Lecture 4

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Macdonald

↑

Schur

↑

GUE

• GUE corner process

GUE M: $N \times N$ Hermitian with $\begin{cases} \text{complex off diag Gaussian} \\ \text{real on diag Gaussian} \end{cases}$

Call $\lambda^{(N)} = (\lambda^{(N)}_N \le \cdots \le \lambda^{(N)}_1) = \text{eig}(M)$

$\lambda^{(k)} = (\lambda^{(k)}_k \le \cdots \le \lambda^{(k)}_1) = \text{eig}\left[M^{[1,k]} \text{ corner}\right]$

λ satisfy Weyl inequalities $\quad \lambda^{(N)} \succeq \lambda^{(N-1)} \succeq \lambda^{(N-2)} \succeq \cdots \succeq \lambda^{(1)}$

$$\begin{array}{ccccccc}
\lambda^{(N)}_N & & \lambda^{(N)}_{N-1} & \cdots & & & \lambda^{(N)}_1 \\
& \le & & \le & & & \le \\
& & \lambda^{(N-1)}_{N-1} & \cdots & & & \lambda^{(N-1)}_1 \\
& & & \ddots & & \le & \\
& & & \le \; \lambda^{(1)}_1 \; \le & & &
\end{array}$$

Vandermonde det.

$$\mathbb{P}(\lambda^{(N)}) = Z^{-1} \cdot \frac{V(\lambda^{(N)})^2}{e^{\Sigma(\lambda^{(N)}_i)^2}} \qquad V(\lambda) := \prod_{i<j} (\lambda_i - \lambda_j)$$

Weyl denominator formula

<u>GUE corner process</u>

$$\mathbb{P}(\lambda^{(N)}, \ldots, \lambda^{(1)}) = \mathbb{P}(\lambda^{(N)}) \cdot \frac{\mathbb{1}_{\lambda^{(N)} \succeq \cdots \succeq \lambda^{(1)}}}{V(\lambda^{(N)})}$$

[HW: $\int \mathbb{1}_{\lambda^{(N)} \succeq \cdots \succeq \lambda^{(1)}} \, d\lambda^{(1)} \cdots d\lambda^{(N-1)} = V(\lambda^{(N)})$]

- What to do

1. Compute correlation functions (using det = V) in terms of minors of correlation kernel (Dyson)

   Determinantal point process $\rightsquigarrow$ relatively easy asymptotics.

2. Introduce dynamics on triangular array which preserves corners process / GUE (up to scaling)

Dyson's Brownian Motion (DBM):

<u>Generator</u> $L = V^{-1} \Delta V$ $\qquad$ $\Delta \sim$ Dirichlet Laplacian

$$\left((Lf)(\lambda) = V^{-1}(\lambda)\ \Delta\big(V(y)f(y)\big)\Big|_{y=x}\right.$$

DBM pushes GUE measure to scaled (by $t^{1/2}$) version.

Can imagine running DBM for each level, but how to couple them to preserve corners process (and interlacing)?

- **Warren's Process** (Not same as eig of evolving GUE matrix)

$\lambda^{(1)} \sim BM$

$\lambda^{(2)}_2 ; \lambda^{(2)}_1 \sim$ BM's reflected on left / right of $\lambda^{(1)}_1$

so on (reflect off lower particles to stay ordered)

<u>Thm</u>: Push forward of GUE corner process after time t is just (marginally) scaled ($t^{1/2}$) version of process.

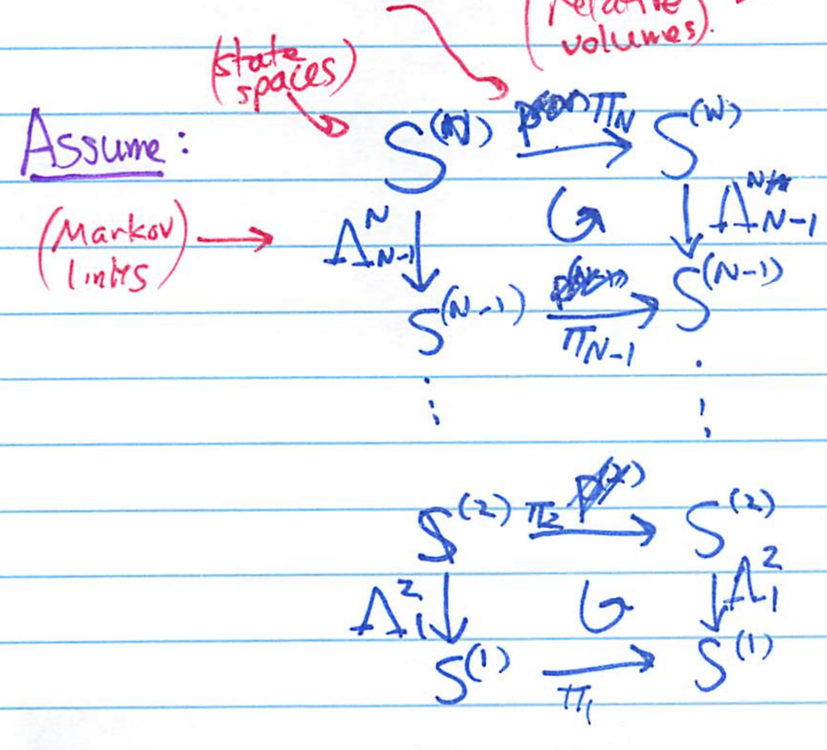
A discrete time version: (Diaconis-Fill / Borodin-Ferrari)

Note $\mathbb{P}(\lambda^{(N)}, \dots, \lambda^{(1)}) = M(\lambda^{(N)}) \cdot \Lambda^{N}_{N-1}(\lambda^{(N)}, \lambda^{(N-1)}) \cdots \Lambda^{2}_{1}(\lambda^{(2)}, \lambda^{(1)})$ (GUE measure: M)

Takes GUE (k to k-1) → $\Lambda^{k}_{k-1}(\lambda^{(k)}, \lambda^{(k-1)}) := \dfrac{V(\lambda^{(k-1)})}{V(\lambda^{(k)})} \cdot \mathbb{1}_{\lambda^{(k)} \succeq \lambda^{(k-1)}}$ (relative volumes)

Markov.

<u>Assume</u>:



- All squares commute
- $\Lambda^{k}_{k-1} : S^{(k)} \to S^{(k-1)}$ markov
- $\pi_N : S^{(k)} \to S^{(k)}$ markov

Define $P : S^{(N)} \times \cdots \times S^{(1)} \to S^{(N)} \times \cdots S^{(1)}$ Markov kernel.

$$P(\lambda \to \mu) = P_{(1)}(\lambda^{(1)} \to \mu^{(1)}) \cdot \prod_{i=2}^{N} \frac{P_i(\lambda^{(i)} \to \mu^{(i)}) \Lambda^{i}_{i-1}(\mu^{(i)} \to \mu^{(i-1)})}{(P_i \Lambda^{i}_{i-1})(\lambda^{(i)} \to \mu^{(i-1)})}$$

(sequential update)

Thm: If $P_{(N)} m = \tilde{m}$ (push-forward top measure)

then $P \cdot m(\lambda^{(N)}) \Lambda^{N}_{N-1}(\lambda^{(N)}, \lambda^{(N-1)}) \cdots \Lambda^{2}_{1}(\lambda^{(2)}, \lambda^{(1)})$

$$= \tilde{m}(\lambda^{(N)}) \Lambda^{N}_{N-1}(\lambda^{(N)}, \lambda^{(N-1)}) \cdots \Lambda^{2}_{1}(\lambda^{(2)}, \lambda^{(1)})$$

Preserves the GUE corner process!

