Outline:
1) Basic facts about algebraic groups
2) Basic facts about group actions on quasi-projective schemes

0. Basic facts about linear algebraic groups

\( k \) a field

\( G \) is a smooth affine group scheme over \( k \)

\[
\begin{align*}
G \times G & \rightarrow G \\
\text{e} : pt & \rightarrow G
\end{align*}
\]

[commutative Hopf algebra \( k[G] \)]

\[
\begin{align*}
k[G] & \rightarrow k[G] \otimes k[G] \\
k[G] & \rightarrow k[a]
\end{align*}
\]

Remark: also characterized via functor of points vs functor \( \text{Rings} \rightarrow \text{groups} \)

Remark: we also discuss group schemes sometimes

\[
\begin{align*}
\text{Eg } \mathbb{G}_{\text{Ln}} & = \text{Spec}(k[x_{ij}][\frac{1}{\det}]) \\
\text{Eg } \mathbb{G}_{m} & = \text{Spec}(k[t^{\pm}]), \quad \Delta : k[t^{\pm}] \rightarrow k[t_{1}^{\pm}] \otimes k[t_{2}^{\pm}] \\
\text{split torus } (\mathbb{G}_{m})^{n}
\end{align*}
\]
split torus \((\mathbb{G}_m)^n\)

E.g. \(G\) is a torus if \(\mathbb{G}_m\) is a torus

Deligne torus: Weil restriction of \((\mathbb{G}_m)_C\) along \(\text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{R})\)

\(\mathcal{A}(\mathbb{R}) = \mathbb{C}^x\), \(\mathcal{A}_C = \mathbb{G}_m \times \mathbb{G}_m\)

Weil restriction compatible w/ base change \(\Rightarrow \mathcal{A} = \text{Spec}(\mathbb{C}[z, \overline{z}]^{\mathbb{C}})\)

takes disjoint unions to products

Generalization - A geometrically reduced finite commutative \(k\)-algebra can\(\text{ Weil restrict } (\mathbb{G}_m)_A \text{ along } \text{Spec}(A) \to \text{Spec}(k),\)

\(\Rightarrow\) Note a action gives embedding \(A^* \hookrightarrow GL(A)\), max'l torus

\(\underline{\text{Representations:}}\) Correct notion is comodule over \(k[G]\):

\[
G \times V \to V \quad \xrightarrow{\otimes} \quad \text{Sym}(V^*) \to \text{Sym}(V^*) \otimes k[G]
\]

\(\xi: V^* \xrightarrow{k\text{-linear}} V^* \otimes k[G] \hookrightarrow \quad \) (must lie in \(V^*\) because geometric points of \(k[G]\) must induce linear maps)

Comodule axioms:
1) \(V^* \to V^* \otimes k[G] \xrightarrow{\otimes} V^* \) is id
2) \(V^* \xrightarrow{\otimes 1} V^* \otimes k[G] \)

\(\xi \downarrow \quad \quad \quad \quad \quad \quad \downarrow \xi \otimes 1\)

\(V^* \otimes k[G] \to V^* \otimes k[G]\) at \(\xi\).
\[ V^* \otimes k[G] \rightarrow V^* \otimes k[G] \otimes k[G] \]

**Ex.** The category of \( G_m \)-modules is equivalent to graded vector spaces over any field of any dimension (for a split torus). **M-graded**

**Ex.** Representations of Deligne-torus are real Hodge structures

**Prop:** Every representation is a union of finite sub reps

**Pf:** write \( \rho(v) = \sum v_i \otimes e_i \) basis for \( k[G] \)

define \( \Delta(e_i) = \sum r_{ijk} e_j \otimes e_k \), \( r_{ijk} \in k \)

associativity \( \Rightarrow \sum r_{ijk} (v_i \otimes e_j \otimes e_k) = \sum_{k} \rho(v_k) \otimes e_k \)

\( \Rightarrow \rho(v_k) = \sum_{i,j} r_{ijk} (v_i \otimes e_j) \Rightarrow \xi v_i \xi^3 \) span a submodule containing \( v \)

