcal{C}$ is closed under composites – this is a formal argument with computing a totalization of a bi-cosimplicial space along the diagonal.

It is also "local" in the following sense: Suppose that $\pi\colon Z \to \operatorname{Spec} R$ is in $\mathcal{C}$, then a map $i\colon Z' \to \operatorname{Spec} R$ is in $\mathcal{C}$ if and only if its base-extension $i_Z\colon Z' \times_{\operatorname{Spec} R} Z \to Z$ is in $\mathcal{C}$.

Finally, if a composite $h = f \circ g$ is in $\mathcal{C}$, then $f \in \mathcal{C}$. A special case is the case of a morphism admitting a section (so that $h = \mathrm{id}$). The general case reduces to this: to show that $f \in \mathcal{C}$, it is enough to show that its base change along h is so. But this base change has a section, induced from the diagonal of f.

Step 3: Case of Z, X classical Suppose that $\pi\colon Z \to X$ has both Z and X an ordinary classical scheme. In this case, $R \to R'$ is a nilpotent surjection of ordinary rings. Filtering by powers of the nilradical shows that it is a composite of (classical) square-zero extensions, which happen to also be ordinary square-zero extensions by [?HigherAlgebra, Section 8.4]. Thus, $\pi \in \mathcal{C}$. (Note that though such π are not preserved by base change, the base change will still be a composite of square-zero extensions.)

Step 4: Reduction to X classical. Suppose that A is a connective algebra and that we write $A = \varprojlim_n A_n$ as an inverse limit of almost perfect A-algebras $\{A_n\}$ satisfying the conditions of Lemma C.9. Then, we claim that nilcompleteness implies that

$$\mathscr{F}(A) \longrightarrow \varprojlim_n \mathscr{F}(A_n)$$

is an equivalence. Indeed, this follows by evaluating the inverse limit

$$\varprojlim_{n,m} \mathscr{F}(\tau_{\leq m} A_n)$$

in two ways – first n then m, and vice-versa – and applying nilcompleteness in each case.

This hypothesis applies to the inverse systems $\{\tau_{\leq n}R\}$ and any connective base change of it, such as

$$R' \simeq (\varprojlim_n \tau_{\leq n}R) \otimes_R R', \qquad R'^{\otimes_R \bullet} = (\varprojlim_n \tau_{\leq n}R) \otimes_R R'^{\otimes_R \bullet}$$

This reduces proving that $\pi \in \mathcal{C}$ to proving that each base change $\pi_{\tau_{\leq n}R} \in \mathcal{C}$.

Furthermore, note that for each n the map $\operatorname{Spec} \pi_0(R) \to \operatorname{Spec} \tau_{\leq n}R$ is a composite of square-zero extensions – thus it is in $\mathcal{C}$. Using locality, we reduce proving $\pi_{\tau_{\leq n}R} \in \mathcal{C}$ to proving $\pi_{\pi_0(R)} \in \mathcal{C}$.

Step 5: Completing the proof Suppose now that $\pi\colon Z = \operatorname{Spec} R' \to X = \operatorname{Spec} R$ is a nil-thickening with X classical. Let $i_0\colon Z_{cl} \to Z$ be the inclusion of the classical part of Z. By Step 3, the composite $\pi \circ i_0$ is in $\mathcal{C}$. By Step 2 ("refinements") this proves that $\pi \in \mathcal{C}$. □

4.1.1. *Closed descent for QC^{cn} and variants.* As an application of the above pattern, we have:

Proposition 4.6. *Suppose that $\pi\colon \mathscr{Z} \to \mathscr{X}$ is a finitely presented, surjective, closed immersion of algebraic stacks. Then, the pullback functor*

$$QC(\mathscr{X})^{cn} \longrightarrow QC(\operatorname{Cech}(\pi))^{cn} = \operatorname{Tot}\left\{QC(\mathscr{Z}^{\times_{\mathscr{X}} \bullet+1})\right\}$$

is an equivalence of ∞-categories. The same holds with QC^{cn} replaced by QC^{acn}, APerf or Perf.

Proof. Note that all the assertions are local on $\mathscr{X}$, so that we may suppose $\mathscr{X} = \operatorname{Spec} R$ is affine. Now we are in a position to apply Lemma 4.1. Let us check the the hypotheses hold.

Note that (i) holds for QC^{cn} because it is left t-complete. In more detail: The claim being local, we work in the affine case on $\operatorname{Spec} R$.

Let us write R_n for $\tau_{\leq n}R$. To see that the restriction map is fully faithful we must show that the map

$$\operatorname{Map}_R(M, N) \longrightarrow \varprojlim_n \operatorname{Map}_{R_n}(M \otimes_R R_n, N \otimes_R R_n)$$

is an equivalence. Using left t-completeness on both sides we may identify this with

$$\varprojlim_k \operatorname{Map}_R(\tau_{\leq k}M, \tau_{\leq k}N) \longrightarrow \varprojlim_{n,k} \operatorname{Map}_{R_n}(\tau_{\leq k}(M \otimes_R R_n), \tau_{\leq k}(N \otimes_R R_n))$$

Next note that the maps

$$\tau_{\leq k}(M) \longrightarrow \tau_{\leq k}(M \otimes_R R_n) \quad \text{for } n \geq k$$

are all equivalences – from this it follows that the previous map is an equivalence.

To see that it is essentially surjective, suppose that

$$\{M_n\} \in \varprojlim_n \{(\tau_{\leq n}R)\text{-mod}^{cn}\}$$

then note that the pushforwards $\{M_n\} \in R\text{-mod}^{cn}$ form an inverse system satisfying the hypotheses of Lemma C.9. In particular, letting $M = \varprojlim_n M_n$ we see that M is connective.

A similar argument shows that (i) holds for $\operatorname{APerf}^{cn}$. Fully faithfullness follows from the above, and to conclude it is enough to note that if the system $\{M_n\}$ consits of modules which are perfect to order k, then the inverse limit M is also perfect to order k. This is because $\tau_{\leq k}M = \tau_{\leq k}M_n$ for $n \geq k$ compatibly with the natural equivalences

$$(R\text{-mod})^{cn}_{\leq k} \simeq (R_n\text{-mod})^{cn}_{\leq k} \quad \text{for } n \geq k$$

Thus if $\tau_{\leq k}M_n$ is compact in $(R_n\text{-mod})^{cn}_{\leq k}$, then the same is true for $\tau_{\leq k}M$.

The case of (i) for each of QC^{acn} and APerf then follows by shifting the modules as needed until they are connective – notice that if we shift M_0 to be connective, the rest will be as well. The case of Perf follows from APerf upon noting that $\operatorname{Perf} \subset \operatorname{APerf}$ may be recognized as the dualizable objects.

Finally, notice that (ii) for each of our four categories follows form [?DAG-IX, Theorem 7.1, Prop. 7.7]. Thus the hypotheses of of Lemma 4.1 hold and our result is proved. □

Corollary 4.7. *Suppose that $\pi\colon \mathscr{Z} \to \mathscr{X}$ is a finitely presented closed immersion of algebraic stacks. Let $\widehat{\mathscr{X}}$ denote the formal completion of $\mathscr{X}$ along the image of $\mathscr{Z}$. Then, the pullback functor*

$$QC(\widehat{\mathscr{X}})^{cn} \longrightarrow QC(\mathrm{Cech}(\pi))^{cn} = \mathrm{Tot}\left\{QC(\mathscr{Z}^{\times_{\mathscr{X}} \bullet+1})\right\}$$

is an equivalence of ∞-categories. The same holds with QC^{cn} replaces by QC^{acn}, APerf*, or* Perf*.*

Proof. Combine the argument of the previous Proposition with Corollary 4.5. □

4.2. **Descent pattern for the h-topology and for (GE) morphisms.** Next, we discuss descent patterns for the derived h-topology. By a (derived) h-cover we will mean a morphism $\pi\colon \mathscr{X}' \to \mathscr{X}$ which is represented by a relative algebraic space, and which is a universal topological submersion. Examples include fppf surjections (since they are universally open), and proper surjections (since they are universally closed).

We will give two, slightly different, descent patterns here. The first is based on the idea that the h-topology is generated by surjective closed immersions, fppf covers, and "abstract blowup squares." In the derived context, closed immersions are no longer monomorphisms – this necessitates modifying the abstract blowup square condition by taking suitable formal completions. We say that a Cartesian diagram of pre-stacks

$$\begin{array}{ccc} \widehat{\mathscr{Y}} & \xrightarrow{i'} & \mathscr{Y} \\ {\scriptstyle\widehat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widehat{\mathscr{X}} & \xrightarrow[i]{} & \mathscr{X} = \mathrm{Spec}\, R \end{array} \tag{10}$$

is an *abstract blowup square* (with affine base) if $\mathscr{X} = \mathrm{Spec}\, R$ for a Noetherian R, π is a proper algebraic space, $\widehat{\mathscr{X}}$ is the completion of $\mathscr{X}$ along a closed subset $|\mathscr{Z}| \subset |\mathscr{X}|$, and $\pi^{-1}(\mathscr{X} \setminus \mathscr{Z}) \to \mathscr{X} \setminus \mathscr{Z}$ is an isomorphism.

Proposition 4.8. *Suppose that $\mathscr{F}$ is a pre-sheaf on locally Noetherian algebraic stacks satisfying:*

(i) $\mathscr{F}$ has descent for surjective finitely presented closed immersions;
(ii) $\mathscr{F}$ has fppf descent;
(iii) For any abstract blowup square (10), the natural map

$$\mathscr{F}(\mathscr{X}) \longrightarrow \mathscr{F}(\widehat{\mathscr{X}}) \times_{\mathscr{F}(\widehat{\mathscr{Y}})} \mathscr{F}(\mathscr{Y})$$

is an equivalence.

Then, $\mathscr{F}$ has descent for h-covers.

Proof. Let $\mathcal{D}$ denote the class of all finitely presented surjections $\pi\colon \mathscr{X}' \to \mathscr{X}$ with locally Noetherian target. Let $\mathcal{C} \subset \mathcal{D}$ consist of those π satisfying, additionally, the condition that for every affine map $\mathscr{Y} \to \mathscr{X}$ the base change $\pi_{\mathscr{Y}}$ satisfies the conclusion of the proposition. We wish to show that $\mathcal{C}$ contains all the morphisms satisfying our hypothesis.

The proof of Lemma 4.1 shows that $\mathcal{C}$ is closed under composition, refinements, and locality along nil-thickenings.

Step 0: Reduction to $\mathscr{X}$ an affine scheme.
Since $\mathscr{F}$ is an fppf sheaf, the assertion is fppf local on $\mathscr{X}$. Thus we may suppose $\mathscr{X}$ is affine.
Step 1: Reduction to $\mathscr{X}$ classical and reduced, $\mathscr{X}'$ classical.
Since $\mathrm{Spec}\, H_0(R)_{red} \to \mathrm{Spec}\, R$ is a surjective closed immersion, it is in $\mathcal{C}$. So by "locality" to show that $\pi \in \mathcal{C}$ it is enough to show that the base change $\pi_{H_0(R)_{red}} \in \mathcal{C}$. That is, we may suppose that $\mathscr{X}$ is classical and reduced. Similarly, by "refinements" we may suppose that $\mathscr{X}'$ is classical.

Step 2: A general descent argument for (classical) h-covers We will prove, by Noetherian induction, that for every closed subscheme $Z \subset \mathscr{X}$ it is the case that all h-coverings having target Z belong to $\mathcal{C}$. Thus we may suppose that this holds for all (classical) proper closed subschemes of $\mathscr{X}$, and we must show that every h-cover $\pi\colon \mathscr{X}' \to \mathscr{X}$ belongs to $\mathcal{C}$.

Since $\mathscr{X}$ is reduced, we may apply "generic flatness" to π to deduce the existence of a dense open $U \subset \mathscr{X}$ such that π_U is flat. We now make the mentioned convenient (but not necessary) reduction: By "platification par eclatement," there exists a closed subscheme $Z \subset \mathscr{X}$ disjoint from U such that the strict transform $\widetilde{p_Z}(\mathscr{X}')$ is flat over $\mathrm{Bl}_Z(\mathscr{Y}')$. More precisely, there is a commutative diagram

$$\begin{array}{ccccc} \mathscr{Y}' = \widetilde{p_Z}(\mathscr{X}') & \hookrightarrow & \mathrm{Bl}_Z(\mathscr{X}) \times_{\mathscr{X}} \mathscr{X}' & \longrightarrow & \mathscr{X}' \\ & \searrow^{\pi'} & \downarrow & & \downarrow \pi \\ & & \mathscr{Y} = \mathrm{Bl}_Z(\mathscr{X}) & \xrightarrow{p_Z} & \mathscr{X} \end{array}$$

such that $\pi' \colon \mathscr{Y}' \to \mathscr{Y}$ is flat. In order to show that $\pi \in \mathcal{C}$ it is enough, by refinement, to show that $p_Z \circ \pi' \in \mathcal{C}$. Note that π' is surjective by the argument of [?Voevodsky-Homology, Prop. 3.1.3], so that it is a flat cover and thus in $\mathcal{C}$. It thus suffices to show that the blowup morphism $p_Z \in \mathcal{C}$.

We now simplify our notation. We will rename p_Z to just π

$$\pi \colon \mathscr{Y} = \mathrm{Bl}_Z(\mathscr{X}) \longrightarrow \mathscr{X}$$

Let $U = \mathscr{X} \setminus Z$, and let I be a defining ideal for $Z \subset \mathscr{X}$.

It is clear that $\pi_U \in \mathcal{C}$ (isomorphism!), and that that $\pi_Z \in \mathcal{C}$ by our inductive hypothesis. By Step 1, $\pi_{R/I^n} \in \mathcal{C}$ for all $n \geq 1$ – we may summarize this by saying that $\pi_{\mathrm{Spf}\, R} \in \mathcal{C}$. It thus suffices, by commuting limits, to show that the pullback induces an equivalences

$$\mathscr{F}(\mathscr{X}) \longrightarrow \mathscr{F}(\widehat{\mathscr{X}}) \times_{\mathscr{F}(\widehat{\mathscr{Y}})} \mathscr{F}(\mathscr{Y})$$

which is guaranteed by our hypothesis (iii). □

4.2.1. *An alternate pattern for "proper enough" descent.* In the cases of APerf, Perf, and geometric stacks we will be able to do a little bit better – we will be able to establish descent for (GE) morphisms. We will say that a Cartesian diagram

$$\begin{array}{ccc} U' & \overset{j'}{\hookrightarrow} & \mathscr{Y}' \\ \downarrow p' & & \downarrow p \\ U & \overset{j}{\hookrightarrow} & \mathscr{Y} \end{array} \tag{11}$$

is a *flat excision square* if $j \colon U \subset \mathscr{Y}$ is a quasi-compact open immersion, $p \colon \mathscr{Y}' \to \mathscr{Y}$ is affine fpqc, and p induces an isomorphism $p^{-1}(Z) \to Z$ for Z the reduced induced structure on $\mathscr{Y} \setminus U$.

We will again formulate our descent pattern generally:

Proposition 4.9. *Suppose that $\mathscr{F}$ is a pre-sheaf on locally Noetherian algebraic stacks satisfying:*

(i) $\mathscr{F}$ has descent for surjective finitely presented closed immersions;
(ii) $\mathscr{F}$ has fppf descent;
(iii) For every flat excision square (11), the natural map $\mathscr{F}(\mathscr{Y}) \longrightarrow \mathscr{F}(U) \times_{\mathscr{F}(U')} \mathscr{F}(\mathscr{Y}')$ is an equivalence.

Let Σ be a collection of finitely presented morphisms of locally Noetherian algebraic stacks that are closed under base change, and such that if $\pi \colon \mathscr{X}' \to \mathscr{X}$ belongs to Σ then each projection $\mathscr{X}'^{\times_{\mathscr{X}} \bullet} \to \mathscr{X}$ belongs to Σ.

(4) Suppose that $\pi \colon \mathscr{X} \to \mathrm{Spec}\, R$ belongs to Σ, that $I \subset \pi_0(R)$ is an ideal, and $\widehat{R}$ is the I-adic completion of R. Then the natural map

$$\mathscr{F}(\mathscr{X} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \widehat{R}) \longrightarrow \mathscr{F}(\mathscr{X} \times_{\mathrm{Spec}\, R} \mathrm{Spf}\, \widehat{R})$$

is an equivalence.

Then $\mathscr{F}$ has descent for every surjective morphism $\pi : \mathscr{X}' \to \mathscr{X}$ in Σ, i.e. the natural map is an equivalence

$$\mathscr{F}(\mathscr{X}') \xrightarrow{\simeq} \mathrm{Tot}\, \mathscr{F}(\mathrm{Cech}(\pi)) \tag{12}$$

Proof. Let $\mathcal{D}$ denote the class of all finitely presented, surjections $\pi \colon \mathscr{X}' \to \mathscr{X}$ with locally Noetherian target. Let $\mathcal{C} \subset \mathcal{D}$ consist of those π satisfying, additionally, the condition that for every affine map $\mathscr{Y} \to \mathscr{X}$ the base change $\pi_{\mathscr{Y}}$ satisfies (12). We wish to show that $\mathcal{C}$ contains all the morphisms satisfying our hypothesis.

The proof of Lemma 4.1 shows that $\mathcal{C}$ is closed under composition, refinements, and locality along nil-thickenings.

Step 0: Reduction to $\mathscr{X}$ affine.
As before, we may reduce to $\mathscr{X}$ an affine scheme.
Step 1: Reduction to $\mathscr{X}$ classical and reduced, $\mathscr{X}'$ classical.
Since $\operatorname{Spec} H_0(R)_{red} \to \operatorname{Spec} R$ is a surjective closed immersion, it is in $\mathcal{C}$. So by "locality" to show that $\pi \in \mathcal{C}$ it is enough to show that the base change $\pi_{H_0(R)_{red}} \in \mathcal{C}$. That is, we may suppose that $\mathscr{X}$ is classical and reduced. Similarly, by "refinements" we may suppose that $\mathscr{X}'$ is classical.

Step 2: Noetherian induction, open-closed-decomposition. We are now in the situation where $\pi\colon \mathscr{X}' \to \mathscr{X}$ is a map of classical stacks, with $\mathscr{X} = \operatorname{Spec} R$ a reduced affine scheme, and the various projections belong to Σ . We will show that the base change $\pi_Z \in \mathcal{C}$ for every closed subscheme $Z \subset \mathscr{X}$ by Noetherian induction – notice also that this really depends only on the underlying closed subset of Z by Step 1. We may thus assume that $\pi_Z \in \mathcal{C}$ for every (classical) proper closed subscheme of $\mathscr{X}$, and we will use this to deduce that $\pi \in \mathcal{C}$.

Since $\mathscr{X}$ is reduced, we may apply "generic flatness" to π to deduce the existence of a dense open $U \subset \mathscr{X}$ such that π_U is flat. Let $I \subset R$ be an ideal cutting out the closed complement $Z = \mathscr{X} \setminus U$, and $\widehat{R}$ the I-adic completion of R. Note that $\pi_U \in \mathcal{C}$ since π has a section flat locally over U. We will show that $\pi_Z \in \mathcal{C}$ and $\pi_U \in \mathcal{C}$ implies that $\pi \in \mathcal{C}$.

Note that by assumption and Step 1, we have that $\pi_{R/I^n} \in \mathcal{C}$. Because $\operatorname{Spf} \widehat{R} = \varinjlim_n \operatorname{Spec} R/I^n$, we informally summarize this by saying $\pi_{\operatorname{Spf} \widehat{R}} \in \mathcal{C}$. We use this to deduce that (12) holds for $\pi_{\operatorname{Spec} \widehat{R}}$, though we do not necessarily claim any base change assertions: Note that the projection maps $\mathscr{X}'^{\times_{X} \bullet} \to \operatorname{Spec} R$ belong to Σ, so that

$$\mathscr{F}\left(\mathscr{X}'^{\times_{X} \bullet} \times_{\operatorname{Spec} R} \operatorname{Spec}(\widehat{R})\right) \longrightarrow \varprojlim_n \mathscr{F}(\mathscr{X}'^{\times_{X} \bullet} \times_{\operatorname{Spec} R} \operatorname{Spec} R/I^n)$$

is an equivalence for each $\bullet \geq 0$. Thus, taking the inverse limit of the equivalence of the Cech nerve for π_{R/I^n} over all n, we obtain that the map to the Cech nerve of $\pi_{\widehat{R}}$ is an equivalence.

Next, observe that for any $\mathscr{Y} \to \operatorname{Spec} R$ the diagram

$$\begin{array}{ccc}
\mathscr{F}(\mathscr{Y}) & \xrightarrow{i^*} & \mathscr{F}(\mathscr{Y} \times_R \widehat{R}) \\
\downarrow{\scriptstyle j^*} & & \downarrow \\
\mathscr{F}(\mathscr{Y} \times_R U) & \longrightarrow & \mathscr{F}(\mathscr{Y} \times_R U \times_R \widehat{R})
\end{array}$$

is a pullback square by hypothesis. In particular, the squares

$$\begin{array}{ccc}
\mathscr{F}(\mathscr{X}'^{\times_{X} \bullet} \times_R R) & \xrightarrow{i^*} & \mathscr{F}(\mathscr{X}'^{\times_{X} \bullet} \times_R \widehat{R}) \\
\downarrow{\scriptstyle j^*} & & \downarrow \\
\mathscr{F}(\mathscr{X}'^{\times_{X} \bullet} \times_R U) & \longrightarrow & \mathscr{F}(\mathscr{X}'^{\times_{X} \bullet} \times_R U \times_R \widehat{R})
\end{array}$$

are pullback diagrams for each $\bullet \geq 0$. Since (12) holds for $\pi_{\widehat{R}}$, for π_U, and for $\pi_{U \times_R \widehat{R}}$ we thus conclude – since limits commute – that it holds for π_R. Thus, we have proven the inductive step. $\square$

4.3. **h-descent theorems for** APerf **and** Perf**.** We will now apply the general patterns developed above to APerf and Perf.

