Calculus 1 Practice Problems

Alex Cowan
cowan@math.columbia.edu

1.* The number of prime numbers less than \( x \) is well approximated by

\[
\int_2^x \frac{dt}{\log t}.
\]

How many prime numbers would you expect to find between \(10^9\) and \(10^9 + 10^7\)?

2. Define

\[
f(x) := \int_{-100}^x e^{-\cos t} (t^2 - 4) \, dt.
\]

For what \( x \) does \( f \) attain a local minimum?

3. Write down a Riemann sum which estimates the value of

\[
\int_{-10}^2 e^{-x^2} \, dx
\]
reasonably well.

4.*

a) A certain random number generator generates

- the number \( \frac{1}{3} \) with probability \( \frac{(\frac{1}{3})^2}{(\frac{1}{3})^2 + (\frac{2}{3})^2 + 1^2} \)
- the number \( \frac{2}{3} \) with probability \( \frac{(\frac{2}{3})^2}{(\frac{1}{3})^2 + (\frac{2}{3})^2 + 1^2} \)
- the number 1 with probability \( \frac{1^2}{(\frac{1}{3})^2 + (\frac{2}{3})^2 + 1^2} \)

If you generate 1000 numbers from this random number generator and add them up, what number do you expect to get?

b) A certain random number generator generates

- the number 0.25 with probability \( \frac{0.25^2}{0.25^2 + 0.3^2 + 0.75^2 + 1^2} \)
- the number 0.5 with probability \( \frac{0.5^2}{0.25^2 + 0.3^2 + 0.75^2 + 1^2} \)
- the number 0.75 with probability \( \frac{0.75^2}{0.25^2 + 0.3^2 + 0.75^2 + 1^2} \)
- the number 1 with probability \( \frac{1^2}{0.25^2 + 0.3^2 + 0.75^2 + 1^2} \)
If you generate 1000 numbers from this random number generator and add them up, what number do you expect to get?

c) For large $n$, A certain random number generator generates

- the number $\frac{1}{n}$ with probability $\frac{(\frac{1}{n})^2}{(\frac{1}{n})^2+(\frac{2}{n})^2+...+1^2}$
- the number $\frac{2}{n}$ with probability $\frac{(\frac{2}{n})^2}{(\frac{1}{n})^2+(\frac{2}{n})^2+...+1^2}$
- ...
- the number 1 with probability $\frac{1^2}{(\frac{1}{n})^2+(\frac{2}{n})^2+...+1^2}$

If you generate 1000 numbers from this random number generator and add them up, what number do you expect to get, approximately?

d) For large $n$, a certain random number generator generates

- the number $\frac{1}{n}$ with probability $\frac{f(\frac{1}{n})}{f(\frac{1}{n})+f(\frac{2}{n})+...+f(1)}$
- the number $\frac{2}{n}$ with probability $\frac{f(\frac{2}{n})}{f(\frac{1}{n})+f(\frac{2}{n})+...+f(1)}$
- ...
- the number 1 with probability $\frac{f(1)}{f(\frac{1}{n})+f(\frac{2}{n})+...+f(1)}$

What is the average value of the output of this random number generator, approximately?

5.* What is the average value of the function $\sin x$ on the interval $[0, \pi]$?

6. Define $f_n(x) := (1 + x)^{\frac{1}{n}}$.
Estimate $f_n(-0.3)$. (Your answer will depend on $n$.)

7.* In this problem we will find a good estimate for $\sqrt{12345}$.

a) Estimate $\sqrt{12345}$ using linear approximation with $f(x) = \sqrt{x}$ and take $a_0 = 10000$ to be your base point. Write down your answer as a single decimal number which you think is correct to at least the 10s place.

b) Let $y_0$ be the answer you found in part a), truncated at the 10s place (meaning replace all digits after the 10s place with zeros). Define $a_1$ to be $y_0^2$. Compute $a_1$ and then approximate $\sqrt{12345}$ using linear approximation with $f(x) = \sqrt{x}$ and $a_1$ as your base point. Write your answer as a single decimal number which you think is correct to at least the 1s place.

c) Let $y_1$ be the answer you found in part b), truncated at the 1s place. Define $a_2$ to be $y_1^2$. Compute $a_2$ and then approximate $\sqrt{12345}$ using linear approximation with $f(x) = \sqrt{x}$ and $a_2$ as your base point.

8.* Estimate $\sqrt{2019}$.