The Warren process is ctns limit of this.

Commutation requires right choice of P, Λ in general.

This procedure will survive at Macdonald process level.

- Schur process

<u>Partition</u>: $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ $\quad \lambda_i \in \mathbb{Z}_{\geq 0}$

$|\lambda| = \sum \lambda_i$, $\ell(\lambda) = \#\{\lambda_i \neq 0\}$. Eg $(4,2,1)$.

<u>Schur symmetric polynomials</u> in $x_1, \dots x_N = X$

$$S_\lambda(X) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{i,j=1}^N}{\det\left(x_i^{N-j}\right)_{i,j=1}^N} \leftarrow V(x)$$

$\{S_\lambda : \ell(\lambda) \leq N\}$ form linear basis of sym. poly in X.

<u>Schur measure</u> (Okounkov '01) $\quad x = (x_1, \dots x_N), \ y = (y_1, \dots, y_N)$

$$\mathbb{P}_{x,y}(\lambda) = \frac{S_\lambda(x)\, S_\lambda(y)}{\Pi(x;y)}$$

supported on $\lambda : \ell(\lambda) \leq N$

- If $x_i, y_j \geq 0$, S_λ's and measure is positive

- $$\Pi(x;y) := \sum_\lambda S_\lambda(x) S_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

— Cauchy identity —

Schur process (Okounkov-Reshetikhin '03)

A measure on $\lambda^{(N)} \succeq \lambda^{(N-1)} \succeq \cdots \succeq \lambda^{(1)}$ partitions

$$\mathbb{P}(\lambda^{(N)},\dots,\lambda^{(1)}) := \frac{S_{\lambda^{(N)}}(x_1,\dots x_N)\cdot S_{\lambda^{(N)}/\lambda^{(N-1)}}(y_N)\cdots S_{\lambda^{(1)}}(y_1)}{\Pi(x;y)}$$

where

$$S_{\lambda/\mu}(u) = u^{|\lambda|-|\mu|}\cdot \mathbb{1}_{\lambda\succeq\mu}$$

Taking all x same and $y_i \equiv 1$, in limit rescale to GUE Corner.

Due to determinant formulas for S_λ can see determinantal point process structure on full triangle. (answers question 1 of how to compute)

Projection on $\lambda^{(k)} \to \dfrac{S_{\lambda^{(k)}}(x_1,\dots x_N)\, S_{\lambda^{(k)}}(y_1,\dots,y_k)}{\Pi(x_1,\dots x_N;\, y_1,\dots,y_k)}$

Question 2: Dynamics. (Example 1)

- Process can be written as

$$m(\lambda^{(N)})\, \Lambda^{N}_{N-1}(\lambda^{(N)}, \lambda^{(N-1)}) \cdots \Lambda^{2}_{1}(\lambda^{(2)}, \lambda^{(1)})$$

with

$$m(\lambda^{(N)}) := \frac{S_{\lambda^{(N)}}(X)\, S_{\lambda^{(N)}}(Y)}{\Pi(X;Y)}$$

$$* \quad \Lambda^{k}_{k-1}(\lambda, \mu) := \frac{S_{\mu}(y_1, \ldots, y_{k-1})}{S_{\lambda}(y_1, \ldots, y_k)}\, S_{\lambda/\mu}(y_k)$$

~~Define Markov Kernel~~

$\Lambda^{k}_{k-1}(\lambda, \mu)$ is markov due to "Branching rule"

$$S_{\lambda}(x_1, \ldots x_N) = \sum_{\mu \leq \lambda} S_{\mu}(x_1, \ldots x_{N-1}) \cdot S_{\lambda/\mu}(x_N)$$

$\Pi_N^{(u)}$

Markov kernel on level N

$$\Pi_N^{(u)}(\lambda^{(N)} \to \mu^{(N)}) := \frac{S_{\mu^{(N)}}(y)}{S_{\lambda^{(N)}}(y)} \cdot \frac{S_{\mu^{(N)}/\lambda^{(N)}}(u)}{\Pi(y;u)}$$

Prop: $\Pi_N^{(u)}$ pushes Schur measure on $\lambda^{(N)}$ forward to Schur measure on $\mu^{(N)}$ with $\{X\} \cup \{u\}$.

Prop: $$\sum_{\mu} \frac{S_{\mu/\lambda}(u)}{\Pi(y;u)} S_\mu(y) = S_\lambda(y)$$

so for $u \geq 0$, $S_\mu(y)$ is eigenvalue 1 eigenfunction (in μ) which is zero outside $\mu \geq \lambda$ $\rightsquigarrow$ Doob h-transform

Prop:

$$\begin{array}{ccc} \lambda^{(N)} & \xrightarrow{\Pi_N^{(u)}} & \mu^{(N)} \\ \Lambda^N_{N-1} \downarrow & \circlearrowright & \downarrow \Lambda^N_{N-1} \\ \lambda^{(N-1)} & \xrightarrow[\Pi_{N-1}^{(u)}]{} & \mu^{(N-1)} \end{array}$$

Don't write

Hw: Follows from

- $\sum_{\varkappa} S_{\varkappa/\nu}(x) S_{\varkappa/\mu}(y) = \Pi(x;y) \sum_{\tau} S_{\mu/\tau}(x) S_{\nu/\tau}(y)$
- $\sum_{\nu} S_{\varkappa/\mu}(x) S_{\mu/\tau}(y) = S_{\varkappa/\tau}(x,y)$

This implies that the sequential update dynamics may be constructed, and that they preserve class of Schur processes.

— Simulation — (discrete space Warren process)

- If we take $u = \varepsilon \searrow 0$, speed up ε^{-1} we get continuous time process. (all $y_i \equiv 1$ so each level rate 1)

- $s_\lambda(\underbrace{\varepsilon, \varepsilon, \ldots, \varepsilon}_{\tau\varepsilon^{-1}}) \to s_\lambda(\rho_\tau)$ Plancherel specialization

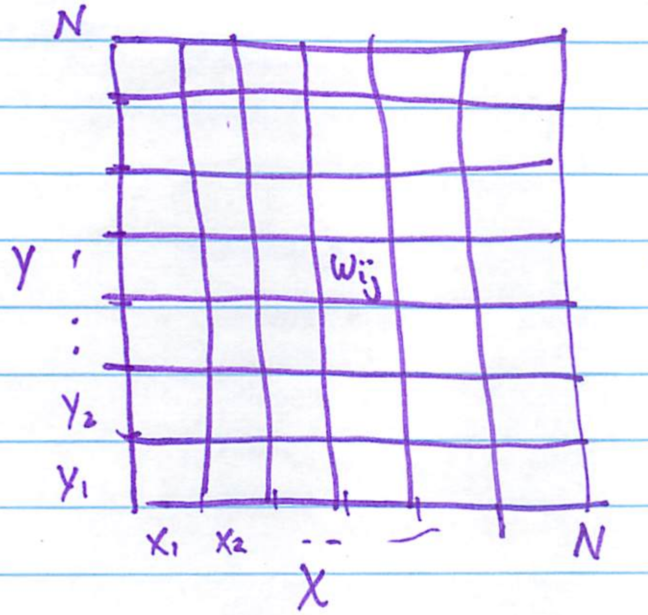
- If initially all $x_i \equiv 0$, measure supported on $\lambda^{(N)} = \cdots = \lambda^{(1)} = \emptyset$.