Lemma 4.10. *Given a flat excision square* (11)*, the natural map*

$$QC(\mathscr{Y}) \longrightarrow QC(U) \times_{QC(U')} QC(\mathscr{Y}')$$

is an equivalence. This is also true with QC replaced by QC^{cn}, QC^{acn}, APerf *or* Perf*.*

Proof. The claim is local on $\mathscr{Y}$, so we will freely suppose that $\mathscr{Y} = \operatorname{Spec} R$ is affine. Let $g = j \circ p' = p \circ j'$. Note that the pullback has a right adjoint

$$(\mathscr{F}_U, \mathscr{F}_{U'}, \mathscr{F}_{\mathscr{Y}'}) \mapsto j_* \mathscr{F}_U \times_{g_* \mathscr{F}_{U'}} p_* \mathscr{F}_{\mathscr{Y}'}$$

Note first that the co-unit map

$$j^* \left(j_* \mathscr{F}_U \times_{g_* \mathscr{F}_{U'}} p_* \mathscr{F}_{\mathscr{Y}'} \right) \to \mathscr{F}_U$$

is an equivalence by base change for the qc.qs. pushforwards occuring here. This implies that the cofiber of the co-unit map

$$p^* \left(j_* \mathscr{F}_U \times_{g_* \mathscr{F}_{U'}} p_* \mathscr{F}_{\mathscr{Y}'} \right) \to \mathscr{F}_{\mathscr{Y}'}$$

is supported on $p^{-1}(Z)$, since its restriction along $(j')^*$ vanishes. Note that, since j and p are t-exact, to verify that this co-unit is an equivalence it suffices to verify it in case each of $\mathscr{F}_U$, $\mathscr{F}_{U'}$, and $\mathscr{F}_{\mathscr{Y}'}$ are connective. In this case, both sides of the co-unit are almost connective. Next, note that the restriction functor

$$QC_{p^{-1}(Z)}(\mathscr{Y}')^{acn} \longrightarrow QC(p^{-1}(Z))^{acn}$$

is conservative. This follows from base change and the fact that $p^{-1}(Z) \to Z$ is an isomorphism.

So, we have proven that the co-unit is an equivalence, so that this right adjoint is fully faithful. It follows also that the unit morphism

$$\mathscr{F} \longrightarrow j_* j^* \mathscr{F} \times_{g_* g^* \mathscr{F}} p_* p^* \mathscr{F}$$

is an equivalence. By fpqc descent, it suffices to verify this after applying each of j^* and p^* – which holds by the above. This completes the proof for QC.

Noting that the properties of being connective, almost connective, almost perfect, and perfect are all fpqc local, we see that the analogus statement holds also for each of the other listed categories. ☐

Lemma 4.11. *Given an abstract blowup square* (10)*, the natural map*

$$\mathrm{APerf}(\mathscr{X}) \longrightarrow \mathrm{APerf}(\widehat{\mathscr{X}}) \times_{\mathrm{APerf}(\widehat{\mathscr{Y}})} \mathrm{APerf}(\mathscr{Y})$$

is an equivalence. This is also true with APerf *replaced by* Perf*.*

Proof. Note that this assertion is flat local on $\mathscr{X}$, so we are free to continue assuming that $\mathscr{X} = \operatorname{Spec} R$ is affine. The proof is similar to the previous lemma. Suppose first that $M \in \mathrm{APerf}(R)$.

Fully-faithfulness will essentially follow from the Theorem on Formal functions, for π: It is enough to show that

$$\begin{array}{ccc} M & \longrightarrow & i_* i^* M = \widehat{M} \\ \downarrow & & \downarrow \\ \pi_* \pi^* M & \longrightarrow & i_* \widehat{\pi}_* \widehat{\pi}^* i^*(M) \underset{\text{formal functions}}{\simeq} i_* i^* \pi_* \pi^* M = \widehat{\pi^* \pi_* M} \end{array}$$

is a pullback diagram in R-mod for all $M \in \mathrm{APerf}(R)$. Note that the cone of the left vertical arrow is an almost perfect R-module which is supported along Z, so that it is already complete. For essential surjectivity, suppose given $M_{\mathscr{Y}}$, $M_{\widehat{\mathscr{X}}}$, and $M_{\widehat{\mathscr{Y}}}$, with an equivalence. Set

$$M := i_* M_{\widehat{\mathscr{X}}} \times_{i_* \widehat{\pi}_* M_{\widehat{\mathscr{Y}}}} \pi_* M_{\mathscr{Y}}$$

we must check that this is almost perfect, and that

$$\pi^* M \simeq M_{\mathscr{Y}} \qquad \text{and} \qquad i^* M \simeq M_{\widehat{\mathscr{X}}}.$$

The second of these is a chase with the Theorem on Formal Functions Theorem 6.5. The first then follows, since the cone of $M \to \pi_* M_{\mathscr{Y}}$ is almost connective and supported along Z – thus to check if it is zero we can pass to the completion by [?DAG-XII, Theorem 5.1.9]. It remains to check that M is almost perfect – for this it is enough to prove that

$$\operatorname{cone}\left(M_{\widehat{\mathscr{X}}} \to \widehat{\pi}_* \widehat{\pi}^* M_{\widehat{\mathscr{Y}}} \right)$$

is almost perfect as an R-module. But $M_{\widehat{\mathscr{X}}} \in \mathrm{APerf}(\widehat{\mathscr{X}})$ is algebrizable – i.e., by the Grothendieck existence theorem for π it is obtained by restriction from something in $\mathrm{APerf}(\mathscr{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R})$. Thus the cone can be identified with the completion of the analogous cone for the algebrized module – and the argument above (with $\operatorname{Spec} R$ replaced by $\operatorname{Spec} \widehat{R}$) implies that the cone is almost perfect over $\widehat{R}$ and supported along Z, thus already complete and almost perfect over R.

Finally, notice that Perf $\subset$ APerf consists of the dualizable objects for the symmetric monoidal structure and that the pullback functors are symmetric monoidal. $\square$

Theorem 4.12. *Suppose that $\mathscr{X}$ is a locally Noetherian algebraic stack, and that $\pi\colon \mathscr{X}' \to \mathscr{X}$ is a finitely presented, qc.qs., surjection satisfying one of the following two conditions:*

(1) Every term of

$$\mathrm{Cech}(\pi) = \left\{ \mathscr{X}'_n := \mathscr{X}'^{\times_{\mathscr{X}} n} = \mathscr{X}' \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \mathscr{X}' \right\}$$

satisfies (GE) over $\mathscr{X}$;

(2) π is an h-covering.

Then the pullback functors determine equivalences

$$(\pi_\bullet)^*\colon \ \mathrm{APerf}(\mathscr{X}) \to \mathrm{Tot}\{\mathrm{APerf}(\mathscr{X}'_n)\}$$

$$(\pi_\bullet)^*\colon \ \mathrm{Perf}(\mathscr{X}) \to \mathrm{Tot}\{\mathrm{Perf}(\mathscr{X}'_n)\}$$

of ∞-categories. In the case of Perf*, we may replace (GE) by (pGE) in (1).*

Proof. We first give a proof assuming hypothesis (2). We will prove the assertion for APerf, since the case of Perf then follows by passing to full subcategories of dualizable objects for the symmetric monoidal structure (note that all the functors involved are in fact symmetric monoidal!). Note that:

- APerf has fppf descent;
- APerf satisfies the condition (ii) of Proposition 4.8 by Proposition 4.6.
- APerf satisfies the condition (i) of Proposition 4.8 by Lemma 4.11.

Thus, applying Lemma 4.1 and Proposition 4.8 completes the proof of this case.

Next, we give a proof assuming hypothesis (1). As before we have fppf descent and descent for surjective closed immersions. Applying Lemma 4.10 implies that APerf has fpqc excision. Note that identifying Perf $\subset$ APerf with the dualizable objects, we see that the same is true for Perf. Let Σ be the collection of morphisms $\pi\colon \mathscr{X}' \to \mathscr{X}$ such that each product $\mathscr{X}'^{\times_{\mathscr{X}} \bullet} \to \mathscr{X}$ satisfies (GE) (resp., (pGE)). Applying Proposition 4.9 shows that APerf (resp., Perf) has descent for surjective morphisms in Σ. $\square$

As a consequence:

Corollary 4.13. *Suppose that $\mathscr{X}$ is a locally Noetherian algebraic stack, and that $\pi\colon \mathscr{X}' \to \mathscr{X}$ is a finitely presented, qc.qs., proper morphism. Let $\widehat{\mathscr{X}}$ denote the completion of $\mathscr{X}$ along the (closed) image of π. Then the pullback functors determine equivalences*

$$\mathrm{APerf}(\widehat{\mathscr{X}}) \xrightarrow{\sim} \mathrm{APerf}(\mathrm{Cech}(\pi)) = \mathrm{Tot}\left\{\mathrm{APerf}(\mathscr{X}'^{\times_{\mathscr{X}} \bullet+1})\right\}$$

$$\mathrm{Perf}(\widehat{\mathscr{X}}) \xrightarrow{\sim} \mathrm{Perf}(\mathrm{Cech}(\pi)) = \mathrm{Tot}\left\{\mathrm{Perf}(\mathscr{X}'^{\times_{\mathscr{X}} \bullet+1})\right\}$$

Proof. Combine the above with the argument given for Corollary 4.5. $\square$

4.4. ***h*-descent theorems for geometric stacks.** The classical h-topology is not sub-canonical, that is the functor represented by an affine scheme is *not* a sheaf for the ordinary h-topology. The situation for the derived h-topology is somewhat better:

Corollary 4.14. *Suppose that R is a derived ring and that $\pi\colon Z = \operatorname{Spec} R' \to X = \operatorname{Spec} R$ is a finitely presented closed immersion. Let $\widehat{X} = \operatorname{Spf} \widehat{R}$ denote the formal completion of X along the image of π, and let $\mathrm{Cech}(\pi) = \{Z^{\times_X \bullet}\}$ be its derived Cech nerve.*

Suppose that $\mathscr{F}$ is representable by an algebraic stack. Then the natural maps

$$\mathscr{F}(\widehat{R}) \longrightarrow \mathscr{F}(\operatorname{Spf} R) \longrightarrow \mathscr{F}(\mathrm{Cech}(\pi)) = \mathrm{Tot}\left\{\mathscr{F}(Z^{\times_X \bullet+1})\right\}$$

are equivalences.

Proof. Note that $\mathscr{F}$ satisfies the hypotheses of Lemma 4.1 – this is the assertion that an Artin stack $\mathscr{F}$ is *nil-complete* and *infinitesimally cohesive* in the notation of Section 2. It is also *integrable* in the notation of Section 2, so that $\mathscr{F}(\widehat{R}) \simeq \mathscr{F}(\operatorname{Spf}(R))$. $\square$

Lemma 4.15. *Any flat excision square (11) is a pushout diagram in the category of geometric stacks.*

Proof. Suppose that $\mathscr{F}$ is an arbitrary geometric stack. We must show that the natural map

$$\mathscr{F}(\mathscr{Y}) \longrightarrow \mathscr{F}(U) \times_{\mathscr{F}(U')} \mathscr{F}(\mathscr{Y}')$$

is an equivalence. In light of the Tannakian formalism (Theorem 2.16) we can identify

$$\mathscr{F}(\mathscr{Y}) \subset \operatorname{Fun}^{L\otimes}(QC(\mathscr{F})^{cn}, QC(\mathscr{Y})^{cn})$$

with the full subcategory consisting of those functors which preserve small colimits and flat objects. Similarly for the other two targets. In light of Lemma 4.10 it is enough to note that colimit diagrams and the property of $\mathscr{F} \in QC(\mathscr{Y})$ being flat are both fpqc local, so can be checked after restricting along each of p and j. □

Lemma 4.16. *Suppose that $\pi\colon \mathscr{X} \to \operatorname{Spec} R$ satisfies (GE), that $I \subset \pi_0(R)$ is an an ideal, and $\widehat{R}$ the I-adic completion of R. Then, the natural map*

$$\mathscr{F}(\mathscr{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}) \longrightarrow \mathscr{F}(\mathscr{X} \times_{\operatorname{Spec} R} \operatorname{Spf} \widehat{R})$$

is an equivalence for every geometric stack $\mathscr{F}$.

Proof. This is essentially contained Proposition 2.17. □

Theorem 4.17. *Suppose that $\mathscr{F}$ is a geometric stack. Then, $\mathscr{F}$ has descent for h-covers and for surjective morphisms all of whose products satisfy (GE).*

Proof. Let Σ refer to the class of all morphisms $\mathscr{X}' \to \mathscr{X}$ such that each product $\mathscr{X}'^{\times_{\mathscr{X}} \bullet} \to \mathscr{X}$ satisfies (GE). (This is what we mean by "all of whose products satisfy (GE).") We will first apply Proposition 4.9 to handle the case of morphisms in Σ. It is well-known that $\mathscr{F}$ has flat descent, verifying (ii); we have seen Corollary 4.14 that $\mathscr{F}$ has descent for surjective closed immersions, verifying (i); Lemma 4.15 verifies (iii); finally, Lemma 4.16 verifies (iv). Thus $\mathscr{F}$ has descent for surjective morphisms in Σ.

In particular, it has descent for all projective morphisms – and thus for all blowups. But the proof of Proposition 4.8 reduces proving h-descent for $\mathscr{F}$ to descent for surjective closed immersions, fppf surjections, and blowups. Thus, $\mathscr{F}$ also has descent for h-covers. □

4.5. **Closure of (GE) and (pGE) under proper maps.** As a corollary, we can deduce:

Corollary 4.18. *Suppose R is a Noetherian derived ring, complete with respect to an ideal $I \subset \pi_0 R$, and let $S = \operatorname{Spec} R$. Suppose that $\mathscr{X}$ is a qc.qs. Noetherian S-stack, and $\pi\colon \mathscr{X}' \to \mathscr{X}$ is surjective morphism, relatively representable by proper algebraic spaces. Suppose that $\mathscr{X}'$ satisfies $(GE)_R$ and that $\mathscr{X}' \times_{\mathscr{X}} \mathscr{X}'$ satisfies $(CP)_R$. Then, $\mathscr{X}$ satisfies $(GE)_R$.*

Proof. Let $S_m = \operatorname{Spec} R_m$ for R_m, $m \geq 0$, as in Proposition C.3. In unfortunately confusing notation, let

$$\pi_m\colon \mathscr{X}' \times_S S_m \longrightarrow \mathscr{X} \times_S S_m$$

be the base change of π. (For the remainder of this proof, we will avoid using subscripts to denote fiber products to avoid ambiguity between base change to S_m and the n-th term of the Cech nerve!)

Note that each π_m satisfies the hypothesis of Theorem 4.12, so that the pullback functor

$$\operatorname{APerf}(\mathscr{X} \times_S S_m) \xrightarrow{\sim} \operatorname{Tot}\left\{\operatorname{APerf}(\mathscr{X}'^{\times_{\mathscr{X}} \bullet} \times_S S_m)\right\}$$

is an equivalence. Taking the inverse limit over all m, and commuting inverse limits, we obtain that the rightmost arrow in the following commutative diagram is an equivalence

$$\begin{array}{ccccc}
\operatorname{APerf}(\mathscr{X}) & \longrightarrow & \operatorname{APerf}(\widehat{\mathscr{X}}) & = & \varprojlim_m \operatorname{APerf}(\mathscr{X} \times_S S_m) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
\operatorname{Tot}\left\{\operatorname{APerf}(\mathscr{X}'^{\times_{\mathscr{X}} \bullet})\right\} & \longrightarrow & \operatorname{Tot}\left\{\operatorname{APerf}(\widehat{\mathscr{X}'^{\times_{\mathscr{X}} \bullet}})\right\} & = & \varprojlim_m \operatorname{Tot}\left\{\operatorname{APerf}(\mathscr{X}'^{\times_{\mathscr{X}} \bullet} \times_S S_m)\right\}
\end{array}$$

Since the vertical maps are equivalences, to complete the proof it suffices to show that the bottom horizontal arrow is an equivalence. Our hypotheses guarantee that each term $\mathscr{X}'^{\times_{\mathscr{X}} \bullet}$, $\bullet \geq 1$, satisfies $(CP)_R$ so that by Proposition 6.8 each of the functors in the totalization, and thus the total functor, is fully faithful. It remains to check essential surjectivity – but indeed, this now follows from essential surjectivity at the zeroth level of the cosimplicial category (i.e., (GE)$_R$ for $\mathscr{X}'$). □

Example 4.19. Suppose that $\mathscr{X}$ admits a proper surjective morphism from a proper algebraic space. Then, each term of the Cech nerve is a proper algebraic space and so satisfies (GE) (since it has finite cohomological dimension, etc.). So for instance if G is a finite group scheme (in arbitrary characteristic), then BG satisfies (GE).

More generally, Olsson proves in [?OlssonProper] that every proper Artin stack admits a proper covering by a projective scheme. This implies that every proper stack satisfies (GE).

5. Cohomologically projective morphisms

In this section we introduce the notion of a *cohomologically projective* morphism between Noetherian geometric stacks $\pi : \mathscr{X} \to \mathscr{S}$ which generalizes the notion of a projective morphism between Noetherian schemes.

First, we require the pushforward along π to have finite cohomological dimension (CD). This is automatic in the case of schemes, and for geometric stacks which are finite type over a scheme over a field of characteristic 0 (See Proposition B.18) but can fail for stacks in finite characteristic. We also require that π satisfy a version of the proper push forward theorem, which we call Property (CP).

Finally, in Definition 5.6, we replace the notion of a relatively ample line bundle for the morphism π with a relatively cohomologically ample, (CA), system of vector bundles $\{V_\alpha\}$. We thus define a *cohomologically projective morphism* to be one satisfying (CD), (CP), and possessing a relatively (CA) system of vector bundles.

A notational remark: For the remainder of this section, we will let

$$\mathrm{Coh}(-) := \mathrm{DCoh}(-)^{\heartsuit}$$

refer to the *abelian* category of coherent sheaves.

5.1. Cohomological properness (CP).

In this section we study a version of the proper push forward theorem for stacks and show that it implies (L) for perfect morphisms of finite cohomological dimension.

Definition 5.1. Let $\mathscr{X}$ be a Noetherian algebraic stack over $\operatorname{Spec} R$, we introduce the property:

$(CP)_R$ For any $F \in \mathrm{DCoh}(\mathscr{X})$ and $i \in \mathbb{Z}$ the sheaf $H_i \circ R\Gamma(F) \in QC(R)^{\heartsuit}$ is coherent.

We say that a morphism $\pi : \mathscr{X} \to \mathscr{S}$ between Noetherian stacks is *cohomologically proper*, or that it satisfies (CP), if for any $\operatorname{Spec} R \to \mathscr{S}$, the base change $\mathscr{X}_R$ satisfies $(CP)_R$.

Lemma 5.2. *Let $S = \operatorname{Spec} R$ be a Noetherian affine scheme and suppose $\mathscr{X}$ is a Noetherian S-stack. Then the following are equivalent to $(CP)_R$*

i) $\pi_*(\mathrm{DCoh}(\mathscr{X})^{\heartsuit}) \subset \mathbb{D}\,\mathrm{APerf}(S)$*;*

ii) $\pi_*(\mathrm{DCoh}(\mathscr{X})) \subset \mathbb{D}\,\mathrm{APerf}(S)$*;*

iii) $\pi_*(\mathbb{D}\,\mathrm{APerf}(\mathscr{X})) \subset \mathbb{D}\,\mathrm{APerf}(S)$*;*

If furthermore $\mathscr{X}$ satisfies (CD) then this is equivalent to

i) $\pi_*(\mathrm{APerf}(\mathscr{X})) \subset \mathrm{APerf}(S)$*;*

ii) $\pi_*(\mathrm{PsCoh}(\mathscr{X})) \subset \mathrm{PsCoh}(S)$*;*

Proof. Note that (i) is equivalent to (iii) by definition. This is equivalent to (ii) by a filtration argument, since every object of $\mathrm{DCoh}(\mathscr{X})$ is in fact bounded in the t-structure since $\mathscr{X}$ is quasi-compact. It is obvious that (iv) implies (iii), so it remains to show the converse:

By Lemma B.9, π preserves quasi-coherence and filtered colimits for bounded above objects. Suppose $F \in \mathrm{PsCoh}(\mathscr{X})_{<\infty}$, we must show that $H_i(\pi_* F)$ is coherent for all i. If F has a single non-vanishing homology sheaf, this follows by (CP)$_R$. If it has finitely many non-vanishing homology sheaves, this follows by induction on the range of non-vanishing sheaves. For the general case, note that

$$H_i(\pi_* F) \simeq H_i(\pi_* \tau_{\geq i} F)$$

and that $\tau_{\geq i} F$ has bounded coherent homology sheaves.

Finally, let us prove the "furthermore" assertion. Since π_* is left t-exact, and right t-exact up to a shift, it is enough to show the following: Suppose $\mathscr{X}$ satisfies $(\mathrm{CP})_R$ and that $F \in \mathrm{PsCoh}(\mathscr{X})$. Then, $H_i(\pi_* F)$ is coherent. Note that

$$H_i(\pi_* F) = H_i(\pi_* \tau_{\geq i} F)$$

since π_* is left t-exact, and that

$$H_i(\pi_* \tau_{\geq i} F) = H_i\left(\pi_* \tau_{\leq i+d} \tau_{\geq i} F\right)$$

where d is the cohomological dimension of π_*. So, we are reduced to the case where F has finitely many non-vanishing homology sheaves – that is $F \in \mathrm{DCoh}(\mathscr{X})$). ☐

This lets us prove the following locality / invariance properties of (CP):

Proposition 5.3. *Let $\pi : \mathscr{X} \to \mathscr{S}$ be a morphism of Noetherian geometric stacks satisfying (CD), and let $S \to \mathfrak{S}$ be a faithfully flat affine morphism from a Noetherian affine scheme S. Then π has property (CP) if and only if $\pi' : \mathscr{X}' := \mathscr{X} \times_{\mathscr{S}} S \to S$ satisfies $(\mathrm{CP})_S$.*

The proof of this proposition follows from the key lemma

Lemma 5.4. *Let S be a Noetherian affine scheme and let $\pi : \mathscr{X} \to S$ be a Noetherian geometric stack. $(\mathrm{CP})_S$ may be checked locally: i.e., if $(\mathrm{CP})_{S'}$ holds after flat affine base change along $S' \to S$, then $(\mathrm{CP})_S$ holds.*

Conversely, if $\mathscr{X}$ satisfies (CD), then $(\mathrm{CP})_S$ is stable under arbitrary affine base change. That is, (CP) is equivalent to $(\mathrm{CP})_S$ in this case.

