Last time: example of stack \( X/G \)

Another way to think about this:

Groupoid scheme \( X \) vs presheaf of groupoids on \( \text{Sch}/k \) (the naive 1-category)

\( \rightsquigarrow \) Fibered category over \( \text{Sch}/k \)

\( \mathfrak{b} : \mathcal{F} \to \text{Sch} \)

\[ \begin{array}{c}
\mathcal{F} \text{ has objects } (U, \delta \in X_0(U)), \text{ maps } \\
\mathfrak{c} : \begin{array}{c}
\downarrow \delta' \\
\downarrow \delta \\
U \to V
\end{array} \\
\text{call it } X_0
\end{array} \]

For any category fibered in groupoids, \( \mathcal{F} \)

a stackification

\[ \mathcal{F} \to \tilde{\mathcal{F}} \]

the map \( \mathcal{F} \to \tilde{\mathcal{F}} \) is universal for maps to stacks

maps in the 2-category of fibered categories are base preserving functors (strictly commute with)
Some key remarks:

1) 2-fiber products of categories

Example 4.30.3. Let $A$, $B$, and $C$ be categories. Let $F : A \to C$ and $G : B \to C$ be functors. We define a category $A \times_C B$ as follows:

1. an object of $A \times_C B$ is a triple $(A, B, f)$, where $A \in \text{Ob}(A)$, $B \in \text{Ob}(B)$, and $f : F(A) \to G(B)$ is an isomorphism in $C$,
2. a morphism $(A, B, f) \to (A', B', f')$ is given by a pair $(a, b)$, where $a : A \to A'$ is a morphism in $A$, and $b : B \to B'$ is a morphism in $B$ such that the diagram

$$
\begin{array}{ccc}
F(A) & \xrightarrow{f} & G(B) \\
| & \downarrow{F(a)} & | \\
F(A') & \xrightarrow{f'} & G(B')
\end{array}
$$

is commutative.

Moreover, we define functors $p : A \times_C B \to A$ and $q : A \times_C B \to B$ by setting

$$p(A, B, f) = A, \quad q(A, B, f) = B,$$

in other words, these are the forgetful functors. We define a transformation of functors $\psi : F \circ p \to G \circ q$. On the object $\xi = (A, B, f)$ it is given by $\psi_\xi = f : F(p(\xi)) = F(A) \to G(B) = G(q(\xi))$.

If $\mathcal{X}_0, \mathcal{X}_1 \to \mathcal{Y}$ are base preserving maps of stacks, then $\mathcal{X}_0 \times \mathcal{X}_1$ is still a stack, with commutative diagram:

$$
\begin{array}{ccc}
\mathcal{X}_0 \times \mathcal{X}_1 & \to & \mathcal{X}_1 \\
| & \downarrow{\text{universal}} & | \\
\mathcal{X}_0 & \to & \mathcal{Y}
\end{array}
$$
3) 2-Yoneda lemma (see Vistoli)

2-YONEDA LEMMA. The two functors above define an equivalence of categories
\[ \text{Hom}_C((C/X), \mathcal{F}) \simeq \mathcal{F}(X). \]

implication: can think of schemes as fibered categories too, no loss of data.

Remark: Can also restrict to affines \( \text{Alg}/k \)

Examples (of many things at once) (skip?)

Let \( X_0 \) be a groupoid scheme.
- regard \( X_0, X_1, X_2 \) as fibered categories
- let \( X \) be cat fibered in groupoids assoc.
- Let \( X \to X^a \) be the stackification

Then consider:

\[ \text{Colim}(X_0) \to X \to X^a \to \text{Sch}/k \]

Categories of sections are all equivalent, so what we defined as \( \text{QCoh}(X) \)
is the category of sections.
equivalent, so what we defined as $\text{Qcoh}(X)$ is the category of sections.

Theorem. TFAE for a cat fibered in groupoids

1) $X \cong (X_0)^a$ for a smooth groupoid $^*$

2) $X \to X \times X$ is representable and smooth surjection $U \to X$ from a scheme

3) $X$ representable smooth surjection $U \to X$

Furthermore, these are equiv. to the same, but with fppf replacing smooth. Such an $X$ is an alg. stack

Remark: Not true that fppf maps admit etale local sections, but there’s a construction, for any fppf map $U \to X$, of a $U' \to U$ s.t. composition $U' \to X$ is smooth surjection.