Simulation has done affine shift to avoid overlapping so $(0,0,0,\ldots) \to (0,-1,-2,-3,\ldots)$.

See TASEP on ~~Right~~ left edge! (step initial cond.)

pushTASEP on right edge!

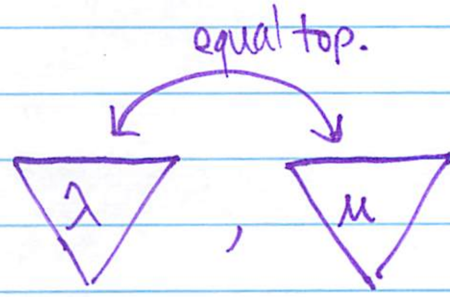
Second application LPP geo. weights



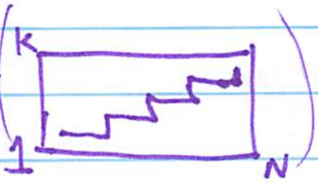
$= M$ (Semi-discrete Poisson limits possible)

$w_{ij} \in \mathbb{Z}_{\geq 0}$

RSK bijection $M \longleftrightarrow$

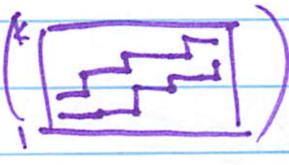


Where $\lambda_1^{(k)} = \max\Big($



$\Big)$

$\lambda_1^{(k)} + \lambda_2^{(k)} = \max\Big($



$\Big)$

$\vdots$

~~Exercise~~ Prop: If $w_{ij} \sim \mathrm{geo}(x_i y_j)$ then

$$\mathbb{P}(\lambda^{(N)}, \ldots, \lambda^{(1)}) = \frac{s_{\lambda^{(N)}}(x)\, s_{\lambda^{(N)}/\lambda^{(N-1)}}(y_N) \cdots s_{\lambda^{(2)}/\lambda^{(1)}}(y_1)}{\Pi(x;y)}$$

Schur process.

Can increase by adding column.
Give DIFFERENT Dynamic

From Schur to Macdonald

(maybe just $k=1$)

Define

$$D_k^N = t^{\frac{k(k-1)}{2}} \sum_{\substack{I \subset \{1,\dots,N\} \\ |I|=k}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q,x_i}$$

$$(T_{q,x_i} f)(x_1,\dots x_N) = f(x_1,\dots, q x_i, \dots x_N)$$

(For now set $t=q$.) Then (from defn)

Exercise $$(D_k^N s_\lambda)(x_1,\dots,x_N) = e_k(q^{\lambda_1} q^{N-1}, \dots, q^{\lambda_N} q^0)\, s_\lambda(x_1,\dots x_N)$$

$$e_k(y_1,\dots y_N) = \sum_{i_1 < \dots < i_k} y_{i_1} \cdots y_{i_k}$$

~~Since s_λ linearly span Sym poly of degree N, shows $\{D_k^N\}_{k=1}^N$ are commuting operators, ~~non~~ self adjoint in inner product in which $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda=\mu}$.~~

Notice degeneracy (for all q s_λ are eigenfunctions)

Macdonald polynomials $P_\lambda(x_1,\dots,x_N) \in \mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ with partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$\left(\mathcal{D}_N^1 f\right)(x_1,\dots,x_N) = \sum_{i=1}^{N} \prod_{j\neq i} \frac{t x_i - x_j}{x_i - x_j} f(x_1,\dots,q x_i,\dots,x_N)$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_N^1 P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \dots + q^{\lambda_N}) P_\lambda .$$

They have many remarkable properties that include orthogonality (dual basis Q_λ), simple reproducing kernel (Cauchy type identity), Pieri and branching rules, index/variable duality, explicit generators of the algebra of (Macdonald) operators commuting with $\mathcal{D}_N^1$, etc.

Reproducing kernel (Cauchy type identity)

$$\Pi(a_1,\dots,a_N;b_1,\dots,b_M) := \sum_{\lambda^{(N)}} P_{\lambda^{(N)}}(a_1,\dots,a_N)\, Q_{\lambda^{(N)}}(b_1,\dots,b_M)$$

$$= \prod_{i,j} \frac{(t a_i b_j; q)_\infty}{(a_i b_j; q)_\infty}$$

If $b_i \equiv \frac{\gamma}{M}$ and $M \to \infty$ then $\Pi(a;b) \to e^{\gamma a_1} \cdots e^{\gamma a_N}$ (Plancherel)

At $q=t$ reduces to Schur function Cauchy identity

$$\sum_{\lambda} S_\lambda(a) S_\lambda(b) = \prod_{i,j} \frac{1}{1-a_i b_j}$$

(Ascending) Macdonald processes are probability measures on interlacing triangular arrays (Gelfand–Tsetlin patterns)

$$
\begin{array}{ccccccccc}
\lambda^{(N)}_N & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_2 & & \lambda^{(N)}_1 \\
& \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & \cdots & & & \lambda^{(N-1)}_1 & \\
& & \ddots & & \cdots & & \iddots & & \\
& & & \lambda^{(2)}_2 & & \lambda^{(2)}_1 & & & \\
& & & & \lambda^{(1)}_1 & & & &
\end{array}
$$

$\lambda^{(N)}_N \geq \lambda^{(N-1)}_{N-1} \leq \lambda^{(N)}_{N-1} \geq \lambda^{(N-1)}_{N-2} \ldots$, $\lambda^{(N)}_2 \geq \lambda^{(N-1)}_1 \leq \lambda^{(N)}_1$, $\lambda^{(2)}_2 \geq \lambda^{(1)}_1 \leq \lambda^{(2)}_1$