**Thm:** Main structural properties

a) \( G \hookrightarrow GL_n \) for some \( k \) (Find a finite subrep of regular rep, containing set of gens)

b) Jordan decomposition for any \( g \in G(\bar{k}), \ g = g_{ss} \cdot g_u \)

\( g_{ss} \) unipotent and then commute
\[ g^s \in G(k) \quad g = g_{ss} \cdot g_u \quad \text{, semistable, unipotent, and they commute} \]

Functional defined in some (and all) linear embeddings

c) \exists maximal connected solvable \( B \hookrightarrow G \Leftrightarrow G/B \) is projective

(in fact \( B \) over \( \mathbb{F}_q \) unique up to conjugacy (Remark on parabolics)

\( \text{d) } \exists \) maximal torus \( T \hookrightarrow G, \quad T_k \hookrightarrow G_k \) is maximal as well, and unique up to conj.

e) \exists maximal unipotent comm. normal subgroup \( R_u(G) \hookrightarrow G \rightarrow H \) \( \text{“unip. radical”} \)

\( \text{Defn} \# G \) reductive \( \iff R_u(G) = 1 \)

\( \text{f) } \) Reductive \( \iff \) linear reductive meaning the category of \( \text{ext} \)'s has \( \text{H} \)-higher (restrictions to torus \( \times \) semisimple)

In char 0: Nagata, conn component \( G_0 \cong (G_\mathfrak{m})^n \) and \( |G/G_0| \) prime to \( p \)

Finite dimensional lin. reps in char 0 are classified by their characters \( \quad \text{irreps by highest weight.} \)
Finite dimensional lin. reps in char 0 are classified by their characters as rep's by highest weight.

**Remark**: If $G$ is split-reductive, any parabolic is standard meaning there is some $P < G$ such that $P = G \cap \mathfrak{g}$ block upper triangular matrices w.r.t. corresponding filtration

Rem's over $F$, $G$ is rational as a variety
Ref's: Borel, linear alg. groups
      Conrad, beginning of reductive gp. schemes
      Milne, algebraic groups

Group actions on schemes:

Def: \( G \times X \rightarrow X \) satisfying axioms:

1) identity \( X \rightarrow G \times X \rightarrow X \)

2) associativity

Important map \( G \times X \rightarrow X \times X \) (or more generally \( \psi_f : G \times T \rightarrow X \times T \))

for any \( T \)-point \( f : T \rightarrow X \)

E.g., for a \( k \)-point, get \( \psi : G \rightarrow X \) as its image is the
orbit, fiber over \( X \) is \( \text{Stab}_x \subset G \), a sub gp

\( \rightarrow \) more generally

\( \text{Stab}_f \rightarrow G \times T \)

\( f \cdot \quad \downarrow \quad \downarrow \psi \)

\( T \rightarrow (f,1) \quad \rightarrow X \times T \)

get inertia group, \( I_X \) as a group scheme

over \( X \)

giving stabilizer of each point

\( \text{Rem}: \) Fiber dimension of \( I_X \rightarrow X \) is

upper semi-continuous because
Kem's fiber dimension of $\mathbf{1}_X \to X$ is upper semi-continuous because it is a locally finite type group scheme.

Most concrete: Actions on affine scheme $X = \text{Spec} A$ is equivalent to giving $k[G]$-comodule structure on $A$ such that $\rho : A \to A \otimes k[G]$ is a map of algebras.

**Lemma:** If $A$ is finite type over $k$, there is a linear rep. $V$ and an embedding $\text{Spec} A \subseteq V$ which is $G$-equivariant.

$\mathcal{P}^G_{\text{ideal}} V$ is the dual of a finite diml invariant sub-rep of $A$ which generates $A$ as an algebra.

**Example:** An affine scheme with $T$ action, where $T$ is split torus, is equiv. to an $M$-graded algebra, where $M$ is char lattice.