Remark: discuss correspondence between groupoid and stack

Example 1: $X \to B$ projective or proper

Tag 06DC From <http://stacks.math.columbia.edu/tag/06DC>

Tag 0372
78.16. The Picard stack

Let $S$ be a scheme. Let $\pi : X \to B$ be a morphism of algebraic spaces over $S$. We define a category $\text{Pic}_{X/B}$ as follows:

(1) An object is a triple $(U, b, \mathcal{L})$, where
   (a) $U$ is an object of $(\text{Sch}/S)_{\text{fppf}}$,
   (b) $b : U \to B$ is a morphism over $S$, and
   (c) $\mathcal{L}$ is in invertible sheaf on the base change $X_U = U \times_{b, B} X$.

(2) A morphism $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ is given by a morphism of schemes $f : U \to U'$ over $B$ and an isomorphism $g : f^* \mathcal{L}' \to \mathcal{L}$.

The composition of $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ with $(f', g') : (U', b', \mathcal{L}') \to (U'', b'', \mathcal{L}'')$ is given by $(f \circ f', g \circ f^*(g'))$. Thus we get a category $\text{Pic}_{X/B}$ and

$$p : \text{Pic}_{X/B} \to (\text{Sch}/S)_{\text{fppf}}, \quad (U, b, \mathcal{L}) \mapsto U$$

Is algebraic, slight modification: Picard scheme, stack of coherent sheaves

Ex 2: Stack of flat families of curves with
   a) geometrically reduced & connected fibers
      of arithmetic genus $h^1(O_E) = 1$

   b) algebraic

b) Flat family of curves $E \to B$
   relatively ample bundle

Separation axiom: we usually work with geometric stacks, meaning
with geometric stacks meaning
\[ \mathcal{X} \to \mathcal{X} \times \mathcal{X} \]
is affine.

\[ \text{Isom}_U(\mathcal{X}, \mathcal{X}) \to \mathcal{X} \]
is this affine?
\[ \Downarrow \quad \Downarrow \quad \Downarrow \]
\[ U \to \mathcal{X} \times \mathcal{X} \]
\[ (\mathcal{X}, \mathcal{X}) \]

No in Ex 2b: look at smooth elliptic curve degenerating to nodal elliptic, so behavior of automorphism groups is weird.

Remark: maps of alg. stacks are repres. iff the maps \( \mathcal{X}(U) \to \mathcal{Y}(U) \) are faithful.

Given homom. \( \varphi : G \to H \) and an equivariant map \( f : X \to Y \)
\[ G \to H \]
\[ Y \to Y \]
\[ X \to X \]
\[ H \times Y \to Y \]
\[ X / G \to Y / H \]
\[ \Rightarrow \text{map of groupoids} \quad \Rightarrow \text{map of stacks} \]

so map is representable iff \( X \times H / H \to Y \)
A map is representable iff $G$ acts freely on $X \times H$.

E.g., if $G \rightarrow H$, then

$$X/G \cong (X \times H)/H$$

And the map is modeled by the $H$-equivariant map $X \times H \rightarrow Y$

$$(x, h) \mapsto h \cdot f(x)$$

**Cor:** $X/G$ is geometric for $X$ separated.

**Ref:**

$X/G \cong X \times (G \times G)/G \times G \cong X \times \mathbb{A}^1 \times G/G \times G$

where $(g_1, g_2) \cdot (h, 1, x) = (g_1 h, g_2, x)$

$\sim (ghq_2^{-1}, 1, g_2 x)$

Diagonal is presented by $G \times G$-equivariant map

$$G \times X \rightarrow X \times X$$

$$(h, x) \mapsto (x, hx)$$

Preimage over any $\text{Spec}(R) \rightarrow X \times X$ is
Theorem 1.1 Let \( X \) be a normal noetherian algebraic stack (over \( \mathbb{Z} \)) whose stabilizer groups at closed points of \( X \) are affine. The following are equivalent.

1. \( X \) has the resolution property: every coherent sheaf on \( X \) is a quotient of a vector bundle on \( X \).
2. \( X \) is isomorphic to the quotient stack of some quasi-affine scheme by an action of the group \( GL(n) \) for some \( n \).

For \( X \) of finite type over a field \( k \), these are also equivalent to:
3. \( X \) is isomorphic to the quotient stack of some affine scheme over \( k \) by an action of an affine group scheme of finite type over \( k \).