$$\lambda^{(m)}_j \in \mathbb{Z}_{\geq 0}$$

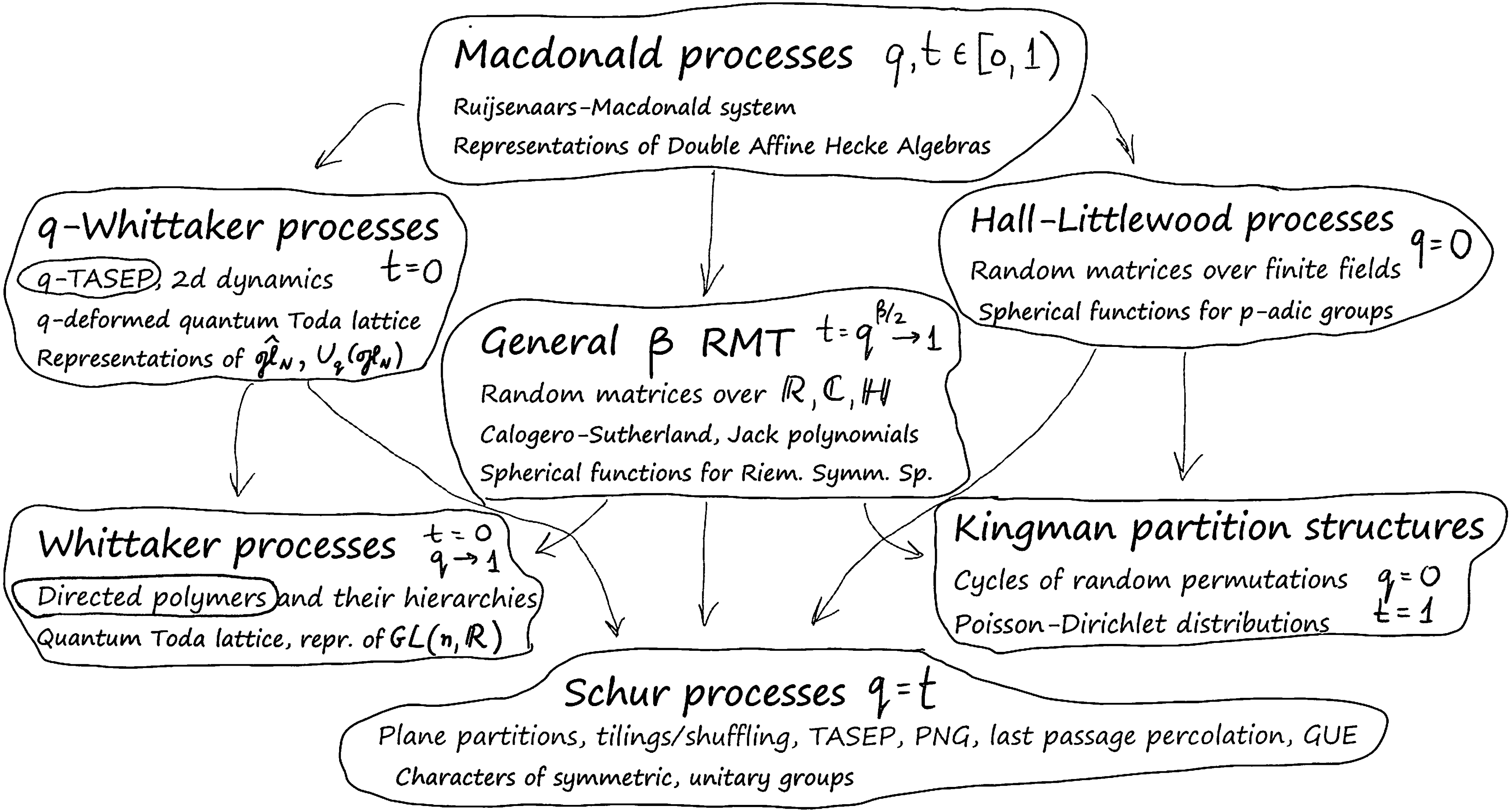
$$\mathbb{P}\left(\lambda^{(N)}\right) = \frac{P_{\lambda^{(N)}}(a_1, \ldots, a_N)\, Q_{\lambda^{(N)}}(b_1, \ldots, b_M)}{\Pi(a_1, \ldots, a_N; b_1, \ldots, b_M)}$$

$\mathbb{P}$ ← $M_N(a_1, \ldots, a_N; b_1, \ldots, b_M)$

P, Q: Macdonald polynomials

Π: normalization constant

$a_1, \ldots, a_N$; $b_1, \ldots, b_M$: two groups of parameters


Macdonald processes $q,t \in [0,1)$
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes $t=0$
q-TASEP, 2d dynamics
q-deformed quantum Toda lattice
Representations of $\widehat{gl}_N$, $U_q(gl_N)$
General β RMT $t=q^{\beta/2}\to 1$
Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. Symm. Sp.
Hall-Littlewood processes $q=0$
Random matrices over finite fields
Spherical functions for p-adic groups
Whittaker processes $t=0$, $q\to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$
Kingman partition structures
Cycles of random permutations $q=0$, $t=1$
Poisson-Dirichlet distributions
Schur processes $q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

We are able to do two basic things:

1. Evaluate averages of a rich class of observables
2. Construct explicit Markov operators that map Macdonald processes to Macdonald processes (with new parameters)

The integrable structure of Macdonald polynomials directly translates into probabilistic content.

By working at a high combinatorial level we avoid analytic issues (eventually need to work hard to take various limits).

Evaluation of averages is based on the following observation. Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

$$\mathcal{D} P_\lambda = d_\lambda P_\lambda .$$

Applying it to the Cauchy type identity $\sum_\lambda P_\lambda(a) Q_\lambda(b) = \Pi(a;b)$ we obtain

$$\mathbb{E}[d_\lambda] = \frac{\mathcal{D}^{(a)} \Pi(a;b)}{\Pi(a;b)} .$$

If all the ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. Contrast with the lack of explicit formulas for the Macdonald polynomials.

Macdonald difference operators $\{\mathcal{D}_N^r\}_{1\le r\le N}$

$$\mathcal{D}_N^r := t^{r(r-1)/2} \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}$$

$\leftarrow (T_{q,x_i} f)(x_1,\dots,x_N) = f(x_1,\dots,qx_i,\dots,x_N)$

Commuting operators all diagonalized by $\{P_\lambda\}_{\ell(\lambda)\le N}$

$$\mathcal{D}_N^r P_\lambda(x_1,\dots,x_N) = e_r(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2},\dots, q^{\lambda_N}t^0)$$

$e_r(y_1,\dots,y_N) = \sum_{i_1<\dots<i_r} y_{i_1}\cdots y_{i_r}$ "elementary symmetric polynomials

Expectations characterize Macdonald meas. (q=t ->cor kernel):

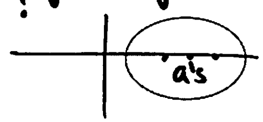
$$\mathbb{E}\Big[\prod_{i=1}^{k} e_{r_i}(q^{\lambda_i}t^{N-i})\Big] = \frac{\mathcal{D}_N^{r_1}\cdots\mathcal{D}_N^{r_k}\,\Pi(a_1,\dots a_N; b_1,\dots,b_M)}{\Pi(a_1,\dots a_N; b_1,\dots,b_M)}$$

Note $\prod(a_1,\dots,a_N;b_1,\dots,b_M) = \prod(a_1;b_1,\dots,b_M)\cdots\prod(a_N;b_1,\dots,b_M)$

Encode products of difference operators as contour integrals

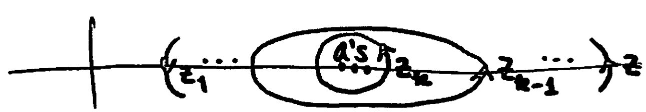
<u>Proposition [Borodin-C '11]</u>: For nice $F(u_1,\dots,u_N) = f(u_1)\cdots f(u_N)$

$$(\mathcal{D}_N^r F)(\vec{a}) = \frac{F(\vec{a})}{(2\pi i)^r r!}\int\cdots\int \det\left(\frac{1}{tz_k - z_\ell}\right)_{k,\ell=1}^{r} \prod_{j=1}^{r}\left(\prod_{m=1}^{N}\frac{tz_j - a_m}{z_j - a_m}\right)\frac{f(qz_j)}{f(z_j)}dz_j$$