Proof. That $(\mathrm{CP})_S$ may be checked on a flat cover of S follows by flat base-change for $\mathrm{PsCoh}(S)_{<\infty}$.

Let us check that if $\mathscr{X}$ satisfies (CD), then $(\mathrm{CP})_S$ is stable under base change. Let $\mathscr{X}' := \mathscr{X} \times_S S'$. The base-change formula shows that $(\mathrm{CP})_{S'}$ holds on the full subcategory of $\mathrm{APerf}(\mathscr{X}')$ spanned by (derived) pullbacks from $\mathrm{APerf}(\mathscr{X})$. Since $\mathscr{X}' \to \mathscr{X}$ is affine, there is the cobar simplicial resolution of any F by pullbacks. By taking shifts, it suffices to prove this for $F \in \mathrm{APerf}(\mathscr{X})_{>d}$, where d bigger than the universal cohomological dimension of π, so that each term in the pushforward simplicial resolution is connective; then, the result follows by noting that $\mathrm{APerf}(S)_{\geq 0}$ is preserved by geometric realizations by an obvious spectral sequence argument. ☐

Proof of Proposition 5.3. One direction follows by definition. Thus we must let $T \to \mathscr{S}$ be any morphism from an affine Noetherian scheme, and show that $(\mathrm{CP})_T$ holds for $\mathscr{X} \times_{\mathscr{S}} T$. Let $T' = T \times_{\mathscr{S}} S$. Because $\mathscr{S}$ is geometric, T' is affine. We now apply the previous lemma twice: first if $(\mathrm{CP})_S$ holds for $\mathfrak{X}' \to S$, then it holds for the base-change to T', second the morphism $T' \to T$ is faithfully flat, so $(\mathrm{CP})_T$ must hold for the base change of $\mathscr{X} \to \mathscr{S}$ to T. ☐

While (CP) is a statement about global sections, it has the following consequence for RHom complexes:

Lemma 5.5. *Suppose that $\pi : \mathscr{X} \to \mathscr{S}$ is a morphism of Noetherian geometric stacks which satisfies (CP), and suppose that $\mathscr{X}$ is perfectly generated after base change to an affine presentation of $\mathscr{S}$. Suppose that $F, G \in QC(\mathscr{X})$ satisfy:*

(1) $H_i(F), H_i(G) \in QC(\mathscr{X})^{\heartsuit}$ are coherent $\mathscr{O}_{\mathscr{X}}$-modules for all i;
(2) $H_i(F) = 0$ for $i \ll 0$;
(3) $H_i(G) = 0$ for $i \gg 0$;

Then,

$$\mathrm{Ext}^n_{\mathscr{X}/\mathscr{S}}(F, G) := H_{-n} \circ \pi_* R\mathscr{H}om_{\mathscr{X}}(F, G) \in QC(\mathscr{S})^{\heartsuit}$$

is coherent for all n. If $\mathscr{X}$ satisfies (CD), then this holds assuming only (1) and (2).

Proof. It is enough to assume that $\mathscr{S} = \operatorname{Spec} R$ is affine and to show the following: $\mathrm{Ext}^n_{\mathscr{X}}(F, G)$ is a coherent R-module whose formation is compatible with flat base-change on R.

Without loss of generality we may suppose that $G \in QC(\mathscr{X})_{<0}$. By Corollary B.12, we can find a homomorphism from a perfect complex $P \to F$ such that $\tau_{\leq n} P \to \tau_{\leq n} F$ is a retract.

It now follows from the existence of the usual t-structure on $QC(\mathscr{X})$, and the connectivity assumption on G, that

$$\mathrm{Ext}^n(F, G) \longrightarrow \mathrm{Ext}^n(P, G) = H_{-n} \circ \pi_* \left(P^{\vee} \otimes G\right)$$

is retract of R-modules, compatible with flat base-change on R. Note that $P^\vee \otimes G$ has bounded above coherent homology sheaves since $P^\vee$ is perfect and G has this property. It follows from Lemma 5.2 that $\pi_* (P^\vee \otimes G)$ has coherent cohomology sheaves as well, and this concludes the proof. $\square$

5.2. **Cohomologically ample (CA) systems.** We introduce an structure for a morphism of stacks which is analogous to specifying a relatively ample bundle for a morphism of schemes.

We will say that a set I is *preordered by the nonnegative integers* if we have an assignment of an integer $\sharp(\alpha) \geq 0$ to each $\alpha \in I$. In this case we say $\alpha \geq \beta$ (respectively $\alpha > \beta$) if $\sharp(\alpha) \geq \sharp(\beta)$ (respectively $\sharp(\alpha) > \sharp(\beta)$). We also recall that a locally free sheaf V on $\mathscr{X}$ is not in $QC(\mathscr{X})^\heartsuit$ unless $\mathscr{X}$ is classical, so we need a more general notion of a surjective homomorphism. For $E, F \in QC(\mathscr{X})_{\geq 0}$ we define a morphism $E \to F$ to be surjective if $\tau_{\leq 0} E \to \tau_{\leq 0} F$ is surjective.

Definition 5.6. Let $\mathscr{X}$ be an algebraic stack over an affine base S, and let $\{V_\alpha\}_{\alpha \in I}$ be a collection of locally free sheaves on $\mathscr{X}$ indexed by a set preordered by the nonnegative integers. We say that $\{V_\alpha\}_{\alpha \in I}$ is a *cohomologically ample (CA) system* if

- (CA1) for all $F \in \mathrm{Coh}(\mathscr{X})$ and $N \geq 0$ there is a surjection onto F from a finite direct sum of sheaves in the collection $\{V_\alpha | \sharp(\alpha) \geq N\}$, and
- (CA2) for all $F \in \mathrm{Coh}(\mathscr{X})$, there exists an N such that $\mathrm{Ext}^i(V_\alpha, F) = 0$ for all $i \geq 1$ and $\sharp(\alpha) \geq N$.

If $\pi : \mathscr{X} \to \mathscr{S}$ is a morphism of algebraic stacks, we say that $\{V_\alpha\}_{\alpha \in I}$ is a *cohomologically ample system relative to* π if for every Noetherian affine scheme T over $\mathscr{S}$, the system $\{V_\alpha|_{\mathscr{X} \times_{\mathscr{S}} T}\}_{\alpha \in I}$ satisfies (CA1) and (CA2) over T.

Example 5.7. If X is a projective scheme over an affine base S, then X admits a cohomologically ample system. We take $I = \{n \geq 0\}$ with $\sharp(n) = n$, and we let $V_n := L^{-n}$, where L is an ample invertible sheaf on X.

Lemma 5.8. *Let $\mathscr{X}$ be an algebraic stack and let $i \colon \mathscr{X}_{cl} \to \mathscr{X}$ be the inclusion of the underlying classical stack. Then $\{V_\alpha\}$ is (CA) a system on $\mathscr{X}$ if and only if $\{i^* V_\alpha\}$ is (CA) on $\mathscr{X}_{cl}$.*

Proof. This follows immediately from the (i^*, i_*) adjunction and the fact that $i_* \colon \mathrm{Coh}(\mathscr{X}_{cl}) \to \mathrm{Coh}(\mathscr{X})$ is an equivalence. $\square$

We also note that with a cohomological dimension and Noetherian hypotheses, the definition of a (CA) system can be weakened significantly

Proposition 5.9. *Let $\mathscr{X}$ be a Noetherian geometric stack satisfying (CD) over an affine Noetherian base S. A system $\{V_\alpha\}_{\alpha \in I}$ of locally free sheaves on $\mathscr{X}$ is (CA) if and only if*

- *(CA1') $\forall F \in \mathrm{Coh}(\mathscr{X})$ and $N \geq 0$, there is an $\alpha \in I$ with $\sharp(\alpha) \geq N$ and $\mathrm{Hom}(V_\alpha, F) = H^0 R\Gamma(F \otimes V_\alpha^*) \neq 0$, and*
- *(CA2') $\forall \beta \in I$, there is an $N \geq 0$ such that $R\Gamma(V_\beta \otimes V_\alpha^*) \in QC(S)_{\geq 0}$ whenever $\sharp(\alpha) \geq N$.*

Furthermore one can equivalently replace $F \in \mathrm{Coh}(\mathscr{X})$ with $F \in \mathrm{APerf}(\mathscr{X})_{\geq 0}$ in the definitions of (CA1) and (CA2).

Remark 5.10. In fact in all of our examples our cohomologically ample systems can be chosen to satisfy the apparently stronger condition that $R^i \pi_* (V_\beta \otimes V_\alpha^*) = 0$ for all $i \geq 1$ and $\alpha > \beta$.

Proof. It is clear that $(CA1 - \mathrm{Aperf}) \Rightarrow (CA1) \Rightarrow (CA1')$ and $(CA2 - \mathrm{APerf}) \Rightarrow (CA2) \Rightarrow (CA2')$, where the 'Aperf' denotes the same definition but with '$F \in \mathrm{Coh}(\mathscr{X})$' replaced by '$F \in \mathrm{APerf}(\mathscr{X})_{\geq 0}$.' We must show the converse implication.

Proof that (CA1') + (CA2') ⇒ (CA2-APerf) :

We introduce the following condition for $p \geq 0$:

- $(CA2)_p$ For all $F \in \mathrm{APerf}(\mathscr{X})_{\geq 0}$, there exists an integer N such that $R\Gamma(F \otimes V_\alpha^*) \in (\mathbb{Z} - Mod)_{>-p}$ for all $\sharp(\alpha) \geq N$.

Our goal is to prove that $(CA_2)_1$ holds, and we will show this by descending induction on $p > 0$. Note that (CD) implies that this holds for some sufficiently large p, providing the base case.

Assume $(CA2)_p$, and note that by considering the long exact homology sequence associated to $(\tau_{>0}F)\otimes V_\alpha^* \to F\otimes V_\alpha^* \to (H_0F)\otimes V_\alpha^* \to$, it suffices to consider $F\in \mathrm{Coh}(\mathscr{X})$ in order to show $(CA2)_{p-1}$. Let $F\in \mathrm{Coh}(\mathscr{X})$. By condition (CA1') we have a nonzero homomorphism $\phi : V_\beta \to F$. This leads to two exact triangles

$$V_\beta \to F \to \mathrm{Cone}(\phi) \to$$
$$\tau_{>0}\,\mathrm{Cone}(\phi) \to \mathrm{Cone}(\phi) \to F_1 := \tau_{\leq 0}\,\mathrm{Cone}(\phi) \to$$

We want to show that $H_{-p+1}R\Gamma(F\otimes V_\alpha^*) = 0$ for $\sharp(\alpha)\gg 0$. For $\sharp(\alpha)\gg 0$ we have $R\Gamma(V_\beta\otimes V_\alpha^*)\in (\mathbb{Z}-Mod)_{\geq 0}$ by (CA1') and $H_{-p+1}R\Gamma(V_\alpha^*\otimes \tau_{>0}\,\mathrm{Cone}(\phi)) = 0$ by hypothesis $(CA2)_p$. Thus from the long exact sequence in homology applied to the exact triangles above, we have

$$H_{-p+1}R\Gamma(F\otimes V_\alpha^*) \simeq H_{-p+1}R\Gamma(\mathrm{Cone}(\phi)\otimes V_\alpha^*) \simeq H_{-p+1}R\Gamma(F_1\otimes V_\alpha^*)$$

whenever $\sharp(\alpha)\gg 0$.

Iterating this argument we get a strictly descending sequence $F \twoheadrightarrow F_1 \twoheadrightarrow F_2 \twoheadrightarrow \cdots$ in $\mathrm{Coh}(\mathscr{X})$ such that $H_{-p+1}R\Gamma(F\otimes V_\alpha^*) \simeq H_{-p+1}R\Gamma(F_i\otimes V_\alpha^*)$ for $\sharp(\alpha)\gg 0$. Because $\mathscr{X}$ is Noetherian, we must have $F_n = 0$ for some n, hence $H^pR\Gamma(F\otimes V_\alpha^*) = 0$.

*Proof that (CA1') + (CA2-*APerf*) ⇒ (CA1-*APerf*) :*

Fix an $F\in \mathrm{APerf}(\mathscr{X})_{\geq 0}$ and an $N\geq 0$. By (CA1') we can find a nonzero morphism $\phi : V_{\alpha_0}\to H_0(F)$ with $\sharp(\alpha_0)\geq N$. By choosing $\sharp(\alpha_0)$ large enough, we can arrange that $\mathrm{Hom}(V_{\alpha_0}, (\tau_{>0}F)[1]) = 0$ by (CA2-APerf), so that $V_{\alpha_0}\to H_0(F)$ extends to a homomorphism $\phi V_{\alpha_0}\to F$ which is nonzero on H_0.

Let $F_1 := \mathrm{Cone}(\phi)\in \mathrm{APerf}(\mathscr{X})_{\geq 0}$, so that we have a surjection $F\twoheadrightarrow F_1$. By (CA2') we have $\mathrm{Hom}(V_\beta, V_{\alpha_0}) = 0$ for $\sharp(\beta)\gg 0$. In particular by repeating the argument for F we can find a morphism $V_{\alpha_1}\to F_1$ which factors through F and is nonzero on H_0.

Iterating this procedure, we get a sequence of morphisms $V_{\alpha_i}\to F$ and a sequence of objects

$$F_n = \mathrm{Cone}(\bigoplus_{i=0}^{n} V_{\alpha_n}\to F)\in \mathrm{APerf}(\mathscr{X})_{\geq 0}$$

Such that there is a strictly descending sequence $H_0(F)\twoheadrightarrow H_0(F_1)\twoheadrightarrow H_0(F_2)\twoheadrightarrow\cdots$. Because $\mathscr{X}$ is Noetherian, we must have $H_0(F_n) = 0$ for some n, hence the morphism $\mathrm{Cone}(\bigoplus V_{\alpha_i}\to F)\in \mathrm{APerf}(\mathscr{X})_{\geq 1}$, which by definition means the morphism $\bigoplus V_{\alpha_i}\to F$ is surjective. □

Remark 5.11. A similar argument should show the following: Suppose that $\mathscr{X}$ is a Noetherian geometric S-stack satisfying (CA) and (CD). Suppose furthermore that for all $\alpha\in I$ and $i\geq 0$, the sheaf

$$H_{-i}\circ \pi_*(V_\alpha)$$

is coherent over $\mathscr{O}_S$. Then, $\mathscr{X}$ sastisfies (CP) over S. We do not need this, so we do not carry this out.

Next we show that cohomological ampleness of a system of locally free sheaves relative to a morphism of finite cohomological dimension between Noetherian, geometric stacks can be checked locally or globally. Furthermore, if the morphism also satisfies $(\mathrm{CP})_S$, the property (CA) can be checked on the fibers of the morphism.

Proposition 5.12. *Let $\pi : \mathscr{X}\to \mathscr{S}$ be a morphism of Noetherian geometric stacks satisfying (CD), and let $\{V_\alpha\}_{\alpha\in I}$ be a system of locally free sheaves preordered by the nonnegative integers. We also fix an affine presentation $S = \mathrm{Spec}(R)\to \mathscr{S}$. Then following are equivalent:*

1. *$\{V_\alpha\}$ is (CA) relative to π, and*
2. *The restriction $\{V_\alpha|_{\mathscr{X}\times_{\mathscr{S}} S}\}$ is (CA).*
3. *For all $\beta\in I$, $\exists N$ such that $\pi_*(V_\beta\otimes V_\alpha^*)\in QC(\mathscr{S})_{\geq 0}$ whenever $\sharp(\alpha)\geq N$, and for all $F\in \mathrm{Coh}(\mathscr{X})$ and all $N\geq 0$, there is an α with $\sharp(\alpha)\geq N$ and $H_0\pi_*(F\otimes V_\alpha^*)\neq 0$.*

Furthermore, if π is (CP) then this is equivalent to

4. *For any closed point $\eta\colon \mathrm{Spec}\,k\to S\to \mathscr{S}$, the system $\{V_\alpha|_{\mathscr{X}\times_{\mathscr{S}}\mathrm{Spec}\,k}\}$ is (CA).*

Note that as a consequence for a Noetherian geometric stack over a Noetherian affine base which satisfies (CD), a system of locally free sheaves is (CA) iff it is (CA) relative to S. The proof of the proposition requires a key lemma.

Lemma 5.13. *Let $S' \to S$ be a morphism of Noetherian affine schemes. Suppose $\mathscr{X}$ is a Noetherian geometric S-stack satisfying (CD) and admitting a (CA) system $\{V_\alpha\}$. Define $f\colon \mathscr{X}' = \mathscr{X} \times_S S' \to \mathscr{X}$; then $\{f^* V_\alpha\}$ is (CA).*

Conversely if $S' \to S$ is faithfully flat and $\{f^* V_\alpha\}$ is (CA) on $\mathscr{X}'$, then $\{V_\alpha\}$ is (CA) on $\mathscr{X}$.

Proof. First check property (CA1): Suppose $F \in \mathrm{Coh}(\mathscr{X})$. The natural map $f^* f_* F \to F$ is surjective since f is affine. Since $\mathscr{X}$ is a Noetherian geometric stack, $f_* F$ is a union of its coherent submodules H_β. Consequently,

$$f^* f_* F = \bigcup_\beta \mathrm{im}\,\{f^* H_\beta \to f^* f_* F\}$$

so that

$$F = \bigcup_\beta \mathrm{im}\,\{f^* H_\beta \to f^* f_* F \to F\}$$

But since F is coherent, it is a compact object in of $QC(\mathscr{X}')^\heartsuit$, so that there is some β for which the natural map

$$f^* H_\beta \to f^* f_* F \to F$$

is surjective. Since $\mathscr{X}$ satisfies (CA), there exists a surjection from a finite sum of V_α onto H_β – applying the right exact functor f^* and composing with the surjection in the previous displayed equation, we obtain a surjection from a finite sum of $f^* V_\alpha$ onto F, as desired. This can be carried out using only V_α with $\sharp(\alpha) \geq N$.

Next, we check property (CA2'): First note that when $\mathscr{X}$ is defined over an affine base, the vanishing of higher global sections in property (CA2') is equivalent to $\pi_*(V_\beta \otimes V_\alpha^*) \in QC(S)_{\geq 0}$. By the base-change formula (Corollary B.16)

$$(R\pi'_*) \circ f^* \left(V_\beta \otimes V_\alpha^*\right) \simeq (R\pi_*) \left(V_\beta \otimes V_\alpha^*\right)|_{S'}$$

So it suffices to show that for each $\beta \in I$ there is an N such that $(R\pi_*) \left(V_\beta \otimes V_\alpha^*\right) \in QC(S)_{\geq 0}$ for $\sharp(\alpha) \geq N$, but this holds by the hypothesis that $\{V_\alpha\}$ is (CA).

Finally we prove the converse by verifying (CA1') and (CA2'): we assume that $S = \mathrm{Spec}\, A$ and $S' = \mathrm{Spec}\, A'$ where A' is a faithfully flat A algebra. Thus $\mathscr{X}' \simeq \underline{\mathrm{Spec}}_{\mathscr{X}} \mathcal{A}'$ where $\mathcal{A}' := \pi^* A'$. By Lazard's theorem [?HigherAlgebra], A' is a filtered colimit of free A-modules, so $f_* \mathcal{O}_{\mathscr{X}'} = \mathcal{A}'$ is a filtered colimit of free $\mathcal{O}_{\mathscr{X}}$-modules.

Note that because $\pi^* A'$ is faithfully flat, a homomorphism $E \to F$ in $QC(\mathscr{X})^\heartsuit$ is nonzero if and only if $\mathcal{A}' \otimes_{\mathcal{O}_{\mathscr{X}}} E \to \mathcal{A}' \otimes_{\mathcal{O}_{\mathscr{X}}} F$ is nonzero. By hypothesis $\{f^* V_\alpha\}$ satisfies (CA1') on $\mathscr{X}'$, so for any $F \in \mathrm{Coh}(\mathscr{X})$ there is a nonzero homomorphism of $\mathcal{A}'$-modules $\mathcal{A}' \otimes V_\alpha \to \mathcal{A}' \otimes F$ which corresponds to a nonzero homomorphism $V_\alpha \to \mathcal{A}' \otimes F$.

This homomorphism need not factor through F; however, writing $\mathcal{A}'$ as a filtered colimit of free $\mathcal{O}_{\mathscr{X}}$ modules of finite rank, we have $V_\alpha \to \mathcal{A}' \otimes F$ factors through a nonzero homomorphism $V_\alpha \to F^{\oplus n}$ for some n. Thus the projection onto one of the factors defines a nonzero homomorphism $V_\alpha \to F$.

By faithfully flat descent, the vanishing of $R^i \pi_*(V_\beta \otimes V_\alpha^*)$ can be checked after restriction to S'. Thus by the base-change formula property (CA2') for $\{f^* V_\alpha\}$ implies (CA2) for $\{V_\alpha\}$. □

Proof of Proposition 5.12. By definition if $\{V_\alpha\}$ is a cohomologically ample system of locally free sheaves relative to π, then $\{V|_{\mathscr{X} \times_{\mathscr{S}} S}\}$ is (CA). For any map $T \to \mathscr{S}$ with T a Noetherian affine scheme, we have $T' := S \times_{\mathscr{S}} T \to T$ is Noetherian, affine, and faithfully flat. Thus if $\{V_\alpha|_{S \times_{\mathscr{S}} \mathscr{X}}\}$ is (CA), Lemma 5.13 implies that $\{V_\alpha|_{T' \times_{\mathscr{S}} \mathscr{X}}\}$ is (CA), and thus also that $\{V_\alpha|_{T \times_{\mathscr{S}} \mathscr{X}}\}$ is (CA).