**Useful Fact:** Every invariant ideal is gen. by homogeneous elements.

**Remark:** Matsushima's theorem: If $G$ is a reductive algebraic group and $H \subseteq G$ an algebraic $G$-subgroup, then $H$ reductive $\iff G/H$ is affine.
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(Alper has proves the same result for lin. reductive fppf group schemes)

transitivity

Then choose $G \rightarrow GL_n$, then $GL_n$ acts on $X = GL_n/G = \text{Spec} A$

by above, have closed imm. $\text{Spec}(A) \hookrightarrow V$ for some $GL_n$-rep $V$

$\Rightarrow$ A reductive $G$, $\exists$ a rep of $GL_n$ and a $v \in V$ with closed orbit such that $G = \text{stab}(v)$

This statement (except for $v$ having closed orbit) is true for any $G$.

Next level of complexity: $V$ linear rep of $G \rightarrow$ action on $\mathbb{P}(V)$

Consider $X \rightarrow \mathbb{P}(V)$ which is equivariant

(rem: we will see later that this is true for any normal proj. variety)

($G \times X \rightarrow \mathbb{P}(V)$ factors through $X$, in which case it does uniquely and induces an action on $X$)

Eg: Homogeneous spaces $G/H$ are always $G$-quasi-projective
Existence of quotients: We don’t quite have the technology yet, but there
is a very general

Thm: If $G \times X \to X \times X$ is a closed immersion, then the sheaf $\mathcal{O}_X$
is an algebraic space

Thm: every algebraic space has an affine open subspace

\[ \Rightarrow \text{For free actions, you can at least form a quotient}\]

\[ \text{for some open subscheme}\]

Special results for $T$-varieties:

Thm (Sumihiro) Let $X \to \mathbb{P}(V)$ be a $T$-equivariant $q$-proj subvariety, then for any point $x \in X$, there exists a $T$-equivariant affine open

containing $x$.

Proof. Case $k = \mathbb{C}$ $X \to \mathbb{P}(V)$ closed $\Rightarrow$ use standard open $\mathbb{P}(V)_f \cap X$ where
Let $T$ act on $X \hookrightarrow \mathbb{P}(V)$, then the functor $R^i \mapsto \text{Map}_T(\text{Spec} R, X)$ is representable by a closed subscheme of $X$.

It is smooth if $X$ is smooth.
\textbf{PF:} can reduce to affine case using Sumihiro, and suffices to assume }k=k, \text{ so } T \text{ is linearly reductive.}

A is } M\text{-graded algebra, let } I \subseteq A \text{ be the ideal gen by } \bigoplus_{x \in M / \mathfrak{g}_0^2} A_x.

Then } Z = \text{Spec}(A/I) \text{ represents the functor. It is smooth because can show that } T \times Z = (T \times X)^T \text{, and can lift non-invariant elements of } m_x / m_x^2 \text{ to elements of } m_x \text{ which define } Z \text{ as a transverse intersection at } x.

\textbf{a)

Thm (Bialynicki-Birula): } X \to P(U) \text{ is } G_m \text{-proj. then the functor}

\begin{align*}
1) \quad & R \mapsto \text{Map}_{G_m}(\text{Spec}(R) \times A^1, X) \text{ is representable by a scheme} \\
2) \quad & \text{the restriction } \text{Map}_{G_m}(\text{Spec}(R) \times A^1, X) \hookrightarrow \text{Map}(\text{Spec}(R) \times \mathbb{G}_m, X) \text{ is an embedding.} \\
3) \quad & \text{if } X \text{ is smooth, then } \text{Map}_{G_m}(\text{Spec}(R) \times A^1, X) \xrightarrow{\sim} \text{Map}_{G_m}(\text{Spec}(R) \times \mathbb{G}_m, X) \text{ is a locally trivial bundle of affine spaces.}
\end{align*}
is a locally trivial bundle of affine spaces

Proof (Idea) Similar to last, can reduce to the affine case. In that case $\text{Spec}(A/I^+)$ represents the functor, where $I^+$ is the ideal generated by elements of positive weight.

Example: $P(V)$ with a linear action of $\mathbb{G}_m$

$\leadsto$ choose eigenbasis, i.e. coordinates s.t. $\mathbb{G}_m$-action is

$[t^{a_0}z_0: \cdots : t^{a_n}z_n]$ with $a_0 \leq \cdots \leq a_n$

$\leadsto$ then fixed loci are eigenspaces for each $a$

$\leadsto$ BB strata look like $[0: \cdots : 0 : 1 : * : * : \cdots]$