Here is another example for powers of first diff. op. at t=0

$$((\mathcal{D}_N^1)^k F)(\vec{a}) = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\int\cdots\int \prod_{1\le A<B\le k}\frac{z_A - z_B}{z_A - qz_B}\prod_{j=1}^{k}\left(\prod_{m=1}^{N}\frac{a_m}{a_m - z_j}\right)\frac{f(qz_j)}{f(z_j)}\frac{dz_j}{z_j}$$

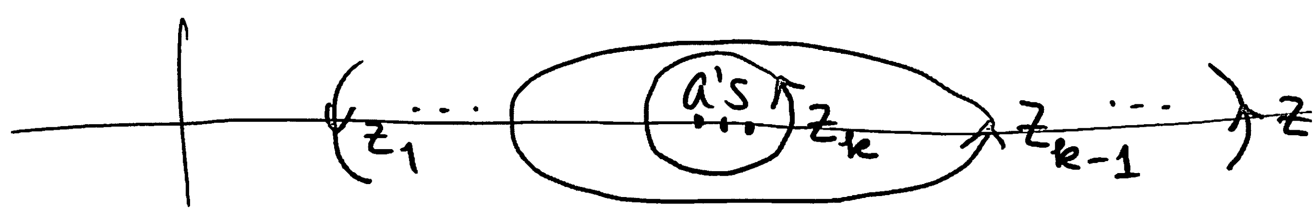


Taking $t=0$, we have that $\mathcal{D}_N^1 P_\lambda = q^{\lambda_N} P_\lambda$

So taking products of the first order Macdonald operators (on different levels) results in the integral representation

$$\mathbb{E}\left[\prod_{i=1}^{k} q^{\lambda_{N_i}^{(N_i)}}\right] \qquad (N_1 \geq N_2 \geq \cdots \geq N_k)$$

$$= \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \int \cdots \int \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left(\prod_{m=1}^{N} \frac{a_m}{a_m - z_j} \right) \frac{\Pi(q z_j ; b)}{\Pi(z_j ; b)} \frac{dz_j}{z_j}$$



Have seen how these led to the q-Laplace transform of $q^{\lambda_N^{(N)}}$

Using another operator (attributed to Noumi) diagonalized by the P_λ, [Borodin-C-Gorin-Shakirov '13] prove (q,t)-Laplace transform Fredholm determinant formula:

$$\mathbb{E}\left[\prod_{i=1}^{N} \frac{(\zeta q^{\lambda_i} t^{N-i+1}; q)_\infty}{(\zeta q^{\lambda_i} t^{N-i}; q)_\infty}\right] = \det\left(I + K^{q,t}_{\zeta,a,b}\right)$$ (Mellin-Barnes type)

At $t=0$ this reduces to the one we have already seen.

Dynamics on Gelfand–Tsetlin patterns comes from the idea of [Diaconis–Fill '90]; in Schur process case [Borodin–Ferrari '08]

<u>Branching rule</u>: $$P_{\lambda^{(N)}}(a_1,\dots,a_{N-1},a_N) = \sum_{\lambda^{(N-1)} \preceq \lambda^{(N)}} P_{\lambda^{(N-1)}}(a_1,\dots,a_{N-1})\, P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)$$

$$\lambda^{(N)}_N \le \lambda^{(N-1)}_{N-1} \le \lambda^{(N)}_{N-1} \le \lambda^{(N-1)}_{N-2} \cdots \lambda^{(N)}_2 \le \lambda^{(N-1)}_1 \le \lambda^{(N)}_1$$

Skew Macdonald polynomial

$$P_{\lambda/\mu}(u) = \begin{cases} \Psi_{\lambda/\mu}\, u^{|\lambda|-|\mu|}, & \lambda \succeq \mu \\ 0, & \text{else} \end{cases} \qquad (\Psi_{\lambda/\mu} \in \mathbb{Q}(q,t))$$

Markov kernel (stochastic link) from level N to N−1

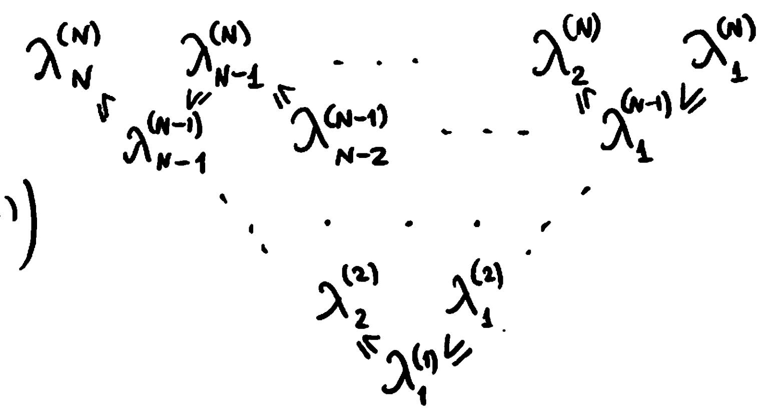
$$\Lambda^{N}_{N-1}\left(\lambda^{(N)}, \lambda^{(N-1)}\right) := \frac{P_{\lambda^{(N-1)}}(a_1,\dots,a_{N-1})\, P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)}{P_{\lambda^{(N)}}(a_1,\dots,a_N)}$$

maps $M_N(a_1,\dots,a_N; b_1,\dots,b_M)$ to $M_{N-1}(a_1,\dots,a_{N-1}; b_1,\dots,b_M)$.

Trajectory of this Markov chain defines the Macdonald process

$$M_{[1,N]}(a_1,\dots,a_N; b_1,\dots,b_M)\left(\lambda^{(N)},\dots,\lambda^{(1)}\right)$$

$$:= M_N(\lambda^{(N)})\Lambda^{N}_{N-1}(\lambda^{(N)},\lambda^{(N-1)}) \cdots \Lambda^{2}_{1}(\lambda^{(2)},\lambda^{(1)})$$



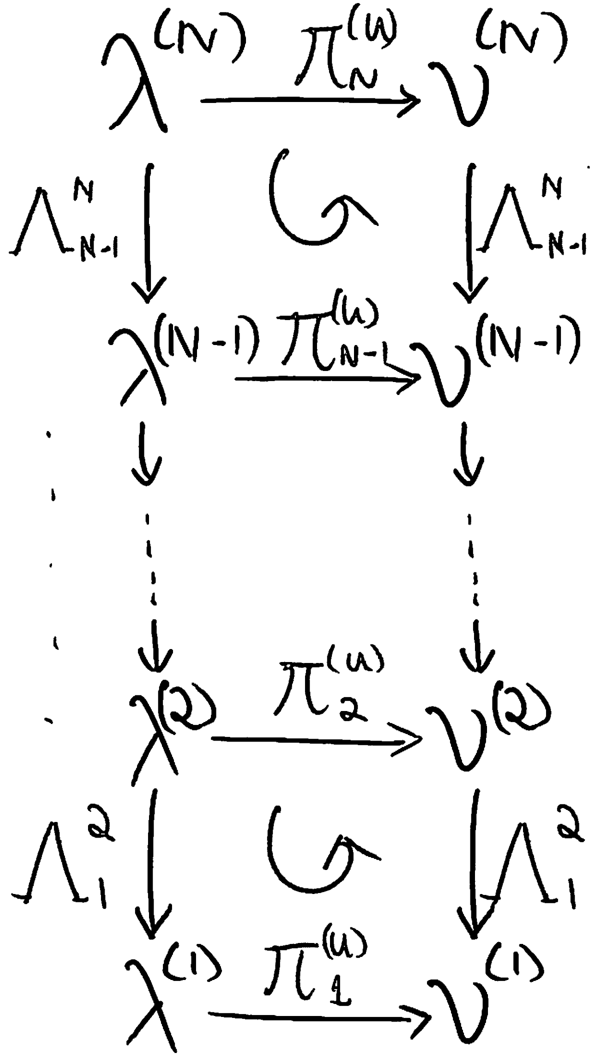
Markov kernel from level N to level N

$$\pi_N^{(u)}\left(\lambda^{(N)}, \nu^{(N)}\right) := \frac{P_{\nu^{(N)}}(a_1,\dots,a_N)}{P_{\lambda^{(N)}}(a_1,\dots,a_N)} \cdot \frac{Q_{\nu^{(N)}/\lambda^{(N)}}(u)}{\Pi(a_1,\dots,a_N;u)}$$