(3) is equivalent to (2), because by faithfully flat descent, $\pi_*(V_\beta \otimes V_\alpha) \in QC(\mathscr{S})_{\geq 0}$ if and only if $\pi_*(V_\beta \otimes V_\alpha)|_S \in QC(S)_{\geq 0}$, and likewise for the condition $H_0 \pi_*(F \otimes V_\alpha^*) \neq 0$. Therefore by base-change these properties correspond exactly to properties (CA1') and (CA2') of Proposition 5.9 applied to the stack $\mathscr{X} \times_{\mathscr{S}} S$.

Now assume that $\pi : \mathscr{X} \to \mathscr{S}$ is (CP), and thus so is $\pi' : \mathscr{X}' := \mathscr{X} \times_{\mathscr{S}} S \to S$. We can check that the fibral criterion (4) implies (CA1') and (CA2) for $\mathscr{X}'$. We must verify that

$$\pi'_* (F \otimes V_\alpha^*) \in QC(S)_{\geq 0} \quad \text{and} \quad H_0 \pi'_* (F \otimes V_\alpha^*) \neq 0$$

for $\sharp(\alpha)$ in certain ranges.

By hypothesis $(\mathrm{CP})_S$, we know that $\pi'_*(F \otimes V_\alpha^*) \in \mathrm{APerf}(S)$. Since S is affine, Nakayama's lemma implies that an object $E \in \mathrm{APerf}(S)$ is connective iff $\eta^* E$ is connective for every closed point $\eta : \operatorname{Spec} k \to S$, and likewise that if E is connective, then $H_0(E) \neq 0$ if and only if $H_0(\eta^* E) \neq 0$ for all closed points.

For $E = \pi'_*(F \otimes V_\alpha^*)$, the base-change theorem implies that

$$\eta^* \pi'_*(F \otimes V_\alpha^*) \simeq R\Gamma(F \otimes V_\alpha^*|_{\mathscr{X}' \times_S \operatorname{Spec} k}).$$

Thus for each closed point, the necessary connectiveness and non-vanishing properties hold as a consequence of (CA1) and (CA2) applied to the object $F|_{\mathscr{X}' \times_S \operatorname{Spec} k} \in \mathrm{APerf}(\mathscr{X}' \times_S \operatorname{Spec} k)$, which hold by Proposition 5.9. □

5.3. **Examples of cohomologically projective morphisms.** Now that we have developed notions of cohomological properness and relatively cohomologically ample systems of vector bundles, we exhibit two large classes of morphisms with these properties.

By a linearly reductive group scheme over an affine base, we mean a smooth group scheme G over $\operatorname{Spec} k$ such that the pushforward (i.e. invariants) functor $QC(BG) \to QC(\operatorname{Spec} k)$ is exact, and such that $QC(BG)$ is generated by locally free sheaves. The typical example is $G \times_{\operatorname{Spec} l} \operatorname{Spec} k$ where k is an algebra over an algebraically closed field l, and G is an algebraic group over l which is reductive (if $\operatorname{char} l = 0$) or of multiplicative type (if $\operatorname{char} l > 0$ or l is not algebraically closed).

Proposition 5.14. *Suppose X is a projective-over-affine scheme over a Noetherian ring k, that G is a linearly reductive k-group scheme, and that G acts on X admitting a G-linearized ample bundle over k. Suppose furthermore that $H^0(X, \mathscr{O}_X)^G$ is coherent over k. Then, $\mathscr{X} = X/G \to \operatorname{Spec} k$ is cohomologically projective.*

Proof. We verify the three properties (CD),(CA), and (CP) in turn:

Verification of (CD): Since $X/G \to BG$ is representable, and G is linearly reductive, we are done.

Verification of (CA): Set $L = \mathscr{O}_X(1)$ to be (the bundle on X/G induced by descent by) the G-linearized ample bundle on X, and let $p : X/G \to BG$. Let $I = \mathbb{Z}_{\geq 0} \times J$, where J indexes a generating set of locally free sheaves ρ on BG, and let $V_{n,\rho} := L^{-n} \otimes p^*(\rho)$ where we use the notation $F(\rho)$ to denote the tensor product of $F \in QC(X/G)$ with the pullback of $\rho \in \mathrm{Irrep}(G)$ regarded as locally free sheaf on BG.

By Proposition 5.9, it suffices to verify $(CA1')$ and $(CA2')$ for the system $\{V_{n,\rho}\}$. Let $p : X/G \to BG$, so that for $F \in \mathrm{Coh}(X/G)$, $p_* F = R\Gamma(X, F)$ regarded as a complex of representations of G. By ordinary ampleness of $\mathcal{O}_X(1)$, the pushforward $p_*(F \otimes L^n) \in QC(BG)_{\geq 0}$ for all $n \gg 0$. Because G is linearly reductive this implies that $R\Gamma(X/G, F \otimes V_{n,\rho}^*) \in QC(\operatorname{Spec} k)_{\geq 0}$ for all ρ and $n \gg 0$.

Furthermore, if $F \neq 0$, then for any n sufficiently large $H_0 R\Gamma(X, F \otimes L^n) \neq 0$. Therefore for any N we can find an $n \geq N$ and a nonzero morphism $\rho \to R\Gamma(X, F \otimes L^n)$ in $QC(BG)$. It follows that $R\Gamma(X/G, F \otimes V_{n,\rho}^*) \neq 0$.

Verification of (CP): Let $Y = \operatorname{Spec} H^0(X, \mathscr{O}_X)$, so that $q \colon X \to Y$ is a projective G-equivariant map. Suppose F is a G-equivariant coherent sheaf on X. We must show that

$$H^i(X, F)^G = H^0(Y, H_{-i} \circ q_* F)^G$$

is coherent over k for each i. By the usual projective pushforward theorem, $H_{-i} \circ q_* F$ is coherent and by functoriality it is G-equivariant.

It is thus enough to show the following: For any G-equivariant coherent $A = H^0(X, \mathscr{O}_X)$-module M, M^G is coherent over A^G (and thus, by our hypotheses, over k). For this, it is enough to assume that A is Noetherian so that every finite subset of A is contained in a finite-dimensional sub-G-representation (by the same argument as with the ϕ_S above).

□

Next we show that any stack which admits a projective good moduli space in the sense of [?Alper] is cohomologically projective. Recall that $\phi : \mathscr{X} \to X$ is a good moduli space morphism if X is an algebraic space, ϕ_* is exact, and $\phi_*(\mathcal{O}_{\mathscr{X}}) = \mathcal{O}_X$. If $\mathscr{X}$ is locally Noetherian, then ϕ is universal for maps to algebraic spaces.

Example 5.15. Let X be a projective variety over a field of characteristic 0. Let G be a reductive group acting on X linearized by an equivariant ample invertible sheaf L. Then the morphism from the stacky GIT quotient to the scheme-theoretic GIT quotient, $X^{ss}(L)/G \to X//_L G$, is a good moduli space morphism.

The following lemma is useful for constructing examples of relatively cohomologically ample systems.

Lemma 5.16. *Let $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$ be (CD) morphisms of Noetherian geometric stacks. Furthermore let f satisfy (CP). If $\{V_\alpha\}_{\alpha \in I}$ is a (CA) system relative to f, and $\{W_\beta\}_{\beta \in J}$ is a (CA) system relative to g, then*

$$\{V_\alpha \otimes f^* W_\beta\}_{(\alpha,\beta) \in I \times J}, \text{ with } \sharp(\alpha,\beta) := \min(\sharp(\alpha), \sharp(\beta))$$

is a (CA) system relative to $g \circ f$.

Proof. According to Proposition 5.9, verifying (CA) is equivalent to considering, for each $F \in \mathrm{APerf}(\mathscr{X})_{\geq 0}$ with $H_0 F \neq 0$, the pushforward

$$E_{\alpha,\beta} = (g \circ f)_*(F \otimes V_\alpha^* \otimes f^* W_\beta^*) \simeq g_*(f_*(F \otimes V_\alpha^*) \otimes W_\beta^*)$$

First we must show that there is an N such that $\sharp(\alpha,\beta) \geq N$ implies $E_{\alpha,\beta} \in QC(\mathscr{Z})^{cn}$. We can choose an N such that $\sharp(\alpha) \geq N$ implies that $f_*(F \otimes V_\alpha^*)$ is connective. Furthermore it lies in $\mathrm{APerf}(\mathscr{Y})_{\geq 0}$ because f satisfies (CP). Thus because $\{W_\beta\}$ is (CA) for g, we can increase our choice of N such that $\sharp(\beta) \geq N$ implies that $E_{\alpha,\beta}$ is connective. Clearly both inequalities hold if $\sharp(\alpha,\beta) = \min(\sharp(\alpha), \sharp(\beta)) \geq N$.

To complete the verification that $\{V_\alpha \otimes f^* V_\beta\}$ is (CA), one must show that for any N there is some $\sharp(\alpha,\beta) \geq N$ with $H_0(E_{\alpha,\beta}) \neq 0$. The argument is the same as that of the previous paragraph. □

Proposition 5.17. *Let $\mathscr{X}$ be a locally Noetherian global quotient stack which admits a good moduli space X. Let $\pi : \mathscr{X} \to S$ be a morphism to a Noetherian scheme such that the corresponding morphism $X \to S$ is projective. Then π is cohomologically projective.*

Proof. Let $\phi : \mathscr{X} \to X$ be a good moduli space morphism with X projective over S. Property (CD) is immediate as the composition of two (CD) morphisms is (CD), and property (CP) follows from the fact that $\phi_* : QC(\mathscr{X}) \to QC(X)$ preserves coherence [?Alper, Theorem 4.16]. Thus we focus on property (CA).

Define $\pi' : X \to S$, and let L be a relatively ample invertible sheaf for π'. Then $\{L^{-n} | n \geq 0\}$ is a (CA) system for π'. Furthermore as $\mathscr{X}$ is a quotient stack, we can choose a set of vector bundles $\{W_i\}_{i \in J}$ which generates $QC(\mathscr{X})$. If we define $I = J \times \mathbb{Z}_{\geq 0}$ and let $V_{i,n} = W_i$, it is immediate that $\{V_{i,n}\}_I$ is a (CA) system for ϕ. Thus by Lemma 5.16 the composition $\pi = \pi' \circ \phi$ admits a (CA) system.

□

6. Establishing Grothendieck Existence and (LL)

The goal of this section will be to reproduce, essentially verbatim, two standard proofs in our context. The first of these is the proof of the Theorem on Formal Functions – which is usually carried out with some properness assumptions, but which we note only depends on certain complexes being almost perfect. The second of these is the proof of Grothendieck existence in the projective case – which we note can be carried out using our property (CA) in place of the existence of an ordinary ample line.

6.1. Theorem on formal functions.

The Theorem on Formal Functions is a strong base-change result in the context of completions. In the derived context, there is an easy intermediate step: Base change results hold to each piece of the completion

Lemma 6.1. *Suppose that $i\colon \operatorname{Spec} R' \to \operatorname{Spec} R$ is a finite, finite Tor amplitude, map of affine schemes, and that $\pi\colon \mathscr{X} \to \operatorname{Spec} R$ is a qc.qs. derived R-stack. Let $\mathscr{X}' = \mathscr{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R'$, $i'\colon X' \to X$ the base-change of i, and $\pi'\colon \mathscr{X}' \to \operatorname{Spec} R'$ the base-change of π. Then, the (derived) base-change map*

$$i^* \pi_* \to (\pi')_* (i')^*$$

is an equivalence.

Proof. Let $p_\bullet\colon U_\bullet = \operatorname{Spec} A_\bullet \to \mathscr{X}$ be a presentation of $\mathscr{X}$ as the geometric realization, in smooth sheaves, of a simplicial diagram of affine schemes along smooth morphisms – such a diagram exists because $\mathscr{X}$ is ∞-quasi-compact. By smooth hyper-descent for QC, we thus have an identification

$$(p_\bullet)^*\colon QC(\mathscr{X}) \simeq \operatorname{Tot}\{QC(\mathscr{X}_\bullet)\}$$

under which π^* identifies with the cosimplicial diagram of pullbacks $(p_\bullet \circ \pi)^*$. Thus, we can compute the right adjoint π_* in terms of this Cech diagram – namely

$$\pi_*(\mathscr{F}) = \operatorname{Tot}\{(p_\bullet \circ \pi)_* p_\bullet^* \mathscr{F}\}$$

Since i is finite and has finite Tor amplitude, we conclude that R' is perfect as R-module (because it is almost perfect of finite Tor amplitude). Thus, i^* commutes with arbitrary homotopy limits (in addition to homotopy colimits). This allows us to reduce to the affine case. □

Combined with Proposition C.3, this allows us to easily prove a derived form of the Theorem on Formal functions Theorem 6.5. We will take a slightly longer path to the Theorem in order to collect some convenient intermediate results.

Lemma 6.2. *Suppose R is a derived ring, $I \subset \pi_0(R)$ finitely generated ideal, $\widehat{R}$ the I-adic completion of R, and $i\colon \operatorname{Spf} R \to \operatorname{Spec} R$ the inclusion. Then, the composite functor*

$$i_* i^*\colon QC(\operatorname{Spec} R) \longrightarrow QC(\operatorname{Spec} R)$$

has finite left t-amplitude (i.e., is left t-exact up to a shift).

Proof. Let $\{R_n\}$ be a tower of perfect R-algebras having Tor amplitude at most d as in Proposition C.3. Then, we may identify $QC(\operatorname{Spec} R) \simeq R\text{-mod}$ and $QC(\operatorname{Spf} R) \simeq \varprojlim_n R_n\text{-mod}$. Under these identifications, the functor $i_* i^*$ is identified with

$$M \mapsto i_* i^*(M) = \varprojlim_n (R_n \otimes_R M)\,.$$

If $M \in R\text{-mod}_{<0}$ then $R_n \otimes_R M \in R\text{-mod}_{<d}$ by hypothesis on R_n. Thus $i_* i^* M \in R\text{-mod}_{<(d+1)}$ since $\varprojlim$ has left Tor amplitude at most 1 (i.e., there is a $\lim^1$ but no more). □

Lemma 6.3. *Suppose R is a derived ring, $I \subset \pi_0(R)$ is a finitely generated ideal, and $\widehat{R}$ the derived I-adic completion of R. Let $i\colon \operatorname{Spf} \widehat{R} \to \operatorname{Spec} R$ be the natural inclusion. Then, the composite*

$$(R\text{-mod})^{I-cplt} \subset R\text{-mod} \simeq QC(\operatorname{Spec} R) \xrightarrow{i^*} QC(\operatorname{Spf} R)$$

is an equivalence. In particular, the co-unit $i^* i_* \to \mathrm{id}$ is an equivalence and the unit $\mathrm{id} \to i_* i^*$ can be identified with the I-adic completion.

Proof. Note that this is proved, under the additional hypothesis that we restrict to *connective* objects on both sides, in [?DAG-XII]Lemma 5.1.10. We will prove this stronger result by leveraging our stronger assupmtions in the form of Lemma 6.2.

We note that i^* has a right adjoint i_*. If we identify $QC(\operatorname{Spec} R) = R\text{-mod}$ and $QC(\operatorname{Spf} R) = \varprojlim_n R_n\text{-mod}$, then i_* is given by the inverse limit in R-modules

$$i_*(\{M_n\}) = \varprojlim_n M_n \in R\text{-mod}.$$

It is enough to prove two assertions:

(1) If $M \in R$-mod, then the natural map

$$M \to i_* i^* M$$

identifies $i_* i^* M$ with the I-adic completion of M. Notice that if M is almost connective, this follows by [?DAG-XII]Remark 5.1.11. Since R-mod is right t-complete we may write $M = \varinjlim \tau_{\leq -k} M$, so that it is enough to show that each of $i_* i^*$ and the I-adic completion functor $M \mapsto \widehat{M}$ both preserve this directed limit. By Lemma C.9, it is enough to show that both $i_* i^*$ and the I-adic completion functor have bounded left t-amplitude. For $i_* i^*$ this was Lemma 6.1, while for the I-adic completion this is [?DAG-XII]Remark 5.11. This shows that $M \to i_* i^* M$ is the I-adic completion for all $M \in R$-mod.

(2) Let $\{R_n\}$ be as in Lemma 6.2. We must show that if $\{M_n\} \in \varprojlim_n R_n\text{-mod}$ then the unit

$$i^* i_* \{M_n\} \longrightarrow \{M_n\}$$

is an equivalence. More concretely, we must show that for each k the natural map

$$R_k \otimes_R \varprojlim_n M_n \longrightarrow M_k$$

is an equivalence. Since R_k is perfect over R, $R_k \otimes_R$ commutes with limits so that we may identify this with the map

$$\varprojlim_n (R_k \otimes_R M_n) \simeq \varprojlim_n ((R_k \otimes_R R_n) \otimes_{R_n} M_n) \longrightarrow M_k.$$

Recall that $\operatorname{Spf} \widehat{R} \simeq \varinjlim_n \operatorname{Spec} R_n$ and that fiber products preserve filtered colimits of (pre-)sheaves, so that we obtain

$$\operatorname{Spec} R_k \times_{\operatorname{Spec} R} \operatorname{Spf} \widehat{R} \simeq \varinjlim_n \operatorname{Spec}(R_k \otimes_R R_n)$$

Furthermore, $\operatorname{Spec} R_k \times_{\operatorname{Spec} R} \operatorname{Spf} \widehat{R} \simeq \operatorname{Spec} R_k$ since $\operatorname{Spec} R_k \to \operatorname{Spec} R$ factors through the monomorphism $\operatorname{Spf} R \to \operatorname{Spec} R$. Thus, the natural map

$$M_k \longrightarrow \varprojlim_n ((R_k \otimes_R R_n) \otimes_{R_n} M_n)$$

is an equivalence by (i). This completes the proof. ☐

Finally, we deduce

Theorem 6.4. *Suppose that $\mathscr{X}$ is a pre-stack and $Z \subset \mathscr{X}$ is a co-compact closed subset. Let $i\colon \widehat{\mathscr{X}} \to \mathscr{X}$ be the inclusion of the completion along Z. Then,*

(1) i^* admits a left adjoint i_+;
(2) The composite

$$QC_Z(\mathscr{X}) \subset QC(\mathscr{X}) \xrightarrow{i^*} QC(\widehat{\mathscr{X}})$$

is an equivalence, with inverse given by i_+ (i.e., the essential image of i_+ is contained in $QC_Z(\mathscr{X})$);
(3) i_+ is of formation local on $\mathscr{X}$, and satisfies the projection formula (i.e., is a functor of $QC(\mathscr{X})$-module categories);

Proof. The proof is essentially the same as that of [?DAG-XII]Theorem 5.1.9 combined with the previous Lemma.

It is enough to prove the claims in case $\mathscr{X}$ is affine, for the claims are local by construction (i.e., because we asked that the formation of i_+ be local).

Suppose now that $\mathscr{X} = \operatorname{Spec} R$ and Z is cut out by a finitely generated ideal $I \subset \pi_0(R)$. Let $(R\text{-mod})^{I-cplt} \subset R\text{-mod}$ be the subcategory of I-complete modules, and $(R\text{-mod})^{I-nil} \subset R\text{-mod}$ the full subcategory of locally I-nilpotent modules. By [?DAG-XII]Prop. 4.2.5 these are equivalent via the completion functor and the "local cohomology" functor Γ_I. Notice that i^* vanishes on I-local modules (i.e., those supported away from Z), so that

$$i^*(M) \simeq i^*(\widehat{M}) \simeq i^*(\Gamma_I(M))$$

It thus follows from Lemma 6.3 and the above mentioned equivalence that the composite

$$(R\text{-mod})^{I-nil} \subset R\text{-mod} \xrightarrow{i^*} QC(\operatorname{Spf} R)$$

is also an equivalence. Let i_+ be the composite of an inverse to this functor with the inclusion $(R\text{-mod})^{I-nil} \subset R\text{-mod}$ – more explicitly, $i_+ = \Gamma_I \circ i_*$. Since the inclusion is left adjoint to $\Gamma_I\colon R\text{-mod} \to (R\text{-mod})^{I-nil}$ it follows that i_* is left adjoint to i^*.

It is easy to check that i_+, so defined, is local on R: given a Cartesian diagram

$$\begin{array}{ccc} \operatorname{Spf} R' & \xrightarrow{i'} & \operatorname{Spec} R' \\ \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spf} R & \xrightarrow[i]{} & \operatorname{Spec} R \end{array}$$

we wish to check that

$$\pi^* i_* \mathscr{F} \longrightarrow (i')^* \widehat{\pi}^* \mathscr{F}$$

is an equivalence for all $\mathscr{F} \in \operatorname{Spf} R$. But by the above we may suppose that $\mathscr{F} = i^* \mathscr{F}'$ for some $\mathscr{F} \in (R\text{-mod})^{I-nil}$, and then notice that $\pi^* \mathscr{F}' \in (R'\text{-mod})^{I'-nil}$. Then, result is then immediate from the equivalence applied upstairs and downstairs.

Finally, the projection formula assertion follows form [?DAG-XII]4.1.22, 4.2.6. ∎

As a result we may deduce the following derived form of the Theorem on Formal Functions. Note the lack of properness or finiteness assumptions. We now establish a version of the Theorem on Formal Functions:

Theorem 6.5 (Theorem on Formal Functions)**.** *Suppose given a Cartesian diagram*

$$\begin{array}{ccc} \widehat{\mathscr{X}} & \xrightarrow{i'} & \mathscr{X} \\ \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle \pi} \\ \widehat{S} & \xrightarrow[i]{} & S \end{array}$$

where S is a derived stack, $\pi\colon \mathscr{X} \to S$ a derived S-stack, and $\widehat{S}$ the completion of S along a co-compact closed subset Z. (It follows that $\widehat{\mathscr{X}}$ is the completion of $\mathscr{X}$ along $\pi^{-1}(Z)$.) Then, the base-change map

$$i^* \pi_* \longrightarrow \widehat{\pi}_* (i')^*$$

is an equivalence.