9. Estimate $\frac{9}{10} \log \frac{9}{10}$.
10. Estimate \( \arctan(100) \).

11. Minimize the function \( x^2 + y^2 \) given the constraint \( x^2 + xy = 2 \). Maximize the function \( x^2 + y^2 \) given the constraint \( x^2 + xy = 2 \).

12. Find all the local and global extrema of the following functions:
   a) \( \log \frac{x}{x} \) on the interval \((0, 5]\).
   b) \( \arctan(2x + \frac{1}{x}) \) for \( x \in \mathbb{R}, x \neq 0 \).
   c) \( \arctan(\log|x^2 - x - 1|) \) for \(-2 < x \leq 4\).
   d) \( \begin{cases} |x|, & -2 < x < 2, x \neq 0 \\ 1, & x = 0 \end{cases} \).

13. a) State the definition of a global maximum, a global minimum, a local maximum, and a local minimum.
   b) Give an example of a function \( f : [0, 1] \to [0, 1] \) which has no global extrema, and only has local extrema at \( x = 0 \) and \( x = 1 \).
   c) Give an example of a continuous function \( f : \mathbb{R} \to \mathbb{R} \) which has at least one local max, at least one local min, and for which \( f'(x) = 0 \) has no solutions.
   d) Give an example of a function \( f : \mathbb{R} \to \mathbb{R} \) which is everywhere differentiable and has no extrema of any kind, but for which there exist distinct \( x_1 \) and \( x_2 \) such that \( f'(x_1) = f'(x_2) = 0 \).

14. Alice is running down the street. Her position is given by
   \[ s(t) := \frac{1}{t}. \]
   a) What is Alice’s average velocity between \( t = 1 \) and \( t = 1.1 \)?
   b) Explain why Alice’s velocity at \( t = 1 \) is defined to be \(-1\).

15. Suppose \( f(3) = 7 \) and \( f(3.03) = 6.99 \). Guess the equation of the line tangent to the curve \( y = f(x) \) at \( x = 3 \).

16. Use the definition of the derivative to prove that \( \frac{d}{dx} x^2 = 2x \).

17. A 10-meter ladder is leaning against the wall of a building, and the base of the ladder is sliding away from the building at a rate of 3 meters per second. How fast is the top of the ladder sliding down the wall when the base of the ladder is 6 meters from the wall?
18. Suppose \( x \) and \( y \) are related via the equation \( x^2 \cos y + 3^y = \frac{2\sqrt{7}}{\pi^2} + \sqrt{3} \), and that \( \frac{dy}{dt} = 2 \). Find \( \frac{dx}{dt} \) when \((x, y) = \left( \frac{2}{\pi}, \frac{\pi}{4} \right) \).

19. Suppose \( p \) and \( q \) are related via the equation \( q \sin(p^2q^2) = p \). At the point \((p, q) = \left( \frac{\pi}{2}, \frac{3}{4} \right) \) it is known that \( \frac{dp}{dt} = 3 \). Find \( \frac{dq}{dt} \) at this point.

20. If \( x \) and \( y \) are related via the equation \( x^2y + 2^y = 80 \), find \( \frac{dx}{dy} \) at the point \((2, 3)\).

21. Let \( f \) be the function depicted below.
   a) Give all values of \( x \) for which \( f'(x) = 0 \) (approximately).
   b) Give all values of \( x \) for which \( f''(x) = 0 \) (approximately).

22. Sketch \( \arctan(x^2) \).

23. Let \( f(x(t)) \) be a differentiable function. Suppose that \( x(t) = e^t \) and that \( \frac{df}{dx} = \frac{1}{1+e^{2t}} \). What is \( \frac{df}{dt} \)?

24. True or false: \((2^3)' = 9 \cdot 2^8\).
25. True or false: \((x^x)' = x \cdot x^{x-1}\).

26. The Lambert W-function is the inverse function of the function \(z \mapsto z e^z\). Prove that
\[
\frac{dW}{dz} = \frac{1}{z + e^{W(z)}},
\]
where \(W\) is the Lambert W-function.

27. Show that
\[
\int_0^\infty \frac{dx}{x^2 + 0.0001 \log \log x} < \frac{1}{M}.
\]

28. True or false:
   a) \(\int_0^1 \sqrt{1 + x^2} \, dx = \sqrt{2} - 1\).
   b) \(\int x^2 \cos x \, dx = \frac{1}{3} x^3 \sin x + C\).