maps $M_N(a_1,\dots,a_N;b_1,\dots,b_M)$ to $M_N(a_1,\dots,a_N;b_1,\dots,b_M,u)$.

Note: $$\sum_{\nu} \frac{Q_{\nu/\lambda}(u)}{\Pi(a_1,\dots,a_N;u)} P_\nu(a_1,\dots,a_N) = P_\lambda(a_1,\dots,a_N)$$

so $P_\nu(a_1,\dots,a_N)$ has eigenvalue 1 and is positive inside $\nu \succcurlyeq \lambda$ and zero outside: (q,t)-deformed Dyson Brownian motion



Multivariate Markov kernel

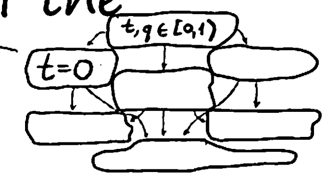
$$\mathbb{P}^{(u)}(\lambda,\nu) := \pi^{(u)}(\lambda^{(1)},\nu^{(1)}) \prod_{k=2}^{N} \frac{\pi_k^{(u)}(\lambda^{(k)},\nu^{(k)})\,\Lambda_{k-1}^{k}(\nu^{(k)},\nu^{(k-1)})}{(\pi_k^{(u)}\Lambda_{k-1}^{k})(\lambda^{(k)},\nu^{(k-1)})}$$

sequentially updates GT-pattern, mapping

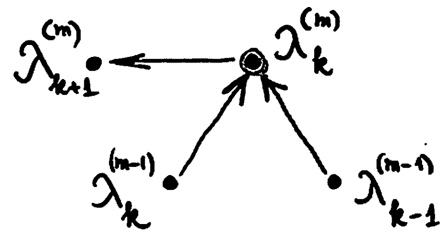
$M_{[1,N]}(a_1,\dots,a_N;b_1,\dots,b_M)$ to $M_{[1,N]}(a_1,\dots,a_N;b_1,\dots,b_M,u)$

Other dynamics also preserve class of Macdonald processes [Borodin-Petrov '13]

Here is an example of a Markov process preserving the class of the q-Whittaker processes (Macdonald processes with $t=0$).



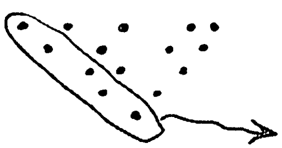
Each coordinate jumps by 1 to the right independently of the others with

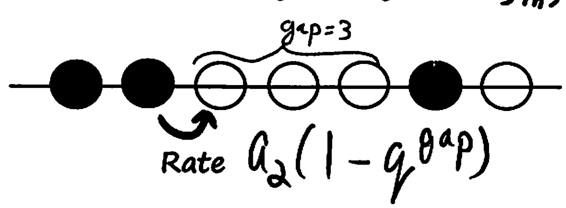


$$\mathrm{rate}\left(\lambda_k^{(m)}\right) = a_m \frac{\left(1-q^{\lambda_{k-1}^{(m-1)}-\lambda_k^{(m)}}\right)\left(1-q^{\lambda_k^{(m)}-\lambda_{k+1}^{(m)}+1}\right)}{\left(1-q^{\lambda_k^{(m)}-\lambda_k^{(m-1)}}\right)}$$