In case S is affine, we may furthermore take global sections and obtain that the map

$$R\Gamma\widehat{(\mathscr{X}, \mathscr{F})} \longrightarrow R\Gamma(\widehat{\mathscr{X}}, \widehat{\mathscr{F}})$$

is an equivalence of complete R-modules for every $\mathscr{F} \in QC(\mathscr{X})$.

Proof. It is enough to prove that the induced map of left adjoints

$$\pi^* i_+ \longrightarrow (i')_+ \widehat{\pi}^*$$

By Theorem 6.4 the assertion is local on S so that we may suppose it affine. Also, by Theorem 6.4 we know that i^* is essentially surjective so that it is enough to show that map

$$\pi^* i_+ i^* \longrightarrow (i')_+ \widehat{\pi}^* i^* \simeq (i')_+ (i')^* \pi^*$$

is an equivalence. By another application of Theorem 6.4 we see that this is true (since $i_+ i^* \simeq \mathrm{id}$ and similarly for i').

The affine statement follows by applying Lemma 6.3. ∎

The ordinary Theorem on Formal functions had extra finiteness and properness hypotheses, and had extra exactness conclusion. This came about because of the following exactness result, which is a slightly derived consequence of the Artin-Rees Lemma.

Lemma 6.6. *Suppose R is a Noetherian derived ring, $I \subset \pi_0(R)$ is a finitely generated ideal, and $\widehat{R}$ the derived I-adic completion of R. Then:*

(1) The natural restriction functor

$$\operatorname{APerf}(\widehat{R}) \longrightarrow \operatorname{APerf}(\operatorname{Spf} R)$$

is a t-exact equivalence of ∞-categories.

(2) *In particular, taking hearts, the natural restriction functor*

$$\mathrm{Coh}(\widehat{R}) \longrightarrow \mathrm{Coh}(\mathrm{Spf}\, R)$$

is an equivalence of abelian categories.

Proof. See [?DAG-XII, $\mathscr{S}$4]. □

6.2. **Fully faithfulness from (CP).** We have the following statements, which are derived (1,2) and underived (3) consequences of the Theorem on Formal Functions:

Proposition 6.7. *Suppose that $\mathscr{X}$ is a Noetherian qc.qs. S-stack, where $S = \mathrm{Spec}\, R$ for a Noetherian ring R complete with respect to an ideal $I \subset \pi_0(R)$. Suppose that $\mathscr{X}$ satisfies (CP)$_R$ and that $G \in \mathrm{APerf}(\mathscr{X})$. Then, the natural map*

$$\mathrm{RHom}_{\mathscr{X}}(F, G) \longrightarrow \mathrm{RHom}_{\widehat{\mathscr{X}}}(i^*F, i^*G)$$

is an equivalence for any $F \in QC(\mathscr{X})$.

Proof. Notice first that since $\mathrm{APerf}(\mathscr{X})$ is left t-complete, we have an equivalence $G = \varprojlim_n \tau_{\leq n} G$. Furthermore, we have seen that $\mathrm{APerf}(\widehat{\mathscr{X}})$ is left t-complete and that i^* is t-exact – so, i^* preserves this inverse limit. Thus, it suffices to prove the assertion under the additional hypothesis that $G \in \mathrm{DCoh}(\mathscr{X})$ is bounded. Suppose for concreteness, shifting as needed, that $G \in \mathrm{APerf}(\mathscr{X})^{cn}_{\leq k}$.

Pick a pro-system $\{R_n\}$ of perfect R-algebras as in Proposition C.3. Let $\mathcal{C} \subset QC(\mathscr{X})$ denote the full subcategory consisting of those F for which the map

$$\mathrm{Map}_{\mathscr{X}}(F, G) \longrightarrow \mathrm{Map}_{\widehat{\mathscr{X}}}(i^*F, i^*G) = \varprojlim_n \mathrm{Map}_{X_n}(i_n^*F, i_n^*G)$$

is an equivalence for our fixed G. Note that $\mathcal{C}$ is closed under extensions and arbitrary colimits. We wish to show that $\mathcal{C} = QC(\mathscr{X})$. Note first that $\mathcal{C} \supset QC(\mathscr{X})_{>k+d}$ since both sides are trivial in this case for t-structure reasons: Indeed, F (resp., i_n^*F) is k-connective (resp., $k+d$-connective), while i_n^*G is k-truncated (resp., $k+d$-truncated). Since $\mathcal{C}$ is closed under extensions, it suffices to show that $QC(\mathscr{X})_{<(k+d)} \subset \mathcal{C}$. Since $\mathcal{C}$ is closed under arbitrary colimits, it is enough by Theorem B.11 to show that $\mathrm{APerf}(\mathscr{X})_{<(k+d)} \subset \mathcal{C}$.

In particular, we are reduced to proving the Proposition in case both F and G are almost perfect. Taking shifts, we may reduce to the case $G \in \mathrm{APerf}(\mathscr{X})_{\leq 0}$ and $F \in \mathrm{APerf}(\mathscr{X})_{\geq 0}$. Let

$$H = \mathrm{RHom}_{\mathscr{X}}^{\otimes_{QC(X)}}(F, G) \in QC(\mathscr{X})$$

be the inner Hom for the monoidal structure on $QC(\mathscr{X})$.

Assuming for now the following three claims about H, we will complete the proof below.

Claims:

(1) *$H \in QC(\mathscr{X})_{\leq 0}$ has coherent homology sheaves, and its formation is fppf local on $\mathscr{X}$;*
(2) *$\mathrm{RHom}_{\mathscr{X}}(F, G) \simeq R\Gamma(\mathscr{X}, H)$ for each n;*
(3) *$\mathrm{RHom}_{\widehat{\mathscr{X}}}(i^*F, i^*G) \simeq R\Gamma(\widehat{\mathscr{X}}, i^*H)$.*

Using the claim, it is enough to show that the map

$$R\Gamma(\mathscr{X}, H) \longrightarrow R\Gamma(\widehat{\mathscr{X}}, i^*H)$$

is an equivalence. By Theorem 6.5, it is enough to show that the left hand side is already I-complete. Every module with coherent homologies is I-complete, so this follows from the first Claim and our assumption that $\mathscr{X}$ has (CP)$_R$.

Proof of claims:
For (i), first note that [?tsd-mf, A.1.1] guarantees that the formation of H is fppf local since F is almost perfect and G is bounded above. As explained in the proof there, this does not require condition $(*)$ from op.cit., since the proof reduces to the affine case. Now the claim that H is co-connective with coherent homology sheaves is flat local, so in proving it we may suppose that $\mathscr{X} = \mathrm{Spec}\, A$ is affine with A Noetherian. In this case, the proof in op.cit. shows that there is a third quadrant spectral sequence converging to the homology groups of H whose starting term consists of finite sums of the homology of G. This shows both that the homology of H is appropriately bounded above and that each homology sheaf is coherent.

For (ii), note that this follows formally from the definition of inner Hom and the (π^*, π_*) adjunction:

$$\pi_* H \simeq \mathrm{RHom}_S(\mathcal{O}_S, \pi_* H) \simeq \mathrm{RHom}_{\mathscr{X}}(\mathcal{O}_{\mathscr{X}}, H) \simeq \mathrm{RHom}_{\mathscr{X}}(\mathcal{O}_X \otimes F, G)$$

For (iii), note that this follows formally from the existence of a left adjoint i_+ satisfiying the projeciton formula:

$$\begin{aligned} R\Gamma(\widehat{\mathscr{X}}, i^* H) &= \mathrm{RHom}_{\widehat{\mathscr{X}}}(\mathcal{O}_{\widehat{\mathscr{X}}}, i^* H) = \mathrm{RHom}_X(i_+ \mathcal{O}_{\widehat{X}}, H) \\ &= \mathrm{RHom}_X(i_+ \mathcal{O}_{\widehat{\mathscr{X}}} \otimes F, G) = \mathrm{RHom}_X(i_+(\mathcal{O}_{\widehat{\mathscr{X}}} \otimes i^* F), G) \\ &= \mathrm{RHom}_X(i^* F, i^* G) \end{aligned}$$

□

We can now prove the fully-faithfullness part of Grothendieck existence for Noetherian stacks satisfying $(\mathrm{CP})_R$:

Proposition 6.8. *Suppose that $\mathscr{X}$ is a Noetherian qc.qs. S-stack, where $S = \operatorname{Spec} R$ for a Noetherian ring R complete with respect to an ideal $I \subset \pi_0(R)$. Suppose furthermore that $\mathscr{X}$ satisfies $(CP)_R$.*
Then,

(1) The natural functor of stable ∞-categories with t-structures

$$\mathrm{APerf}(\mathscr{X}) \longrightarrow \mathrm{APerf}(\widehat{\mathscr{X}})$$

is fully-faithful.

(2) For each $m \geq 0$, the functor on $(m+1, 1)$-categories

$$\mathrm{Coh}^m(\mathscr{X}) \longrightarrow \mathrm{Coh}^m(\widehat{\mathscr{X}})$$

is fully-faithful.

Proof. Note that (2) follows from (1), since the functor is t-exact. Note that (1) follows from Proposition 6.7. □

6.3. Essential surjectivity from (CA) and (CP).

Lemma 6.9. *Suppose that $\mathscr{X}$ is quasi-compact, and that the conclusion of Proposition 6.8 holds.*
Then, (GE) is equivalent to requiring that $\mathrm{Coh}(\mathscr{X}) \to \mathrm{Coh}(\widehat{\mathscr{X}})$ be essentially surjective.

Proof. One implication is clear, so we suppose that $\mathscr{X}$ is quasi-compact, satisfies the assumptions of Proposition 6.8, and that $\mathrm{Coh}(\mathscr{X}) \to \mathrm{Coh}(\widehat{\mathscr{X}})$ is essentially surjective and we will prove that it satisfies (GE). Suppose $\widehat{\mathscr{F}} \in \mathrm{APerf}(\widehat{\mathscr{X}})$, we must show that $\widehat{\mathscr{F}}$ lies in the essential image of i^*. Since $\mathrm{APerf}(\widehat{\mathscr{X}})$ is left t-complete, we have

$$\widehat{\mathscr{F}} \simeq \varprojlim_n \tau_{\leq n} \widehat{\mathscr{F}}$$

Note first that it is enough to show that each $\tau_{\leq n}\widehat{\mathscr{F}}$ is in the essential image of i^*: Indeed, suppose that each $\tau_{\leq n}\widehat{\mathscr{F}} \simeq i^* \mathscr{F}_n$ for some $\mathscr{F}_n \in \mathrm{APerf}(\mathscr{X})$. The t-exactness and conservativity of i^* guarantees that $\mathscr{F}_n$ is n-truncated; fully-faithfullness of i^* allows us to lift the morphisms in the tower to the $\mathscr{F}_n$; and left t-completness of $\mathrm{APerf}(\mathscr{X})$ guarantees that $\varprojlim_n \mathscr{F}_n$ exists in $\mathrm{APerf}(\mathscr{X})$. Then, Lemma C.9 guarantees that this inverse limit is preserved by i^*, so that $i^* \varprojlim_n \mathscr{F}_n \simeq \widehat{\mathscr{F}}$.

Since $\mathscr{X}$ is assumed quasi-compact, we have that $\mathrm{APerf}(\widehat{\mathscr{X}})$ is left t-bounded. Thus, we are reduced to the case where $\widehat{\mathscr{F}}$ is t-bounded. Since i^* is fully faithful, $\mathrm{APerf}(\mathscr{X})$ has all finite limits and colimits, and i^* preserves these – an induction on the degrees in which $\widehat{\mathscr{F}}$ has non-trivial homology reduces us to the case where $\widehat{\mathscr{F}}$ lies in the heart. But this is exactly what we assumed, thereby completing the proof. □

Finally, we have the following Theorem. The proof is just the usual proof of Grothendieck existence for projective morphisms.

Theorem 6.10. *Suppose that $\mathscr{X}$ is a Noetherian qc.qs. S-stack, where $S = \operatorname{Spec} R$ for a Noetherian ring R complete with respect to an ideal $I \subset \pi_0(R)$. Suppose furthermore that $\mathscr{X}$ satisfies (CP)$_R$ and admits a (CA) system.*

Then, $\mathscr{X}$ satisfies (GE). That is,

(1) The natural functor of stable ∞-categories with t-structures

$$\operatorname{APerf}(\mathscr{X}) \longrightarrow \varprojlim_n \operatorname{APerf}(\mathscr{X}_n)$$

is an equivalence.

(2) For each $m \geq 0$, the induced functor of $(m+1,1)$-categories

$$\operatorname{Coh}^m(\mathscr{X}) \longrightarrow \varprojlim_n \operatorname{Coh}^m(\mathscr{X}_n)$$

is an equivalence.

Proof. In light of previous the Lemma, it remains to show that the functor in (2) is essentially surjective. This depends only on the underlying classical scheme of R and the underlying classical stack of $\mathscr{X}$: Since the properties (CP)$_R$, (CA), and (CD) pass to underlying classical substacks, we may assume without loss of generality that S and $\mathscr{X}$ are classical.

Let $\mathscr{I}^n = I^n \cdot \mathscr{O}_{\mathscr{X}} \subset \mathcal{O}_{\mathscr{X}}$ be the induced ideal sheaf for all n. We may identify the reduction functor with

$$\operatorname{Coh}(\mathscr{X}) \ni G \mapsto \{G/\mathscr{I}^n\}_n \in \varprojlim_n \operatorname{Coh}(\mathscr{X}_n)$$

Claim: We claim that every object

$$\{F_n\}_n \in \varprojlim_n \operatorname{Coh}(\mathscr{X}_n)$$

admits an epimorphism from an object in the essential image of the reduction functor.

Assuming the claim, we complete the proof: Recall that we have already established that both categories are abelian and the functor is exact. Picking a surjection onto $\{F_n\}$ and then a surjection onto its kernel, we can write $\{F_n\}$ as a cokernel

$$\{F_n\}_n \simeq \operatorname{cok}\left\{\{G/\mathscr{I}^n\} \xrightarrow{\phi_n} \{G'/\mathscr{I}^n\}\right\}$$

By fully-faithfullness, ϕ_n is obtained by reduction from a map $\phi\colon G \to G'$, and by exactness we are done

$$\{F_n\} \simeq \operatorname{cok}\{\phi_n\} \simeq \{(\operatorname{cok}\phi)/\mathscr{I}^n\}$$

Proof of claim: Suppose $\{F_n\}_n \in \varprojlim_n \operatorname{Coh}(\mathscr{X}_n)$. We may identify the inverse system with the following data:

(1) A family of coherent sheaves F_n on $\mathscr{X}$ such that $\mathscr{I}^n$ annihilates F_n;
(2) A surjective map $\phi_n\colon F_n \to F_{n-1}$ whose kernel is $\mathscr{I}^{n-1} \cdot F_n$.

Introducing the associated gradeds:
Set

$$R_{gr} = \bigoplus_n I^n/I^{n+1} \qquad \text{and set} \quad S_{gr} = \operatorname{Spec} R_{gr}$$

and similarly

$$\mathscr{X}_{gr} = (\mathscr{X} \times_S S_{gr})_{cl} = \operatorname{Spec}_{\mathscr{X}} \mathscr{I}^n/\mathscr{I}^{n+1}$$

We will be interested in

$$F_{gr} = \bigoplus_n (\ker\phi_n) = \bigoplus_n (\mathscr{I}^{n-1} \cdot F_n)$$

as a graded quasi-coherent sheaf on $\mathscr{X}_{gr}$

Finiteness for the associated graded:
We claim that F_{gr} is in fact coherent over $\mathscr{X}_{gr}$. The assertion is local on $\mathscr{X}_{gr}$ and hence on $\mathscr{X}$, so that we may assume that $\mathscr{X} = \operatorname{Spec} A$ is affine. Locally $\{F_n\}$ is an I-adic-system of A-modules, so that

$$\varprojlim_n F_n \in \operatorname{Coh}(\widehat{A_I})$$

and F_{gr} is the associated graded for its I-adic filtration. A finite set of generators for this module provides a finite set of generators for this associated graded, proving that F_{gr} is coherent.

Using (CA) to find a (liftable) surjection from the associated graded:
Let $\{V_\alpha\}$ be a (CA) system of vector bundles relative to S. By the definition of (CA), we find that their restrictions $V_\alpha|_{\mathscr{X}_{gr}}$ also satisfy the conditions (CA1) and (CA2) for $\mathscr{X}_{gr}$ over S_{gr}.

Note that (CA2) allows us to "erase Ext-s": More precisely, there exists

$$\mathrm{Ext}^1_{\mathscr{X}_{gr}}(V_\alpha|_{\mathscr{X}_{gr}}, F_{gr}) = \mathrm{Ext}^1_{\mathscr{X}}(V_\alpha, \oplus_n(\ker\phi_n)) = 0$$

for all α with $\#(\alpha) \geq N$. Applying (CA1), we may pick a surjection onto F_0 from a sheaf G which is a finite sum of sheaves of the form V_α with $\#(\alpha) \geq N$. So, G is a coherent sheaf on $\mathscr{X}$ which surjects onto F and which satisfies

$$\mathrm{Ext}^1_{\mathscr{X}}(G, \ker\phi_n) = 0$$

for all n.

Considering the short exact sequences

$$0 \to \ker\phi_n \to F_n \to F_{n-1} \to 0$$

the vanishing of the Ext^1-s allows us to lift r_0: that is, we may construct a compatible family of maps

$$r_i \colon G \to F_i \in \mathrm{Coh}(\mathscr{X}), \qquad i \geq 0$$

Indeed, the obstruction encountered at each stage lives in $\mathrm{Ext}^1_{\mathscr{X}}(G, \ker\phi_n) = 0$. By Nakayama's Lemma, each r_i is surjective. So,

$$\{G/\mathscr{I}^n\} \to \{F_n\}$$

is the desired epimorphism. □

6.4. **Methods of establishing (L).** Our methods for establishing (L) come in two flavors. The first applies to perfect morphisms of finite cohomological dimension and finite Tor-amplitude, and the second approach uses a descent result.

Proposition 6.11. *Let $\mathscr{X}$ and $\mathscr{S}$ be geometric stacks, and let $f : \mathscr{X} \to \mathscr{S}$ be a perfect morphism of finite Tor-dimension and satisfying (CD). Then the following are equivalent*

(1) f satisfies (CP),
(2) for any Noetherian affine scheme T over $\mathscr{S}$, $f'_* : QC(\mathscr{X} \times_{\mathscr{S}} T) \to QC(T)$ preserves perfect objects, and
(3) f satisfies (L).

Proof. First note that because (1) and (3) are properties defined after base change to a Noetherian affine scheme, it suffices to prove the equivalence for the non-universal versions of these properties, $(\mathrm{CP})_S$ and $(\mathrm{L})_S$, in the case where $\mathscr{S} = S$ is a Noetherian affine scheme, and $\mathscr{X}$ is perfect.

Proving (1) ⇒ (2):
We use the fact that $M \in \mathrm{APerf}(S)$ is perfect if and only if it has finite Tor-amplitude [?HigherAlgebra, Proposition 8.2.5.23]. Let $F \in \mathrm{Perf}(\mathscr{X})$ and let $G \in QC(S)^\heartsuit$. Because F is perfect and f has finite Tor-dimension, there is some n, independent of G, such that $F \otimes f^*G \in QC(\mathscr{X})_{<n}$. The projection formula, Corollary B.16, then implies that

$$f_*(F) \otimes G \simeq f_*(F \otimes f^*G) \in QC(S)_{<n},$$

and hence $f_*(F) \in \mathrm{APerf}(S)$ has Tor-amplitude $< n$.

Proving (2) ⇒ (1):
Let $F \in \mathrm{DCoh}(\mathscr{X})^\heartsuit$ and fix d larger than the universal cohomological dimension of f. Because $\mathscr{X}$ is perfect, Corollary B.12 implies that there is a homomorphism from a perfect object $P \to F$ such that $\tau_{\leq d}P \to F$ is a retract. It follows that

$$\tau_{\leq 0}f_*(P) \simeq \tau_{\leq 0}f_*(\tau_{\leq d}P) \to \tau_{\leq 0}f_*(F) \simeq f_*(F)$$

is a retract, where the first equivalence follows from (CD), and the second because f_* is left t-exact. Thus $f_*F \in \mathrm{DCoh}(S)$.

Proving (2) ⇒ (3):
If M and N are perfect objects, then

$$\begin{aligned} Hom_S((f_*(M^\vee))^\vee, N) &\simeq Hom_S(N^\vee, f_*(M^\vee)) \\ &\simeq Hom_{\mathscr{X}}(f^*(N^\vee), M^\vee) \\ &\simeq Hom_{\mathscr{X}}(M, (f^*(N^\vee))^\vee) \end{aligned}$$

Because f^* is a tensor functor, $(f^*(N^\vee))^\vee \simeq f^*N$. It follows from this and the fact that every $N \in QC(S)$ can be written as a colimit of perfect objects that $f_+(M) \simeq (f_*(M^\vee))^\vee$ exists when M is perfect. By hypothesis every $M \in QC(\mathscr{X})$ is a filtered colimit of perfect objects M_i, so $f_+(M) = \varinjlim f_+(M_i)$ also exists.

Proving (3) ⇒ (2):
Assume that the left adjoint $f_+ : QC(\mathscr{X}) \to QC(S)$ for f^* exists. Because f^* is continuous, f_+ must preserve compact objects for purely formal reasons, and the compact objects are precisely the perfect objects by hypothesis. A calculation similar to the one above shows that $(f_+(M^\vee))^\vee \simeq f_*(M)$ for $M \in \mathrm{Perf}(\mathscr{X})$. Hence f_* preserves perfect objects. □

We also note some consequences of a morphism $f : \mathscr{X} \to S$ being flat and satisfying Property (L) in terms of the abelian categories $QC(\mathscr{S})^\heartsuit$ and $QC(\mathscr{X})^\heartsuit$.