29. True or false: There exists a function \(f\) and real numbers \(a < b < c\) such that \(f\) has a vertical asymptote at \(b\) and \(\int_a^c f(x) \, dx\) exists.

30. a) Show that the function \(x^5 - 2x^2 + x - 1\) has a root in the interval \([0, 2]\).
    b) Explain why every 5th degree polynomial has at least one real root.

31. Prove that there’s a real number \(x\) such that \(2^x = 50x - 235\).

32. A Tibetan monk leaves the monastery at 7:00 am and takes his usual path to the top of the mountain, arriving at 7:00 pm. The following morning, he starts at 7:00 am at the top and takes the same path back, arriving at the monastery at 7:00 pm. Show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

33. Let \(f: \mathbb{R} \to \mathbb{R}\) be a function whose derivative is continuous everywhere.
   a) Suppose there exist two points \(x_0\) and \(x_1\) with \(x_0 < x_1\) and \(f(x_0) > f(x_1)\). Prove that there exists an \(x^*\) such that \(f'(x^*) < 0\).
   b) Suppose that \(f'(x) > 0\) for all \(x\) in the interval \((a, b)\) Using the result from part a), prove that for all \(c_1\) and \(c_2\) in \((a, b)\), if \(c_1 < c_2\) then \(f(c_1) < f(c_2)\).
   c) Suppose that \(f'(x) > 0\) for all \(x\) in the interval \((a, b)\) Using the fundamental theorem of calculus, prove that for all \(c_1\) and \(c_2\) in \((a, b)\), if \(c_1 < c_2\) then \(f(c_1) < f(c_2)\).

34. Let \(f(x) = 2x + 1\) if \(x < 1\) and \(-x^2 + ax + b\) if \(x \geq 1\). For what choice(s) of \(a\) and \(b\) will \(f\) be:
   a) Continuous at \(x = 1\)?
   b) Differentiable at \(x = 1\)?

35. Evaluate the following limits:
a) \[ \lim_{x \to -1} \frac{x^2 - 1}{x^2 + 3x + 2} \]

b) \[ \lim_{x \to \infty} \frac{3x^5 + 6x}{4x^3 - 3x^2 + 2} \]

c) \[ \lim_{x \to \pi} \frac{\tan(x)}{x} \]

d) \[ \lim_{x \to 0} \frac{e^x - 1 - x}{\cos(x) - 1} \]

e) \[ \lim_{x \to 0} \frac{\cos(5x) - 1}{2^x - 1 - x \log 2} \]

f) \[ \lim_{x \to 0} \frac{x^2 e^x}{e^{3x} - 1 - 3x} \]

g) \[ \lim_{x \to \infty} x^4 e^{-x} \]

h) \[ \lim_{n \to \infty} \begin{cases} 1 + \frac{1}{n} & \text{if } n \text{ is even} \\ 1 - \frac{1}{n} & \text{if } n \text{ is odd} \end{cases} \]

i) \[ \lim_{h \to 0} \frac{1}{h} \int_{2}^{2+h} \tan \sqrt{7} \, dt \]
36. Differentiate the following functions with respect to $x$:

a) $\frac{1}{x^{\frac{5}{3}}} + 17 \cdot 10^x + t$

b) $(\log(x^3 + 1))^4$

c) $x^2e^x \sin x$

d) $\frac{x}{x^2 + 1}$

e) $(\log x + x^4 + 1)^{\frac{3}{2}} + ((\arctan x)^2 - 2x + 3)^2 - x + 88$

f) $2x^2 \arcsin(3^x)$

g) $\frac{4x^2}{\sqrt{e^{2x^2} - 2}}$
37. Evaluate the following integrals:

a) \[ \int_{-2}^{2} (3x^5 + 2x^3 - x + 1) \, dx \]

b) \[ \int xe^{\arcsin r} \, \sqrt{1 - r^2} \, dr \]

c) \[ \int_{3}^{4} 2^{2x} \, 2^y \, dy \]

d) \[ \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2 \sin(sin \, u) \cos \, u \, du \]

e) \[ \int \frac{\sqrt{\log t}}{t} \, dt \]

f) \[ \int_{-\infty}^{\infty} \frac{dx}{1 + 4x^2} \]

g) \[ \int_{-\infty}^{\infty} \sin x \, dx \]

h) \[ \int_{-\infty}^{\infty} \frac{\sin x}{e^{-x^2}} \, dx \]

i) \[ \int_{0}^{1} \sqrt{s} \, ds \]

j) \[ \int_{1}^{\infty} t^{-t} \, dt \]