<u>Simulation</u>

The set of coordinates $\left\{\lambda_m^{(m)}-m\right\}_{m>1}$ forms q-TASEP





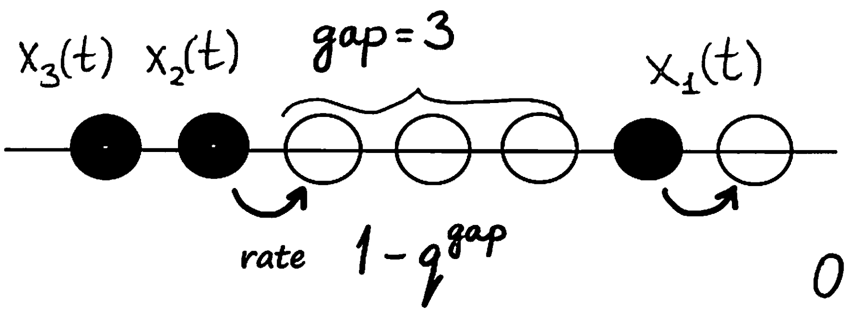
$$q^{\lambda_m^{(m)}} = q^{x_m(t)+m}$$

[O'Connell-Pei '12] give different dynamics with q-TASEP marginal

After time τ, Plancherel specialization with $\gamma = \tau$, hence

$$\Pi(a_1, \dots, a_N ; \mathrm{Plan}_\tau) = \prod_{i=1}^{N} e^{\tau a_i}$$

$a_i \equiv 1$



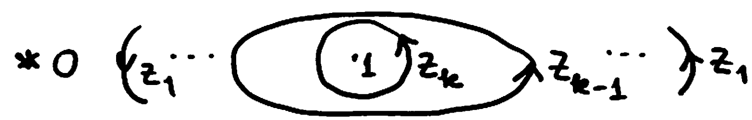
$0 < q < 1$

$$\boxed{q^{\lambda^{(m)}_m} = q^{x_m(t)+m}}$$

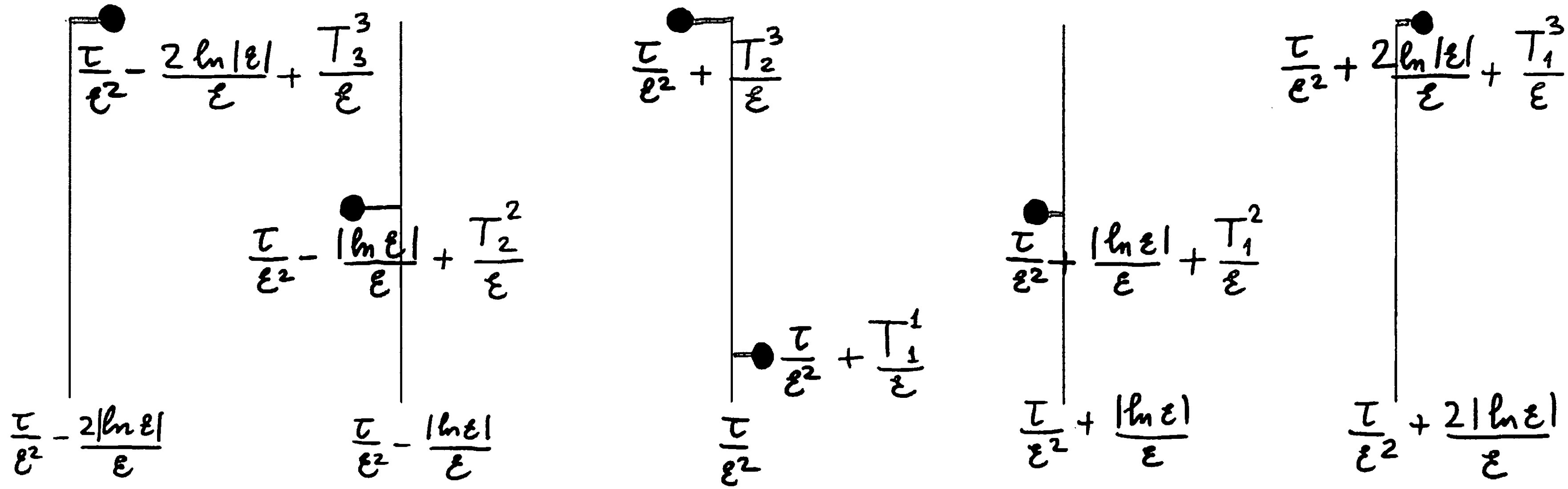
<u>Theorem [Borodin-C '11], [B-C-Sasamoto '12]</u> *For q-TASEP with* $\{x_n(0) = -n\}_{n \geq 1}$

$$\mathbb{E}\left[q^{(x_{N_1}(\tau)+N_1) + \dots + (x_{N_k}(\tau)+N_k)} \right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)\tau z_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j}$$

$(N_1 \geq N_2 \geq \dots \geq N_k)$



As $q = e^{-\varepsilon} \to 1$, at large times τ/ε^2, with zero initial conditions, low rows of the triangular array behave as



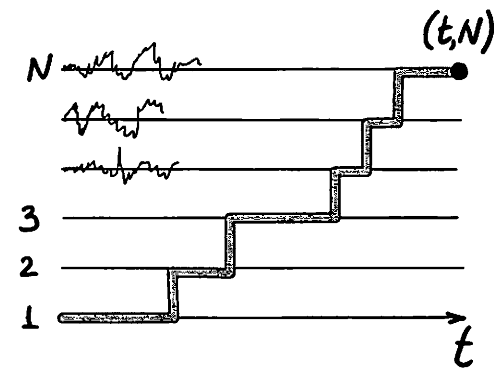
The real array $\{T_j^m\}_{1 \le j \le m}$ is distributed according to the Whittaker process, and T_1^N or $-T_N^N$ is distributed as $\log Z(\tau, N)$. The Whittaker process and its connection to polymers is due to [O'Connell '09].

Partition function for a semi-discrete directed random polymer
[O'Connell-Yor '01]

$$Z(t,N) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{B_1(0,s_1)+B_2(s_1,s_2)+\cdots+B_N(s_{N-1},t)} \, ds_1 \cdots ds_{N-1}$$

$B_1, \ldots, B_N$ independent Brownian motions

$$B_k(\alpha,\beta) := B_k(\beta) - B_k(\alpha) = \int_\alpha^\beta \dot{B}_k(x)\,dx$$



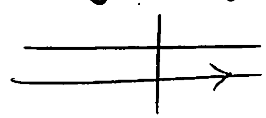
Whittaker measure arises under geometric lifting of RSK (tropical)

Also: log-gamma polymer [C-O'Connell-Seppalainen-Zygouras '11]

Different dynamics than Diaconis-Fill! Fixed level -> quantum Toda

[O'Connell '09] proved a Laplace transform formula

$$\mathbb{E}\left[e^{-\varsigma Z(\tau,N)}\right] = \int \cdots \int \prod_{j,k=1}^{N} \Gamma(i\nu_j) \prod_{j=1}^{N} \varsigma^{-i\nu_j} e^{-\frac{\tau}{2}\nu_j^2} m_N(d\nu)$$



Skylanin measure

$$m_N(d\nu) = \frac{d\nu_1 \cdots d\nu_N}{(2\pi i)^N N!} \prod_{j \neq k}^{N} \frac{1}{\Gamma(i\nu_k - i\nu_j)}$$

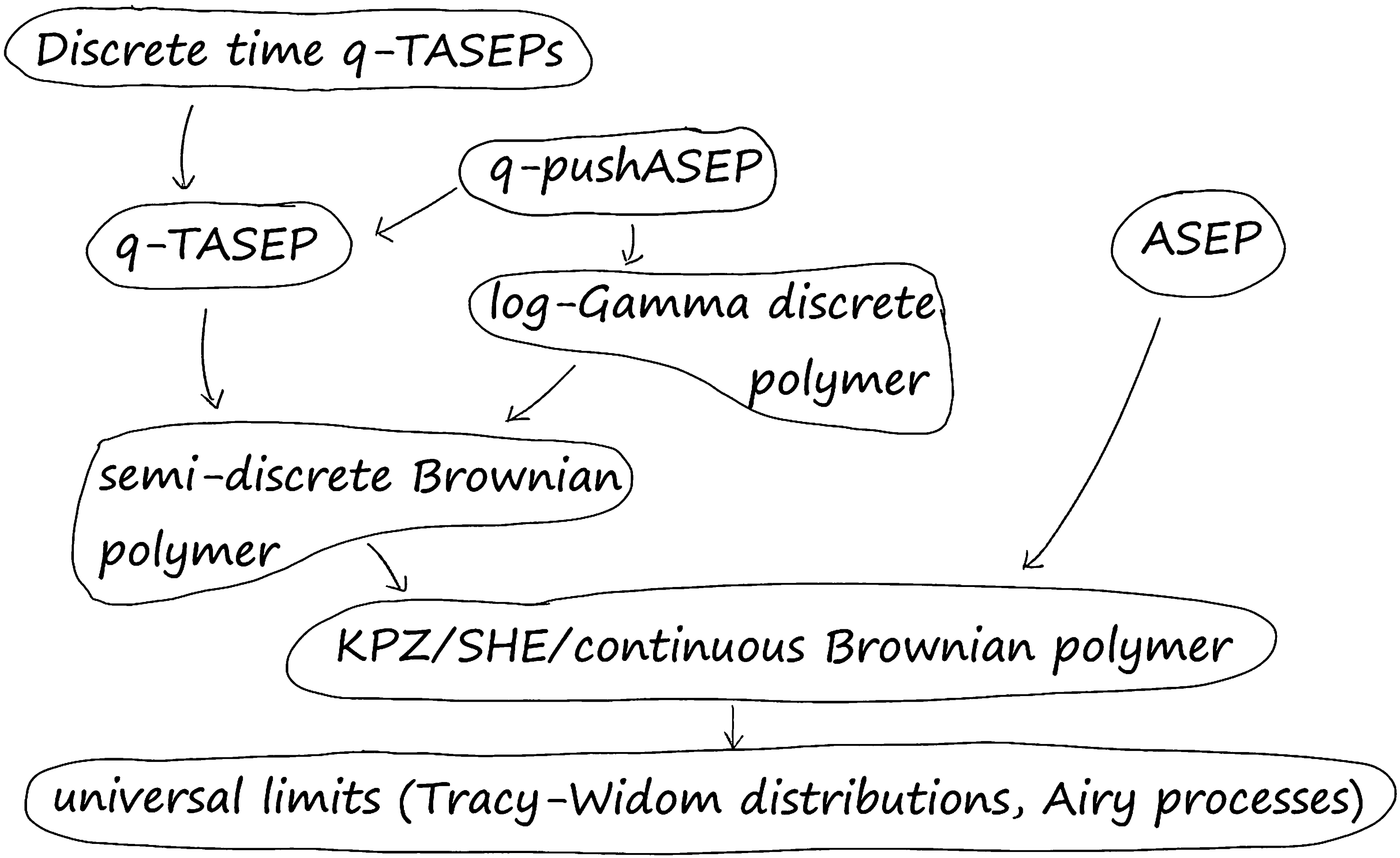
Initially unclear how to take asymptotics of this.