Lemma 6.12. *Let $f : \mathscr{X} \to \mathscr{S}$ be a flat morphism of algebraic stacks such that $f^* : QC(\mathscr{S}) \to QC(\mathscr{X})$ admits a left adjoint f_+. Then f_+ is right t-exact, and $H_0 \circ f_+$ is a left adjoint for the functor $f^* = H_0 \circ f^* : QC(\mathscr{S})^\heartsuit \to QC(\mathscr{X})^\heartsuit$.*

Proof. First we note that f_+ must be right t-exact. If $M \in QC(\mathscr{X})_{\geq 0}$, then for all $N \in QC(\mathscr{S})_{<0}$, $\mathrm{Hom}_{\mathscr{S}}(f_+(M), N) \simeq \mathrm{Hom}_{\mathscr{S}}(M, f^*N) = 0$ because f^* is t-exact.

Now for $M \in QC(\mathscr{X})^\heartsuit$ and $N \in QC(\mathscr{S})^\heartsuit$, we have $\mathrm{Hom}_{\mathscr{S}}(f_+(M), N) \simeq \mathrm{Hom}_{\mathscr{S}}(H_0(f_+(M)), N)$ because f_+ is right t-exact, so $H_0 \circ f_+$ is left adjoint to $H_0 \circ f^*$, because $f^* \simeq H_0 \circ f^*$ on $QC(\mathscr{S})$. □

In order to establish (L) for non-perfect morphisms, we now prove that descent (L) satisfies certain descent properties.

Lemma 6.13. *Let $\pi : \mathscr{X} \to \mathscr{S}$ be a flat morphism of algebraic stacks. Then the following are equivalent*

(1) $\pi^* : QC(\mathscr{S}) \to QC(\mathscr{X})$ admits a left adjoint, and
(2) $\pi^* : QC(\mathscr{S})_{\leq 0} \to QC(\mathscr{X})_{\leq 0}$ admits a left adjoint.

Furthermore if $\mathscr{X}$ is Noetherian and qc.qs., we have

(3) $\pi^* : \mathrm{APerf}(\mathscr{S}) \to \mathrm{APerf}(\mathscr{X})$ admits a left adjoint, and
(4) $\pi^* : \mathrm{APerf}(\mathscr{S})_{\leq 0} \to \mathrm{APerf}(\mathscr{X})_{\leq 0}$ admits a left adjoint.

In this case the left adjoint on APerf *is just the restriction of the left adjoint on QC.*

Proof. First we prove the equivalence between (1) and (2):

Assume that $\pi^* : QC(\mathscr{S}) \to QC(\mathscr{X})$ admits a left adjoint, then $\tau_{\leq 0} \circ \pi_+$ is a left adjoint for $QC(\mathscr{S})_{\leq 0} \to QC(\mathscr{X})_{\leq 0}$. By applying the shift functor we have adjoints $\pi_+^{\leq n} : QC(\mathscr{S})_{\leq n} \to QC(\mathscr{X})_{\leq n}$. For $F \in QC(\mathscr{X})$, an examination of the functor correpresented by $\pi_+^{\leq n}(\tau_{\leq n} F)$ shows that $\tau_{\leq n} \pi_+^{\leq n+1}(\tau_{\leq n+1} F) \simeq \pi_+^{\leq n+1}(\tau_{\leq n} F)$. Therefore we can define

$$\pi_+(F) = \varprojlim \pi_+^{\leq n}(\tau_{\leq n} F).$$

We compute

$$\begin{aligned} \mathrm{RHom}_R(\pi_+(F), G) &= \varprojlim_n \mathrm{RHom}_{\mathscr{S}}(\pi_+(F), \tau_{\leq n} G) \\ &= \varprojlim_n \mathrm{RHom}_{\mathscr{S}}(\pi_+^{\leq n}(F), \tau_{\leq n} G) \\ &= \varprojlim \mathrm{RHom}_{\mathscr{X}}(F, \tau_{\leq n} \pi^* G) \simeq \mathrm{RHom}_{\mathscr{X}}(F, G) \end{aligned}$$

Thus π_+ is a left adjoint for π^*.

The statements for APerf:

The proof that (3) and (4) are equivalent is identical to the previous argument. Showing that (4) is equivalent to (2) formal. For any qc.qs. Noetherian stack $\mathscr{X}$, we have $QC(\mathscr{X})_{\leq 0} = \operatorname{Ind}\operatorname{APerf}(\mathscr{X})_{\leq 0}$ by Theorem B.11. The functor $\pi^* : QC(\mathscr{S})_{\leq 0} \to QC(\mathscr{X})_{\leq 0}$ is continuous and restricts to a functor $\operatorname{APerf}(\mathscr{S})_{\leq 0} \to \operatorname{APerf}(\mathscr{X})_{\leq 0}$. It follows that π^* on $QC_{\leq 0}$ admits a left adjoint if and only if π^* on $\operatorname{APerf}_{\leq 0}$ admits a left adjoint. □

Lemma 6.14. *Let $\phi : \mathscr{S}' \to \mathscr{S}$ be an affine fppf morphism and let $\pi : \mathscr{X} \to \mathscr{S}$ be a flat morphism. Consider the base change $\pi' : \mathscr{X}' := \mathscr{X} \times_{\mathscr{S}} \mathscr{S}' \to \mathscr{S}'$. Then π^* admits a left adjoint if and only if $(\pi')^*$ admits a left adjoint.*

Proof. By Lemma 6.13, it suffices to prove the claim for the categories $QC(\bullet)_{\leq 0}$. Assume that $\pi^* : QC(\mathscr{S})_{\leq 0} \to QC(\mathscr{X})_{\leq 0}$ admits a left adjoint π_+. Then for any $F \in \operatorname{DCoh}(\mathscr{X})_{\leq 0}$ we have that

$$\begin{aligned} \operatorname{RHom}_{\mathscr{S}'}(\pi_+(F)|_{\mathscr{S}'}, G) &\simeq \operatorname{RHom}_{\mathscr{X}}(F, \pi^*\phi_* G) \\ &= \operatorname{RHom}_{\mathscr{X}}(F, (\phi')_*(\pi')^* G) \end{aligned}$$

Hence $(\pi')_+(F|_{\mathscr{X}'}) \simeq \pi_+(F)|_{\mathscr{S}'}$. Furthermore, because $\mathscr{X}' \to \mathscr{X}$ is fppf and affine, the category $QC(\mathscr{X}')_{\leq 0}$ is compactly generated by objects of the form $F|_{\mathscr{X}'}$ with $F \in \operatorname{DCoh}(\mathscr{X})_{\leq 0}$ (see Theorem B.11). It follows that $(\pi')_+$ is defined on all of $QC(\mathscr{X})_{\leq 0}$.

Conversely, suppose that $(\pi')^*$ admits a left adjoint. Let $\mathscr{S}'_\bullet \to \mathscr{S}$ be the Cech nerve of ϕ, and let $\mathscr{X}'_\bullet \to \mathscr{X}$ be the Cech nerve of $\mathscr{X}' \to \mathscr{X}$ (it is a base change of $\mathscr{S}'_\bullet$). By the previous argument each $\mathscr{X}'_n \to \mathscr{S}'_n$ admits a functor $(\pi_n)_+$. By faithfully-flat descent we have $\phi_\bullet^* : QC(\mathscr{S}) \to \operatorname{Tot} QC(\mathscr{S}'_\bullet)$ is an equivalence, and likewise for $(\phi'_\bullet)^*$. Under these identifications it is straightforward to check that $(\pi_\bullet)_+ : \operatorname{Tot} QC(\mathscr{X}'_\bullet) \to QC(\mathscr{S}'_\bullet)$ is left adjoint to $(\pi_\bullet)^*$. □

Proposition 6.15. *Let $\mathscr{Y} \xrightarrow{f} \mathscr{X} \xrightarrow{g} \mathscr{S}$ be locally finitely presented morphisms of algebraic stacks where f is surjective and g is flat. If either of the following holds:*

- *f is flat, and f and $f \circ g$ satisfy (L), or*
- *All of the stacks are Noetherian qc.qs., f satisfies (GE), and every level of the Cech nerve $\mathscr{Y}_n := \mathscr{Y} \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \mathscr{Y}$ satisfies (L) over $\mathscr{S}$,*

then g satisfies (L).

Proof. Under the first set of hypotheses:

Let $f_n : \mathscr{Y}_n = \mathscr{Y} \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \mathscr{Y} \to \mathscr{X}$ be the n^{th} level of the Cech nerve of $f : \mathscr{Y} \to \mathscr{X}$. For any $F \in QC(\mathscr{X})$ we have a simplicial diagram in $QC(\mathscr{X})$ whose n^{th} level is $(f_n)_+ \circ f_n^* F$. Then

$$\begin{aligned} \operatorname{RHom}_{\mathscr{S}}(\operatorname{Tot}^{\oplus}\{(f_\bullet)_+ f_\bullet^* F\}, G) &\simeq \operatorname{Tot} \operatorname{RHom}_{\mathscr{X}_\bullet}(\{f_\bullet^* F\}, \{f_\bullet^* G\}) \\ &\simeq \operatorname{RHom}_{\mathscr{X}}(F, G) \end{aligned}$$

where $\operatorname{Tot}^{\oplus}$ denotes the colimit of the simplicial diagram, and the second equivalence follows from faithfully flat descent. Thus the canonical map $\operatorname{Tot}^{\oplus}\{(f_\bullet)_+ f_\bullet^* F\} \to F$ is an equivalence. One can now check that $\operatorname{Tot}^{\oplus}\{(g \circ f_\bullet)_+ f_\bullet^* F\}$ correpresents the functor $\operatorname{Hom}_{\mathscr{X}}(F, g^*(\bullet))$. Hence $g_+(F) = \operatorname{Tot}^{\oplus}\{(g \circ f_\bullet)_+ f_\bullet^* F\}$.

Under the second set of hypotheses:

We again form the Cech nerve. By hypothesis we have a level-wise left adjoint for $(g \circ f_\bullet)_+$. For any $F \in \operatorname{APerf}(\mathscr{X})$ and $G \in \operatorname{APerf}(\mathscr{S})$, we compute

$$\begin{aligned} \operatorname{RHom}_{\mathscr{S}}(\operatorname{Tot}^{\oplus}\{(g \circ f_\bullet)_+ f_\bullet^* F\}, G) &\simeq \operatorname{Tot} \operatorname{RHom}_{\mathscr{X}_\bullet}(\{f_\bullet^* F\}, \{f_\bullet^* g^* G\}) \\ &\simeq \operatorname{RHom}_{\mathscr{X}}(F, f^* G) \end{aligned}$$

Where the second equality follows from the APerf-descent theorem Theorem 4.12, which guarantees that $(f_\bullet)^*$ is fully faithful. We have therefore constructed a left adjoint for $f^* : \operatorname{APerf}(\mathscr{S}) \to \operatorname{APerf}(\mathscr{X})$, and by Lemma 6.13 this implies that $f^* : QC(\mathscr{S}) \to QC(\mathscr{X})$ admits a left adjoint. □

Appendix A. BG for reductive G in finite characteristic

In this section we demonstrate some methods for extending our results to positive characteristic, where many of the methods above fail because even quotient stacks can fail to be perfect and morphisms between quotient stacks can fail to satisfy (CD). Our main result is the following

Theorem A.1. *Let k be a field with $\operatorname{char} k > 0$, and let G be a smooth k-group whose connected component is reductive. Then the morphism $BG \to \operatorname{Spec} k$ satisfies (GE) and (L).*

Remark A.2. In characteristic 0, Theorem A.1 and the key intermediate result Proposition A.6 are almost covered by our previous results. The stack BG is cohomologically projective. Also, for a Borel subroup B in characteristic 0 one can show that (L) follows from Proposition 6.11 and the existence of a full exceptional collection (Corollary 3.17) in $\operatorname{Perf}(BB)$ implies (pGE).

First we consider a smooth affine k-group B along with a one-parameter subgroup $\lambda : \mathbb{G}_m \to B$ such that

(†) the centralizer of λ, L, is linearly reductive over k and the λ-weights in the adjoint representation $\mathfrak{b}$ are nonnegative.

We have a surjective group homomorphism $B \to L$ with kernel U, the subgroup attracted to the identity under conjugation by $\lambda(t)$ as $t \to 0$. Thus we have $B = U \rtimes L$. We will use BB_R to denote $\operatorname{Spec} R \times BB$.

Lemma A.3. *Let R be a Noetherian k-algebra, and let $F \in QC(BB_R)^\heartsuit$, then regarding F as an R-module graded with respect to λ, the submodule $F_{\geq w}$ spanned by elements with λ-weight $\geq w$ is naturally a B_R-equivariant submodule. If F is coherent as an R-module, then so is $F_{\geq w}$.*

Proof. Note that it suffices to consider discrete R. The projection $BB \to BL$ is a trivial BU gerbe, and we can identify it with the classifying stack for the smooth affine relative group scheme $U/L \to BL$, where L acts on U by conjugation. Writing $U = \operatorname{Spec} A$, A is an L-equivariant coalgebra, and we identify $QC(BB_R)^\heartsuit$ with the category of L-equivariant R-comodules over the R-coalgebra $R \otimes A$. Under the conjugation action of λ, A obtains a non-positive grading $A = \bigoplus_{w \leq 0} A_w$ with $A_0 = k$ and A_w finite dimensional for all w. Thus the image of the comultiplication map $F_{\geq w} \to A \otimes F$ must land in the R-submodule $A \otimes F_{\geq w}$. □

Fix a Noetherian k-algebra R, and consider the simplicial resolution $\mathscr{X}_\bullet$ of BB_R given by the Cech nerve of $BL_R \to BB_R$, so we have

$$\mathscr{X}_n \simeq \operatorname{Spec} R \times U^n / L.$$

By faithfully flat descent $F \simeq \operatorname{Tot}\{(p_\bullet)_* F_\bullet\}$ where $p_n : \mathscr{X}_n \to BB_R$ is the projection and $F_n \simeq p_n^* F$. It follows that $\pi_* F \simeq \operatorname{Tot}\{(\pi_\bullet)_* F_\bullet\}$ where $\pi_n : \mathscr{X}_n \to \operatorname{Spec} R$ is the projection.

Lemma A.4. *Let R be a Noetherian k-algebra, let $QC(BB_R)^{acn}_{\lambda<h} \subset QC(BB_R)$ be the full subcategory of almost connective objects such that $H_i(F)$ is supported in λ-weight $< h$ for all i. Then for any $F \in QC(BB_R)^{acn}_{\lambda<h}$ the canonical morphism*

$$\operatorname{Tot}((\pi_\bullet)_* F_\bullet) \to \operatorname{Tot}_{\leq m}((\pi_\bullet)_* F_\bullet) \tag{13}$$

is an equivalence for $m \geq 2h$, where the latter denotes the limit of the m^{th} coskeleton of the cosimplicial diagram computing $\pi_* F_\bullet$.

Proof. Note that both sides of Equation 13 commute with limits in F, so because $QC(BB_R)$ is left t-complete it suffices to consider $F \in QC(BB_R)^\heartsuit$. Such an object is pushed forward from $QC(BB_{\pi_0 R})^\heartsuit$, so it suffices to consider the case when R is discrete because the pushforward from $QC(\operatorname{Spec} \pi_0 R)$ to $QC(\operatorname{Spec} R)$ commutes with totalization.

Note that $(\pi_n)_* F_n \in QC(R)^\heartsuit$, because $\operatorname{Spec} R \times U^n$ is affine and L is linearly reductive. Thus we may use the Dold-Kan correspondence to write the totalization as a complex

$$\operatorname{Tot}\{(\pi_n)_* F_n\} \simeq M^0 \to M^1 \to \cdots$$

where

$$M^n = \operatorname{coker}\left(\bigoplus_{i=1}^{n} (\pi_{n-1})_* F_{n-1} \xrightarrow{\delta_i^n} (\pi_n)_* F_n \right)$$

and the differential $\delta^n : M^{n-1} \to M^n$ is induced by δ_0^n. Furthermore $\mathrm{Tot}_{\leq m}$ is isomorphic to the naive truncation of that complex.

The fact that $A_0 \simeq k$ and F_0 has highest λ-weight $< h$ implies that $(\pi_n)_* F_n \simeq (A^{\otimes n} \otimes F_0)^L$ is spanned by simple tensors $a_1 \otimes \cdots \otimes a_n \otimes f$ with $a_i = 1$ for all but at most $h-1$ factors. If n is sufficiently large, then there must be an $i > 0$ with $a_i = a_{i+1} = 1$. It follows from this and the fact that the boundary maps are induced by the comultiplication on A that this element is in the image of a boundary map. Hence $M^n = 0$. □

Corollary A.5. *Let R be a Noetherian k-algebra, the pushforward $\pi_* : QC(BB_R) \to QC(\mathrm{Spec}\, R)$ maps $\mathrm{DCoh}(BB_R)$ to $\mathrm{DCoh}(\mathrm{Spec}\, R)$ and $\mathrm{Perf}(BB_R)$ to $\mathrm{Perf}(R)$. In particular $BB \to \mathrm{Spec}\, k$ satisfies (CP).*

Proof. By the previous lemma, it suffices to show that every coherent (resp. perfect) $F \in QC(BB_R)$ is contained in $QC(BB_R)^{acn}_{\lambda<h}$ for some h and that $(\pi_n)_* : QC(\mathscr{X}_n) \to QC(R)$ preserves bounded coherent (resp. perfect) objects. The first claim is immediate, because as a $\mathbb{G}_m$-equivariant R-module F splits as a finite direct sum of λ-weight eigensheaves. The second claim follows from the fact that if F_0 is coherent (resp. perfect) as an R-module then

$$(\pi_n)_* F_n \simeq (A^{\otimes n} \otimes F_0)^L \simeq ((\bigoplus_{w \geq -h} A_w)^{\otimes n} \otimes F_0)^L$$

is a coherent (resp. perfect) R-module because L is linearly reductive.

□

Proposition A.6. *Let BB be a smooth affine k-group satisfying the property (†) above. Then $BB \to \mathrm{Spec}\, k$ satisfies (GE) and (L).*

Proof. Let R be a complete local Noetherian ring. The fully faithfulness part of $(\mathrm{GE})_R$ follows from Corollary A.5 and Proposition 6.7. Thus by Lemma 6.9 it suffices to show that $\mathrm{Coh}(BB_R) \to \varprojlim \mathrm{Coh}(BB_{R_n})$ is essentially surjective.

Note that the canonical filtration $\cdots \subset F_{\geq w+1} \subset F_{\geq w} \subset \cdots \subset F$ induced by Lemma A.3 is compatible with the restriction functor $H_0(i_n^*) : \mathrm{Coh}(\mathscr{X}_{n+1}) \to \mathrm{Coh}(\mathscr{X}_n)$ in the sense that

$$(H_0 \circ i_n^* F)_{\geq w} = H_0 \circ i_n^*(F_{\geq w})$$

because the inclusion $F_{\geq w} \subset F$ is a summand once we forget the B-action.

Let $\{F_n\} \in \mathrm{APerf}(\widehat{\mathscr{X}})^\heartsuit$ be an inverse system. Because the t-structure on $\mathrm{APerf}(\widehat{\mathscr{X}})$ is fppf-local on $\mathscr{X}$, as is $\mathrm{Coh}^0(\mathscr{X}_n)$, we see that the natural functor $\mathrm{APerf}(\widehat{\mathscr{X}})^\heartsuit \to \mathrm{Coh}^0(\widehat{\mathscr{X}})$ mapping $\{F_n\} \to \{H_0(F_n)\}$ is an equivalence. Thus from the inverse system of fppf-locally split exact sequences $0 \to H_0(F_n)_{\geq w} \to H_0(F_n) \to H_0(F_n)_{<w} \to 0$ we get an fppf-locally split short exact sequence in $\mathrm{APerf}(\mathscr{X})^\heartsuit$, which we denote $0 \to \{F'_n\} \to \{F_n\} \to \{F''_n\} \to 0$. Thus if $\{F'_n\}$ and $\{F''_n\}$ are in the image of $\mathrm{APerf}(\mathscr{X}) \to \mathrm{APerf}(\widehat{\mathscr{X}})$, it follows from the fully faithfulness of this functor that

$$\{F_n\} = \mathrm{Cofib}(\{F''_n[-1]\} \to \{F'_n\})$$

is as well.

It thus suffices to assume that $\{F_n\} \in \mathrm{APerf}(\widehat{\mathscr{X}})^\heartsuit$ is such that $H_0(F_n)$ is concentrated in a singe weight w. However, for any ring, pullback functor $\mathrm{Coh}(BL_R) \to \mathrm{Coh}(BB_R)$ induces an equivalence between the subcategories of objects concentrated in weight w. It follows that $\{F_n\}$ is canonically pulled back from an inverse system in $\mathrm{APerf}(\widehat{BL_R})$, and because $(\mathrm{GE})_R$ holds for BL_R, we can find an $F \in \mathrm{APerf}(BB_R)$ such that $\hat{i}^* F = \{F_n\}$.

Proof of (L):

Let R be a Noetherian ring. By Corollary A.5 the pushforward $(\pi_R)_* : QC(BB_R) \to QC(\mathrm{Spec}\, R)$ preserves perfect objects. Thus following the proof of Proposition 6.11, it follows that $(\pi_R)_+(F) = (\pi_R)_*(F^\vee)^\vee$ exists for all perfect M.

If $F \in \mathrm{Perf}(\mathscr{X})$ is perfect and $G \in QC(\mathrm{Spec}\, R)_{\leq n}$, then

$$\mathrm{RHom}_{\mathscr{X}}(\tau_{\leq n} F, \pi_R^* G) \simeq \mathrm{RHom}_{\mathscr{X}}(\tau_{\leq n}(\pi_R)_+(F), G)$$

Thus we have a partially defined left adjoint to $\pi^* : QC(R)_{\le n} \to QC(\mathscr{X})_{\le n}$ defined on the category of objects of the form $\tau_{\le n}F$ for perfect F. If $\mathscr{X}$ is perfectly generated, then these objects compactly generate $QC(\mathscr{X})_{\le n}$, hence we have a left adjoint, which we denote $(\pi_R)_+^{\le n} : QC(\mathscr{X})_{\le n} \to QC(R)_{\le n}$. Lemma 6.13 now shows that $(\pi_R)^*$ admits a left adjoint. □

Proof of Theorem A.1. First note that by Corollary 4.18, and Lemma 6.14 it suffices to prove the claim after an étale change of base field. Thus we may assume that the connected component of the identity, $G^\circ \subset G$ is split-reductive.

Let $B \subset G$ be a Borel subgroup. By the structure theory of split-reductive groups, we can find a one-parameter subgroup $\lambda : \mathbb{G}_m \to B$ satisfying the properties (†). $X = G/B$ is a smooth projective variety so that, $BB = X/G \to BG$ is a representable, smooth, projective morphism. Proposition 6.15 implies that in order to prove (L) for BG, it suffices to show that $BB \to \operatorname{Spec} k$ satisfies (L). Corollary 4.18 implies that in order to show (GE) for BG, it suffices to show (GE) for BB and to show that $BB \times_B GBB \simeq X/B$ satisfies (CP). The theorem now follows from Proposition A.6 and Corollary A.5. □

Appendix B. Recollections on quasi-coherent complexes on derived stacks

B.1. Useful subcategories of $QC(X)$.

Definition B.1. Suppose that $\mathscr{X} \in Fun(\mathrm{CAlg}, \mathcal{S})$ is an arbitrary pre-stack. Then, we can define:

- The symmetric monoidal ∞-category $QC(\mathscr{X})$ of *quasi-coherent complexes*:
$$QC(\mathscr{X}) = \varprojlim_{\eta \in \mathscr{X}(R)} R\text{-mod}$$
i.e., a quasi-coherent complex $F \in QC(\mathscr{X})$ is the coherent assignment to each pair of an $R \in \mathrm{CAlg}$ and an R-point $\eta\colon \operatorname{Spec} R \to \mathscr{X}$ of an R-module $F_\eta \in R$-mod. It carries a t-structure, having connective objects $QC(X)_{>0}$ consisting precisely of those F such that $F_\eta \in (R\text{-mod})_{>0}$ is connective for all pairs (R, η). To a morphism $f\colon \mathscr{X} \to \mathscr{Y}$ of pre-stacks, one has a pullback functor $f^*\colon QC(\mathscr{Y}) \to QC(\mathscr{X})$. For more details, consult [?DAG-VIII, 2.7].
- There is a symmetric monoidal ∞-category $\mathrm{Perf}(\mathscr{X}) \subset QC(\mathscr{X})$ of *perfect complexes*:
$$\mathrm{Perf}(\mathscr{X}) = \varprojlim_{\eta \in \mathscr{X}(R)} \mathrm{Perf}\, R$$
i.e., this is the full subcategory spanned by those F such that $F_\eta \in R$-mod is perfect for all pairs (R, η). Recall that an R-mod is perfect if it is in the smallest subcategory of R-mod closed under cones, shifts, and retracts. By [?DAG-VIII, 2.7.28], $\mathrm{Perf}(\mathscr{X}) \subset QC(\mathscr{X})$ consists precisely of the *dualizable objects* with respect to the symmetric monoidal structure on $QC(\mathscr{X})$.
- For any derived ring $R \in \mathrm{CAlg}$, one can define a full stable sub-category $\mathrm{APerf}\, R \subset R$-mod of *almost perfect* R-modules – it consists precisely of those R-modules M such that $\tau_{<\ell} M \in (R\text{-mod})_{<\ell}$ is compact for each $\ell \in \mathbb{Z}$. (This is one derived version of the usual notion of a *pseudo-coherent module*, where now the module is not required to live in a single homological degree.) We define $\mathrm{APerf}(\mathscr{X}) \subset QC(\mathscr{X})$ to be the full subcategory spanned by those F such that $F_\eta \in R$-mod is almost perfect for each pair (R, η).

Remark B.2. Note that $QC(\mathscr{X})$ is an ∞-category, and does *not* denote the usual abelian category of quasi-coherent sheaves on the functor $\mathscr{X}$. Nevertheless, one also has the symmetric monoidal (abelian) category $QC(\mathscr{X})^\heartsuit$ given by the heart of the t-structure. Since pullback is right t-exact, there is a truncation functor
$$\left(\varprojlim_{\eta \in \mathscr{X}(R)} R\text{-mod}\right)^\heartsuit \longrightarrow \varprojlim_{\eta \in \mathscr{X}(R)} \left((R\text{-mod})^\heartsuit\right) \qquad F_\eta \mapsto H_0(F_\eta)$$
and one can check that this is a fully faithful embedding. In all the cases of interest to us (e.g., $\mathscr{X}$ is a geometric stack) one can show that it is in fact an equivalence so that $QC(\mathscr{X})^\heartsuit$ identifies with the abelian category of quasi-coherent sheaves on the classical pre-stack $\mathscr{X}_{cl}$. Due to this, we can refer to $H_i(F) \in QC(\mathscr{X})^\heartsuit$ for $F \in QC(\mathscr{X})$ as the *homology sheaves* of F.

Definition B.3. Suppose that $\mathscr{X} \in Fun(\mathrm{CAlg}, \mathcal{S})$ is a *locally Noetherian pre-stack*. This means that it is Kan extended from a functor $\mathscr{X}^N \in Fun(\mathrm{CAlg}^{Noeth}, \mathcal{S})$. Then, we can furthermore define:

- There is a full subcategory $\mathrm{PsCoh}(\mathscr{X}) \subset QC(\mathscr{X})$ consisting precisely of those $F \in QC(X)$ such that $H_i(F_\eta) \in QC(R\text{-mod})^\heartsuit$ is a *coherent* $H_0(R)$-module for all pairs (R, η) and all $i \in \mathbb{Z}$. We'll call such a complex "psuedo-coherent," though really only its truncations are.
- We define $\mathrm{APerf}(\mathscr{X}) \subset QC(\mathscr{X})$ to consist precisely of F such that $H_i(F_\eta) \in QC(R\text{-mod})^\heartsuit$ is coherent for all i and such that $H_i(F_\eta) = 0$ for $i \gg 0$ (the bound depending on (R, η)) for all pairs (R, η). One can chek that this is in fact equivalent to the previous definition [?HigherAlgebra, 8.2]. The importance of $\mathscr{X}$ being locally Noetherian, and this alternate description is that it ensures that $\mathrm{APerf}(\mathscr{X})$ carries a unique t-structure for which the inclusion to $QC(\mathscr{X})$ is t-exact. (There is an analogous definition with the opposite bound – but we do not need it here.)
- We define $\mathbb{D}\,\mathrm{APerf}(\mathscr{X}) \subset QC(\mathscr{X})$ to consist precisely of F such that $H_i(F_\eta) \in QC(R\text{-mod})^\heartsuit$ is coherent for all i and such that $H_i(F_\eta) = 0$ for $i \ll 0$ (the bound depending on (R, η)) for all pairs (R, η). Our notation is motivated by the following: In the case that $\mathscr{X}$ admits a Grothendieck dualizing complex, then Grothendieck duality provides an anti-equivalence $\mathbb{D}\,\mathrm{APerf}(\mathscr{X}) \simeq \mathrm{APerf}(\mathscr{X})^{op}$.
- We define $\mathrm{DCoh}(\mathscr{X}) \subset QC(\mathscr{X})$ to consist precisely of those F such that $H_i(F_\eta) \in QC(R\text{-mod})^\heartsuit$ is coherent for all i and such that $H_i(F_\eta) = 0$ for all but finitely many i (depending on (R, η)) for all pairs (R, η). (This is a variant of the *bounded coherent* category.) As with APerf, there is a unique t-structure on $\mathrm{DCoh}(\mathscr{X})$ such that the inclusion to $QC(\mathscr{X})$ is t-exact.

Remark B.4. The definition of the t-structure on an arbitrary pre-stack is formally convenient, but using the t-structure is only practical when $\mathscr{X}$ is a stack. For instance, if $\pi\colon U = \operatorname{Spec} R \to \mathscr{X}$ is an fppf atlas then π^* is t-exact – in particular, $F \in QC(\mathscr{X})$ is connective (resp., co-connective) if and only if $\pi^* F$ is so. For instance, we see that $\pi^* H_i(\mathcal{O}_X) = H_i(\pi^*\mathcal{O}_X) = H_i(R)$. Note that if R has infinitely many non-vanishing homologies, then it follows that $\operatorname{Perf} X$ is not contained in $\operatorname{DCoh} X$ since $\mathcal{O}_X$ is perfect but has infinitely many non-vanishing homology sheaves.

Remark B.5. Note that the hearts $\mathrm{DCoh}(\mathscr{X})^\heartsuit \simeq \mathrm{APerf}(\mathscr{X})^\heartsuit$ coincide with the ordinary abelian category of coherent complexes on the locally Noetherian functor $\mathscr{X}_{cl}$. In contrast to DCoh, the categories APerf have natural pullback functors and symmetric monoidal functor inherited from QC. Furthermore, it is clear from the first definition that $\mathrm{Perf}(\mathscr{X}) \subset \mathrm{APerf}(\mathscr{X})$ for any locally Noetherian pre-stack $\mathscr{X}$ – in contrast to classical algebraic geometry, the analogous statement may fail for $\mathrm{DCoh}(\mathscr{X})$ since $H_i(\mathcal{O}_X)$ may be non-vanishing in infinitely many degrees.

B.2. **Quasi-coherent pushforwards, and base-change.** Next, let's learn to pushforward quasi-coherent complexes.

Definition B.6. Suppose that $f\colon \mathscr{X} \to \mathscr{Y}$ is an arbitrary map of pre-stacks. Then, $f^*\colon QC(\mathscr{Y}) \to QC(\mathscr{X})$ is a colimit-preserving functor between presentable ∞-categories. It follows from the general theory in [?HigherTopos] that it admits a right adjoint. We denote this right adjoint by f_*.

Remark B.7. It is a priori non-obvious that $QC(\mathscr{X})$ has anything to do with sheaves of modules in some ∞-topos, or that the pushforward defined above has anything to do with a pushforward of sheaves. Nevertheless, this is true if $\mathscr{X}$ (resp., $\mathscr{X} \to \mathscr{Y}$) is nice enough. We do not dwell on this point, but the interested reader may consult e.g., [?DAG-VIII, 2.7.18] for the case of $\mathscr{X}$ a Deligne-Mumford stack and the étale ∞-topos.

We recall the following convenient separation axiom:

Definition B.8. We say that a pre-stack $\mathscr{X}$ is a *geometric stack* if $\mathscr{X}$ has affine diagonal, and there exists a smooth surjection $\pi\colon U = \operatorname{Spec} R \to \mathscr{X}$. In this case, the Cech nerve of π exhibits $\mathscr{X}$ as a colimit (in smooth sheaves) of affine schemes

$$\mathscr{X} \xleftarrow{\sim} \| U \leftarrow U \times_{\mathscr{X}} U \leftarrow \cdots \|$$

There is a relative version of this notion: We say that a morphism $f\colon \mathscr{X} \to \mathscr{Y}$ is a map of pre-stacks if for every map $S = \operatorname{Spec} R \to \mathscr{Y}$ the base-change pre-stack $\mathscr{X}_S$ is a geometric stack.

One might expect that the above, being much stronger than the usual "quasi-compact and quasi-separated" separation axiom, would guarantee that quasi-coherent pushforward as defined above is well-behaved. In general this is quite false: Take

$$f\colon \mathscr{X} = B\mathbb{Z}/p \to \operatorname{Spec} \mathbf{F}_p$$

Then, the functor f_* will not preserve filtered colimits; will not be compatible with arbitrary base-change; and will generally not be pleasant. Nevertheless, one has the following positive result:

Lemma B.9. *Suppose that $f\colon \mathscr{X} \to \mathscr{Y}$ is a quasi-compact and quasi-separated morphism of derived 1-stacks. Then,*

- *For any (homologically) bounded above object $F \in QC(\mathscr{X})_{<\infty}$, the formation of f_*F commutes with base-change along maps $\mathscr{Y}' \to \mathscr{Y}$ of finite Tor amplitude (and in particular flat maps).*
- *f_* preserves filtered colimits (equivalently, infinite sums) in $QC(\mathscr{X})_{<n}$ for each n (i.e., for uniformly bounded above colimits).*

Sketch. It is enough, by the definition of QC as extended from affines, to verify this in case $\mathscr{Y} = \operatorname{Spec} R$ is affine. In this case, our assumptions guarantee that there is a smooth hypercover $U_\bullet = \operatorname{Spec} B_\bullet \to \mathscr{X}$ of $\mathscr{X}$ by affine schemes. In this case $B_\bullet$ is a cosimplicial object of CAlg together with a coaugmentation to R.

By fppf descent for QC, the pullback induces an equivlence

$$QC(\mathscr{X}) \simeq \operatorname{Tot}\{B_\bullet\text{-mod}\} \qquad F \mapsto M_\bullet$$

and the pullback functor is nothing but tensoring up $B_\bullet \otimes_R (-)$ so that it follows that its right adjoint is given the usual Cech construction

$$\operatorname{Tot}\{B_\bullet\text{-mod}\} \ni M_\bullet \mapsto \operatorname{Tot} M_\bullet \in R\text{-mod}$$

Next, note that $F \in QC(\mathscr{X})_{<0}$ if and only if $M_\bullet \in (R\text{-mod})_{<0}$ for all $\bullet$. For such objects, the resulting spectral sequence of a totalization is a a (convergent) third quadrant spectral sequence. The formation of this spectral sequence is evidently compatible with filtered colimits and flat base change; a slight elaboration gives the case of finite Tor dimension base change. □

We will see more positive results in a later subsection where, rather than changing the sheaf, we instead require that our stacks have finite cohomological amplitude.

B.3. **Enough coherent / perfect complexes.** Now, we introduce several axioms that say that $QC(\mathscr{X})$ is "generated by small objects" for various notions o small:

Definition B.10. Suppose that $\mathscr{X}$ is a pre-stack. We say that

- $\mathscr{X}$ is a *perfect stack* if $\mathscr{X}$ is geometric, and $\operatorname{Perf}(\mathscr{X})$ coincides with the subcategory of compact objects in $QC(\mathscr{X})$. This is equivalent to asking that the natural functor $\operatorname{Ind}\operatorname{Perf}(\mathscr{X}) \to QC(\mathscr{X})$ be an equivalence.
- $\mathscr{X}$ is *perfectly generated* if every object of $QC(\mathscr{X})$ can be written as a filtered colimit of objects of $\operatorname{Perf}(\mathscr{X})$. (This is weaker than $\mathscr{X}$ being perfect in two ways: It does not mention geometricity, and does not require that the object of $\operatorname{Perf}(\mathscr{X})$ be compact.)

These admit relative versions. Suppose that $f\colon \mathscr{X} \to \mathscr{Y}$ is a map of pre-stacks. We say that

- f is a *relative geometric stack* (resp., relative perfect stack, relatively perfectly generated) if for every map $S = \operatorname{Spec} R \to \mathscr{Y}$ the base-change $\mathscr{X}_S$ is a geometric stack (resp., perfect stack, perfectly generated).

We record here some results towards these generation properties:

Theorem B.11. *Suppose that $\mathscr{X}$ is a Noetherian qc.qs. algebraic stack. Then,*

1. *([?LMB00]) $QC(\mathscr{X})^\heartsuit$ is compactly-generated by $\operatorname{DCoh}(\mathscr{X})^\heartsuit$.*
2. *([?DrinfeldGaitsgory]) The subcategory $QC(\mathscr{X})_{<0} \subset QC(\mathscr{X})$ is compactly-generated, with compact objects precisely $\operatorname{DCoh}(\mathscr{X})_{<0}$.*
3. *For each $d \geq 0$, the subcategory $QC(\mathscr{X})_{\geq 0, <d} \subset QC(\mathscr{X})$ is compactly-generated, with compact objects precisely $\operatorname{DCoh}(\mathscr{X})_{\geq 0, <d}$.*

Proof. See [?LMB00, Prop. 15.4] for (i), [?DrinfeldGaitsgory] for (ii), and (iii) follows similarly. □

Corollary B.12. *Let $\mathfrak{X}$ be a perfectly generated stack, then for any $F \in \mathrm{APerf}(\mathscr{X})$ and any $n \in \mathbb{Z}$, there is a $P \in \mathrm{Perf}(\mathscr{X})$ and a homomorphism $P \to F$ such that $\tau_{\leq n} P \to \tau_{\leq n} F$ is a retract.*

Proof. Write F as a filtered colimit $\varinjlim F_i$ with $F_i \in \mathrm{Perf}(\mathscr{X})$. Then $\tau_{\leq m} F = \varinjlim \tau_{\leq m} F_i$ is a compact object of $QC(\mathscr{X})_{\leq m}$, as are the $\tau_{\leq m} F_i$. It follows that the identity homomorphism $\tau_{\leq m} F \to \tau_{\leq m} F$ factors through $\tau_{\leq m} F_i$ for some i, and as a consequence, $\tau_{\leq m} F$ is a retract of $\tau_{\leq m} F_i$ for this i. □

Theorem B.13. *Suppose that $\mathscr{X} = W/G$ is a global quotient stack for a linear algebraic group G acting on a quasi-projective scheme W. Then,*

(1) $\mathscr{X}$ is geometric and perfectly-generated.
(2) In characteristic 0, $\mathscr{X}$ is a perfect stack.

Proof. See [?BFN] for a discussion of this and other examples of perfect stacks. ¡¡Still have to show that $\mathscr{X}$ is generated by connective perfect objects¿¿ □

B.4. **More pushforward and base-change.**

Definition B.14. We say that a morphism $f : \mathscr{X} \to \mathscr{Y}$ of pre-stacks is of *cohomological dimension at most d* if for any $F \in QC^{\heartsuit}(\mathscr{X})$ we have $f_* F \in QC(\mathscr{X})_{\geq -d}$. (Note that this depends only on the induced morphism of underlying classical stacks.)

We say that a morphism $f : \mathscr{X} \to \mathscr{Y}$ of pre-stacks is *universally of finite cohomological dimension* (or, *satisfies (CD)* for short) if there is some d for which this condition is satisfied for the base-change of f along any morphism of pre-stacks $\mathscr{Y}' \to \mathscr{Y}$.

Proposition B.15. *Suppose that $\mathscr{Y}$ is a geometric stack, and that $f : \mathscr{X} \to \mathscr{Y}$ is a relative quasi-compact and quasi-separated stack.*

Then, the following are equivalent

(1) f is universally of cohomological dimension at most d;
(2) for any flat morphism $S = \operatorname{Spec} R \to \mathscr{Y}$, the base-change $f_S : \mathscr{X}_S \to S$ is of cohomological dimension at most d;
(3) f is of cohomological dimension at most d;
(4) f_* preserves filtered colimits, takes $QC(\mathscr{X})_{>0}$ into $QC(S)_{>-d}$, and its formation is compatible with arbitrary base-change.

We will prove Proposition B.15 at the end of this subsection. But first, let us record some consequenes:

Corollary B.16. *Suppose that $f : \mathscr{X} \to \mathscr{Y}$ is a quasi-compact and quasi-separated morphism, with $\mathscr{Y}$ a geometric stack, and that f is of cohomological dimension at most d. Then:*

(1) $f_* : QC(\mathscr{X}) \to QC(\mathscr{Y})$ preserves filtered colimits
(2) the formation of f_* is compatible with arbitrary base-change;
(3) f_* and f^* satisfy the projection formula, i.e., the natural morphism $f_*(F) \otimes G \to f_*(F \otimes f^* G)$ is an equivalence for all F, G.

Proof. Note that (i) and (ii) are part of the equivalence of the previous Proposition.

We must prove (iii): First, repeatedly applying (ii) and the fact that pullbacks are symmetric monoidal, we see that the claim is local on $\mathscr{Y}$ so that we may suppose that $\mathscr{Y} = \operatorname{Spec} R$. Pick a hypercover $U_\bullet = \operatorname{Spec} B_\bullet \to \mathscr{X}$, so that $QC(\mathscr{X}) = \mathrm{Tot}\{B_\bullet\text{-mod}\}$. Suppose that $M_\bullet \in \mathrm{Tot}\{(B_\bullet\text{-mod})_{\geq d}\}$ corresponds to F and that $N \in QC(\mathscr{Y}) = (R\text{-mod}) \geq d$ corresponds to G. We must verify that the natural map

$$\mathrm{Tot}\{M_\bullet\} \otimes_R N \longrightarrow \mathrm{Tot}\left\{M_\bullet \otimes_{B_\bullet} (B_\bullet \otimes_R N)\right\}$$

is a quasi-isomorphism.