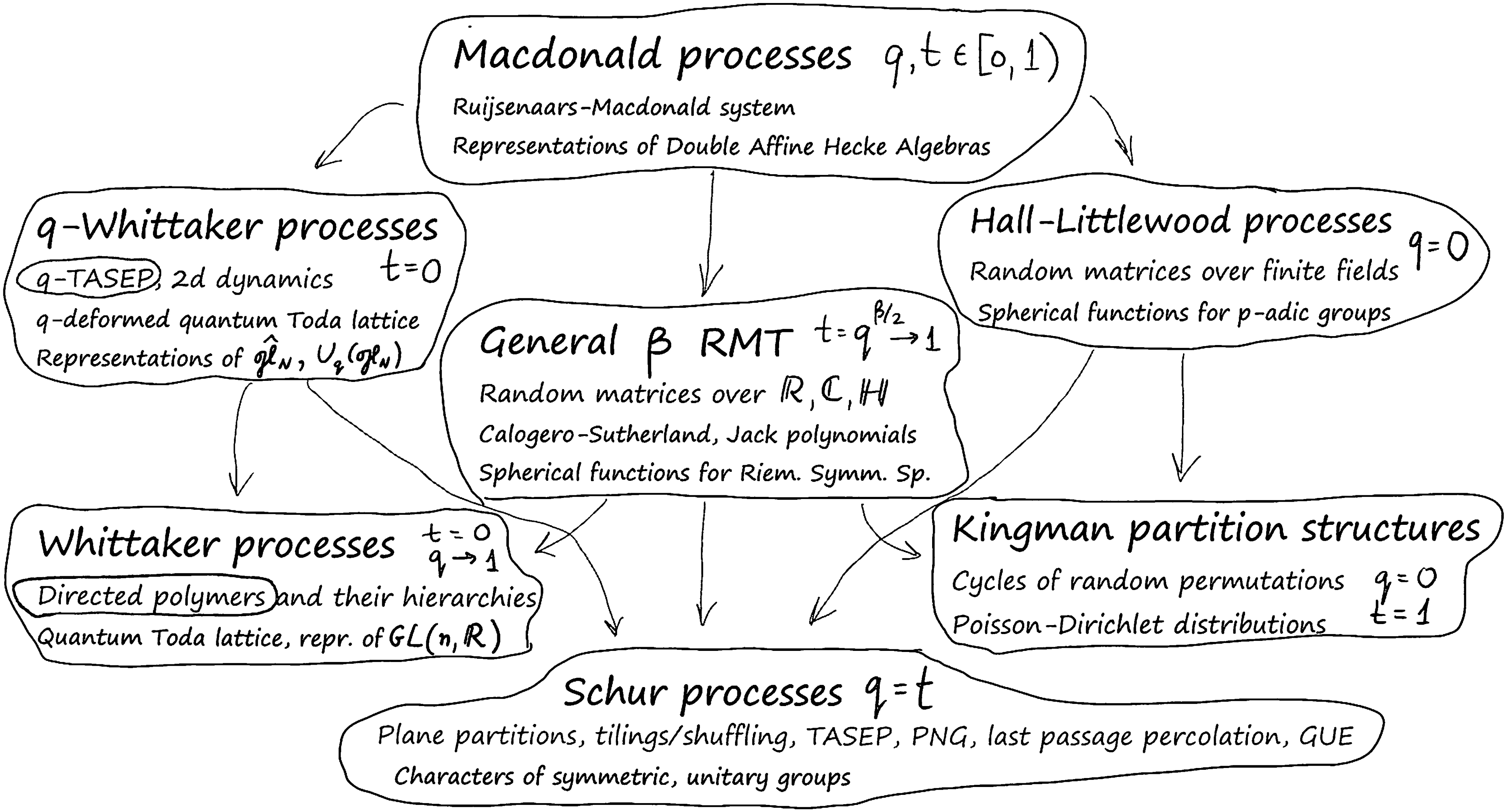
[Borodin-C '11] developed general machinery to compute observables; led to Fredholm determinant formula.

[Borodin-C-Remenik '12] show equivalence of two formulas.

To summarize:

- ASEP and q-TASEP are important systems in the KPZ universality class, which can be scaled to the KPZ equation
- Macdonald processes are a source of integrable probabilistic models
- Generalize Schur processes but are not determinantal
- Integrability from structural properties of Macdonald polynomials (lead to nice Markov dynamics and concise formulas for averages)
- Turning averages into asymptotics remains challenging
- Rigorous replica trick developed for q-TASEP and ASEP
- Nested contour integral ansatz formulas for ASEP moments suggest search for new structure parallel to Macdonald processes


Discrete time q-TASEPs
q-pushASEP
q-TASEP
ASEP
log-Gamma discrete polymer
semi-discrete Brownian polymer
KPZ/SHE/continuous Brownian polymer
universal limits (Tracy-Widom distributions, Airy processes)


Macdonald processes q,t ∈ [0,1)
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes t=0
q-TASEP, 2d dynamics
q-deformed quantum Toda lattice
Representations of ĝl_N, U_q(gl_N)
Hall-Littlewood processes q=0
Random matrices over finite fields
Spherical functions for p-adic groups
General β RMT t=q^{β/2}→1
Random matrices over ℝ, ℂ, ℍ
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. Symm. Sp.
Whittaker processes t=0 q→1
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of GL(n,ℝ)
Kingman partition structures
Cycles of random permutations q=0
Poisson-Dirichlet distributions t=1
Schur processes q=t
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

A few directions:

- General initial data (diagonalize many-body systems)?
- Symmetries (half-space polymers, Koornwinder processes)?
- Multipoint/multitime asymptotics?
- Other solvable systems (e.g. discrete time q-PushASEP/ASEP)?
- RSK type correspondences at (q,t) level?
- ASEP 2+1 extension (analog of Macdonald processes)?
- Higher versions of Macdonald processes?
- Other degenerations?
- KPZ fixed point / equation universality?

Reviews:

Lectures on integrable probability (arXiv:1212.3351)

The Kardar-Parisi-Zhang equation and universality class (arXiv:1106.1596)

Two ways to solve ASEP (arXiv:1212.2267)

Articles:

From duality to determinants for q-TASEP and ASEP (arXiv:1207.5035)

Macdonald processes (arXiv:1111.4408)