Let $\mathcal{C} \subset R$-mod denote the full subcategory consisting of those $N \in R$-mod for which the preceding map is a quasi-isomorphism for all $M_\bullet$. Note that $\mathcal{C}$ is closed under cones, shifts, and retracts since both f_*, f^*, and $\otimes$ preserve these operations up to quasi-isomorphism. Next, note that $\mathcal{C}$ is closed under filtered colimits, because all three operations preserve filtered colimits by (i). Finally, observe that $R \in \mathcal{C}$. But the smallest subcategory R-mod containing R and closed under cones, shifts, and filtered colimits is all of R-mod. This completes the proof. □

Corollary B.17 ("Base change for p_+"). *Suppose we are given a Cartesian square of stacks*

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{q'} & \mathscr{X} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \mathscr{S}' & \xrightarrow[q]{} & \mathscr{S} \end{array}$$

such that $\mathscr{S}$ is geometric and q is relatively quasicompact and quasiseparated and universally of cohomological dimension at most d. Assume that $p^* : QC(\mathscr{S}) \to QC(\mathscr{X})$ admits a left adjoint p_+ and likewise for $(p')^*$. Then, there is a natural isomorphism

$$q^* p_+ \simeq (p')_+ (q')^*$$

Proof. It is enough to give a natural isomorphism of their right adjoints – for this we take the base-change isomorphism $p^* q_* \simeq (q')_* (p')^*$ for this Cartesian square from B.16. □

In characteristic zero, it turns out that (CD) is very often satisfied:

Proposition B.18. *Suppose that S is a Noetherian characteristic zero scheme, and that $\mathscr{X}$ is a finite-type S-stack such that the automorphisms of its geometric points are affine. Then, $\mathscr{X}$ satisfies (CD) over S for some d.*

Proof. This is [?DrinfeldGaitsgory]. The sketch is: One can show that S has a finite stratification by global quotient stacks, and a straightforward argument shows that having finite cohomological dimension is stable under open-closed decompositions. It thus sufices to prove the result for global quotient stacks. This follows by noting that quasi-compact and quasi-separated algebraic spaces have finite cohomological dimension, and that reductive groups (e.g., GL_n) are linearly reductive in characteristic zero. □

Now we turn to the proof of Proposition B.15:

Lemma B.19. *Suppose $\mathscr{X}$ is an S-stack. Then, property (CD_d), for fixed integer d, may be checked on a flat affine cover of S. Furthermore, it is stable under affine base-change.*

Proof. Affine maps are t-exact, so the property is clearly stable under affine base change. Flat base-change shows that if it holds over a flat cover $S' \to S$, then it holds over S. □

Proof of Proposition B.15: It is clear that (1) implies (2) implies (3). The previous Lemma shows that (3) implies (2) implies (1). It is clear that (4) implies (3), so it is enough to prove that (1-3) implies (4).

If we can prove that the formation of f_* is compatible with flat base-change in the affine case, then we can apply faithfully flat descent to reduce (4) to the case of $S = \operatorname{Spec} A$. Thus it is enough to prove (4), including the compatibility with base-change, in the case where $S = \operatorname{Spec} A$.

Claim: Assuming (3), $R\Gamma(\mathscr{X}, -)$ takes $QC(\mathscr{X})_{>0}$ into $A\text{-mod}_{>-d}$.

Assuming the claim, we complete the proof:

Let us show first show that the Claim together with Lemma B.9 implies that f_* is compatible with filtered colimits and flat base change on S. Indeed, both statements can be checked at the level of homology groups, and the Claim implies that

$$H_i \circ f_* = H_i \circ f_* \circ \tau_{\leq i+d}$$

thereby reducing us to the situation of Lemma B.9. Similarly, to show that $f_*(QC(\mathscr{X})_{>0}) \subset QC(S)_{>-d}$ it is enough to show that $H_i \circ f_*(F) = 0$ for all $F \in QC(\mathscr{X})_{>0}$ and $i \leq -d$. For each fixed i the above argument reduces us to consider F bounded, and then using the t-structure and shifting reduces us to the case of F in the heart – which is precisely (3).

Finally, we must show that the formation of f_* is compatible with arbitrary base-change. In light of the above, it is enough to consider the case of an affine base-change. Suppose that $S' = \operatorname{Spec} R' \to \operatorname{Spec} R$ is arbitrary, and let $f' \colon \mathscr{X}' = \mathscr{X} \times_S S' \to S'$ be the base-change of S. We must show that the natural map

$$R' \otimes_R R\Gamma(\mathscr{X}, F) \longrightarrow R\Gamma(\mathscr{X}', F|_{\mathscr{X}'}) = R\Gamma(\mathscr{X}, R' \otimes_R F)$$

is an equivalence; note that here we have used the affine version of the projection formula, which may be checked locally. Note that the underlying complex on both sides depends on R' only as an R-module, rather than an R-module. Let $\mathcal{C} \subset R$-mod denote the full subcategory consisting of those M for which $M \otimes_R R\Gamma(\mathscr{X}, F) \to R\Gamma(\mathscr{X}, M \otimes_R F)$ is an equivalence. It is easy to see that $R \in \mathcal{C}$ and that $\mathcal{C}$ is closed under finite colimits and retracts, so that $\mathrm{Perf}\, R \subset \mathcal{C}$. Since $R\Gamma(\mathscr{X}, -)$ preserves filtered colimits, we see that $\mathcal{C}$ is closed under filtered colimits in R-mod so that $\mathcal{C}$ is all of R-mod. This completes the proof. □

Appendix C. Recollections on formal completions

We follow [?DAG-XII, 5.1.1] in viewing the formal completion of a stack as a functor of points (rather than say a pro-ringed topos, etc.):

Definition C.1. Suppose that $\mathscr{X}$ is a pre-stack. Let $|\mathscr{X}|$ denote the underlying Zariski topological space of points of $\mathscr{X}$, and $\mathscr{Z} \subset |\mathscr{X}|$ a closed subset. We say that $\mathscr{Z}$ is co-compact if the inclusion of the complement $\mathscr{X} - \mathscr{Z} \to \mathscr{X}$ is a quasi-compact open immersion.

Given a pre-stack $\mathscr{X}$ and a co-compact $\mathscr{Z} \subset |\mathscr{X}|$ define the *formal completion of* $\mathscr{X}$ *along* $\mathscr{Z}$ to be the following pre-stack

$$\widehat{\mathscr{X}}(R) := \mathscr{X}_{\widehat{\mathscr{Z}}}(R) := \{\eta \in \mathscr{X}(R) \colon \text{such that } \eta \text{ factors set-theoretically through } \mathscr{Z}\}$$

One special case is where $\mathscr{Z}$ is gotten from an ideal sheaf $\mathscr{I} \subset H_0(\mathcal{O}_X)$:

Definition C.2. Suppose that $\mathscr{X}$ is a stack and that $\mathscr{I} \subset H_0(\mathcal{O}_{\mathscr{X}})$ is a locally finitely generated ideal sheaf. Then, $\mathscr{I}$ determines a co-compact closed subset $\mathscr{Z}(\mathscr{I}) \subset |\mathscr{X}|$ as follows: Since fppf morphisms are topological quotients and $\mathscr{X}$ is a stack, it is enough to describe the pre-image of $\mathscr{Z}(\mathscr{I})$ under each fppf morphism $\eta\colon \mathrm{Spec}\, R \to \mathscr{X}$; in this case, $\eta^*(\mathscr{I}) \subset H_0(R)$ is an ideal sheaf and we may set $\eta^{-1}\mathscr{Z}(\mathscr{I}) = \mathrm{Supp}(\mathrm{Spec}\, H_0(R)/\eta^*(\mathscr{I}))$. By the completion of $\mathscr{X}$ along $\mathscr{I}$ we will mean the completion of $\mathscr{X}$ along $\mathscr{Z}(\mathscr{I})$.

In case $\mathscr{X}$ (and hence $\mathscr{Z}$) is affine, we have an explicit description of $\widehat{X}$:

Proposition C.3. *Suppose that $R \in \mathrm{CAlg}$ and that $I \subset \pi_0(R)$ a finitely generated ideal. Let $\mathscr{X} = \mathrm{Spec}\, R$ and $\widehat{\mathscr{X}} = \mathrm{Spf}\, R$ its I-adic completion. Then, there exists a tower*

$$\cdots \to R_2 \to R_1 \to R_0$$

in CAlg *such that there is an equivalence of pre-stacks*

$$\mathrm{Spf}\, R = \varinjlim_n \mathrm{Spec}\, R_n.$$

Furthermore, we may suppose that $H_0(R_i) \to H_0(R_{i-1})$ is surjective for each i, and that each R_n is perfect as R-module with Tor-amplitude uniformly bounded by the number of generators of I.

Proof. This is [?DAG-XII, Lemma 5.1.5]. Note that the proof goes over essentially unchanged in simplicial commutative rings, but that now the rings R_n can be assumed to be *perfect* rather than merely *almost perfect*: This follows from the construction of the algebras denote $A(x)_n$ in op.cit., and the fact that in the universal example $\mathbb{Z}$ is a perfect module over $\mathbb{Z}[x]$ of Tor dimension explicitly bounded by 1, thanks to the Koszul resolution.

Note that this is, essentially, the one statement in this paper that depends strongly on the fact that we are working with simplicial commutative rings rather than E_∞ rings. We'll come back to its consequences in the following Proposition. □

Remark C.4. We will often find ourselved considering the following relative variant of Proposition C.3: Suppose that $S = \mathrm{Spec}\, R$, $I \subset \pi_0(R)$ a finitely generated ideal, and $\widehat{\mathscr{S}}$ the assocaited completion. Let $\pi\colon \mathscr{X} \to S$ be an S-stack, $\pi^{-1}(I) \subset H_0(\mathcal{O}_{\mathscr{X}})$ the induced (locally finitely generated) ideal sheaf on $\mathscr{X}$, and $\widehat{\mathscr{X}}$ the associated completio. Then, the natural map

$$\widehat{\mathscr{X}} \xrightarrow{\sim} \mathscr{X} \times_S \widehat{S}$$

is an equivalence (just use the functor-of-points descriptions on both sides). Furthermore, if $\{R_n\}$ is a tower as in Proposition C.3, then there is an equivalence

$$\widehat{\mathscr{X}} \xleftarrow{\sim} \varinjlim_n \mathscr{X} \times_S \operatorname{Spec} R_n$$

since fiber products preserve filtered colimits in pre-sheaves. We will generally write write $i_n\colon \mathscr{X}_n = \mathscr{X} \times_S \operatorname{Spec} R_n \to \mathscr{X}$ for the base-changed closed immersions.

A natural question is how this compares to the classical notion of completion. To this end, we have:

Proposition C.5. *Suppose that R is a Noetherian* classical *commutative algebra and that $I \subset \pi_0(R)$. Let $\operatorname{Spf} R$ denote the (derived) completion of $\operatorname{Spec} R$ along I, and let $\operatorname{Spf}^{cl} R$ denote Kan extension of the (classical) pre-stack $\varinjlim_n \operatorname{Spec} R/I^n$. Then, the natural morphism*

$$F\colon \operatorname{Spf}^{cl} R \longrightarrow \operatorname{Spf} R$$

is an equivalence.

Proof. Note first that the natural maps $\operatorname{Spec} R/I^n \to \operatorname{Spec} R$ factor uniquely through $\operatorname{Spf} R$ – this determines the natural morphism of the proposition. Let R_n be the Koszul-type algebra killing the n-th powers of a finite set of generators for the ideal $I \subset A$ (in dg-language this would be $R[B_1, ..., B_r]/dB_i = f_i^n$) – it satisfies the conditions of Proposition C.3 by the proof of [?DAG-XII, Lemma 5.1.5]. Note that $\operatorname{Spf}^{cl} R \simeq \varinjlim \operatorname{Spec} H_0(R_n)$ since $I^{(n-1)r+1} \subset (f_1^n, \dots, f_r^n) \subset I^n$. We must thus show that the natural map

$$\varinjlim \operatorname{Spec} H_0(R_n) \longrightarrow \varinjlim \operatorname{Spec} R_n$$

is an equivalence. There is almost an argument for this in [?DAG-XII, Lemma 5.2.17] – we must show that the map of pro-objects in (almost perfect) commutative R-algebras $\{R_n\} \to \{H_0(R_n)\}$ is a pro-equivalence. By the Lemma, we have that

$$\{\tau_{\leq k} A_n\} \longrightarrow \{H_0(A_n)\}$$

is an equivalence of pro-objects in (almost perfect) R-modules for all k.

We now expound on the above remark on needing simplicial commutative rings: We need the fact that if $I \subset R$ is generated by r elements, then each R_n may be assumed to be $(r+1)$-truncated – this is true because $R_n \simeq R \otimes_{k[x_1,\dots,x_r]} k$ so that this follows from the bound on the Tor dimension of k over $k[x_1, \dots, x_r]$ gotten by considering the Koszul resolution.[13]Taking $k > r+1$ in the previous displayed equation thus completes the proof of the pro-equivalence, at the level of R-modules. Furthermore, the module level statement does in fact imply the algebra statement (i.e., potential issues with pro-algebras vs algebras in pro-objects don't get in the way) because we may restricted to $(r+1)$-truncated algebras (c.f., the proof of [?DAG-XII, Lemma 6.3.3]). ☐

Since fppf morphisms are topological quotient morphisms, we can use this to deduce:

Corollary C.6. *Suppose that $\mathscr{X}$ is a Noetherian classical stack, and that $\mathscr{I} \subset \mathcal{O}_{\mathscr{X}}$ is an ideal sheaf. Then, the derived completions of $\mathscr{X}$ along $\mathscr{I}$ is the Kan extension (from classical rings to derived rings) of the classical completion of $\mathscr{X}$ along $\mathscr{I}$.*

Remark C.7. Note that the rings R_n appearing in Proposition C.3 are *not* unique, in contrast to the usual R/I^n that we are used to from classical completions – in particular, they need not globalize. This is related to the fact that our notion of completion $X^{\wedge}_{Z}$ actually depended only on the underlying subset of $\mathscr{Z}$ and not on the choice of structure sheaf etc.

There *is* a notion of completion along a closed immersion $\mathscr{Z} \to \mathscr{X}$ in derived algebraic geometry more analogous to the usual $\mathscr{I}$-adic completion, i.e., it gives rise to a canonically defined pro-algebra $\widehat{\mathcal{O}_{\mathscr{X}}} = \text{“}\varprojlim_n\text{''} \mathcal{O}_{\mathscr{X}_n}$ such that $H_0 \mathcal{O}_{\mathscr{X}_n} = H_0(\mathcal{O}_{\mathscr{X}})/\mathscr{I}^n$ where $\mathscr{I} = \ker\{H_0(\mathcal{O}_{\mathscr{X}}) \to H_0(\mathcal{O}_{\mathscr{Z}})\}$. However this construction is somewhat involved, especially outside of characteristic zero, and since we do not need it we will not discuss it further.

[13]In the case of E_∞ algebras we would have to replace $k[x_1, \dots, x_r]$ by the much more complicated free E_∞-algebra on r generators.

C.1. **Almost perfect complexes and coherent sheaves.** In this section we will use both abelian categories – of *coherent* sheaves, and coherent sheaves on formal completions – and infinity categories with t-structures – of *almost perfect* complexes. In fact, we will also consider some n-categories that interpolate between the two. In contrast to the full quasi-coherent categories, these all have the following pleasant behavior with respect to completion:

Proposition C.8. *Suppose $\mathscr{X}$ is a locally Noetherian derived stack, $\mathscr{Z} \subset |\mathscr{X}|$ a co-compact closed subset, and $\widehat{\mathscr{X}}$ the completion of $\mathscr{X}$ along $\mathscr{Z}$. Then,*

1. *The (ordinary) category $\mathrm{Coh}(\widehat{\mathscr{X}})$ is abelian. For $f\colon \mathscr{X}' \to \mathscr{X}$ a flat morphism from another Noetherian stack, let $\widehat{\mathscr{X}'}$ denote the completion of $\widehat{\mathscr{X}'}$ along $f^{-1}(\mathscr{Z})$; then, the induced pullback functor $\mathrm{Coh}(\widehat{\mathscr{X}}) \to \mathrm{Coh}(\widehat{\mathscr{X}'})$ is exact. The exactness of sequences in $\mathrm{Coh}(\widehat{\mathscr{X}})$ may be checked on a flat cover.*
2. *The (∞-) category $\mathrm{APerf}(\widehat{\mathscr{X}})$ admits a t-structure. For $f\colon \mathscr{X}' \to \mathscr{X}$ a flat morphism from another Noetherian stack, the induced pullback functor $\mathrm{APerf}(\widehat{\mathscr{X}}) \to \mathrm{APerf}(\widehat{\mathscr{X}'})$ is t-exact. The property of being connective/co-connective in $\mathrm{APerf}(\widehat{\mathscr{X}})$ may be checked on a flat cover of $\mathscr{X}$. In the affine case, the t-structure is described in Lemma 6.6 below.*
3. *The heart of the t-structure in (2) identifies with $\mathrm{Coh}(\widehat{\mathscr{X}})$. More generally, the connective n-truncated objects $\mathrm{APerf}(\widehat{\mathscr{X}})^{cn}_{\leq n}$ naturally identifies with $\mathrm{Coh}^n(\widehat{\mathscr{X}})$.*
4. *The t-structure on $\mathrm{APerf}(\widehat{\mathscr{X}})$ is left t-complete, and (if $\mathscr{X}$ is quasi-compact) right t-bounded. There is a natural equivalence*
$$\mathrm{APerf}(\widehat{\mathscr{X}})^{cn} \xrightarrow{\sim} \varprojlim_n \mathrm{APerf}(\widehat{\mathscr{X}})^{cn}_{\leq n} \simeq \varprojlim_n \mathrm{Coh}^n(\widehat{X})$$

Proof. For (1), see e.g., [?Conrad-FormalGaga, $\mathscr{S}$1-2] – note that this is a statement at the level of classical stacks.

For (2), note first that the claim is fppf local on $\mathscr{X}$: See [?DAG-XII, Proposition 5.2.4, Remark 5.2.13] for the case where $\mathscr{X}$ is Deligne-Mumford and étale descent. It remains to show that the t-structure is, in fact, fppf local in the affine case: Note that the proofs of Lemmas 5.2.5 and 5.2.7 of op.cit. applies verbatim with étale replaced by fppf.

For (3) and (4), we apply the last points of the following Lemma. □

We record some convenient facts on left t-complete t-structures that we will use.

Lemma C.9.

1. *Suppose that $\mathcal{C}$ is a stable ∞-category with a t-structure. Then, $\mathcal{C}$ is left t-complete if and only if the following condition holds: Suppose given a tower $\mathscr{F}_0 \leftarrow \mathscr{F}_1 \leftarrow \cdots$ in $\mathcal{C}$ such that for every $k \geq 0$, the tower $\tau_{\leq k}(\mathscr{F}_n) \in \mathcal{C}_{\leq k}$ is eventually constant. Then, the tower has an inverse limit $\mathscr{F}$, and for every $k \geq 0$ the natural map $\tau_{\leq k}\mathscr{F} \to \tau_{\leq k}\mathscr{F}_n$ is an equivalence for $n \gg 0$ (depending on k).*
2. *Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is an exact and right t-exact functor, and that $\{\mathscr{F}_n\}$ is a tower in $\mathcal{C}$ satisfying the above conditions. Then, the tower $\{F(\mathscr{F}_n)\}$ in $\mathcal{D}$ satisfies the above conditions as well.*
3. *Suppose given a limit diagram $i \mapsto \mathcal{C}_i$ and $\mathcal{C} \to \mathcal{C}_i$, of small stable ∞-categories with t-structures such that all the functors are exact and right t-exact. If each $\mathcal{C}_i$ is left t-exact, then so is $\mathcal{C}$.*
4. *Suppose given a diagram $i \mapsto \mathcal{C}_i$ of small stable ∞-categories with t-structures, such that all the functors are exact and t-exact. Then, there is a unique t-structure on the limit $\mathcal{C} := \varprojlim_i \mathcal{C}_i$ such that the natural functors $\mathcal{C} \to \mathcal{C}_i$ are t-exact. In this case, for each $n \geq 0$ the natural functor*
$$\mathcal{C}^{cn}_{\leq n} \longrightarrow \varprojlim_i (\mathcal{C}_i)^{cn}_{\leq n}$$
is an equivalence.

Proof. For (1), note first that left t-completeness is equivalent to the analogous assertion for the tower $\tau_{\leq n}\mathscr{F}_n$. In particular, one direction is obvious. It remains to suppose that $\mathcal{C}$ is left t-complete and prove the desired condition. Consider the double-tower $\{\tau_{\leq m}\mathscr{F}_n\}_{m,n}$. We will show that it has a limit, and evaluate this limit in two different ways "rows-then-columns" and the tranpose.

In one direction, we have that

$$\mathscr{F}'_m := \varprojlim_n \tau_{\leq m}\mathscr{F}_n$$

exists since the diagram is eventually constant by hypothesis. Furthermore, $\mathscr{F}'_m \simeq \tau_{\leq m}\mathscr{F}'_n$ for any $n \geq m$ by construction. Thus

$$\mathscr{F} := \varprojlim_m \varprojlim_n \tau_{\leq m}\mathscr{F}_n = \varprojlim_m \mathscr{F}'_m$$

exists, and $\mathscr{F} \to \tau_{\leq m}\mathscr{F}'_m$ induces an equivalence on $\tau_{\leq m}$, since $\mathcal{C}$ is left t-complete. So, the inverse limit over the whole double-tower exists and is also equal to $\mathscr{F}$. Computing this in the other direction, we note

$$\varprojlim_m \tau_{\leq m}\mathscr{F}_n = \mathscr{F}_n$$

since $\mathcal{C}$ is left t-complete, so that

$$\mathscr{F} \simeq \varprojlim_n \varprojlim_m \tau_{\leq m}\mathscr{F}_n = \varprojlim_n \mathscr{F}_n.$$

The assertion on $\tau_{\leq k}\mathscr{F} \to \tau_{\leq k}\mathscr{F}_n$ for $n \gg 0$ by comparing it to $\tau_{\leq k}\mathscr{F} \to \tau_{\leq k}\mathscr{F}'_n$ for $n \geq k$. This completes the proo fof (1).

For (2), it is enough to show that F preserves $\tau_{\leq k}$-equivalences to $\tau_{\leq k}$-equivalences. This follows from the fact that F preserves extension sequences and k-connective objects.

For (3), we apply the criterion in (1). Notice that a putative limit diagram in $\mathcal{C}$ which gives a limit diagram in each $\mathcal{C}_i$ is itself a limit diagram (though the converse need not hold in general!), so that (2) completes the proof.

For (4), note that the t-structure is characterized by stating that an object of $\mathcal{C}$ is connective (resp. co-connective) if and only if this is true of its image in each $\mathcal{C}_i$. Since the transition functors are t-exact, the truncation functors on each $\mathcal{C}_i$ pass to the limit to provide truncation functors on $\mathcal{C}$. It is thus straightforward to check that this is a t-structure, and the desired description of the connective n-truncated objects.

Notice also that (3) and (4) are essentially contained in [?DAG-XII, Remark 5.2.9]. □