Universal phenomena in random systems

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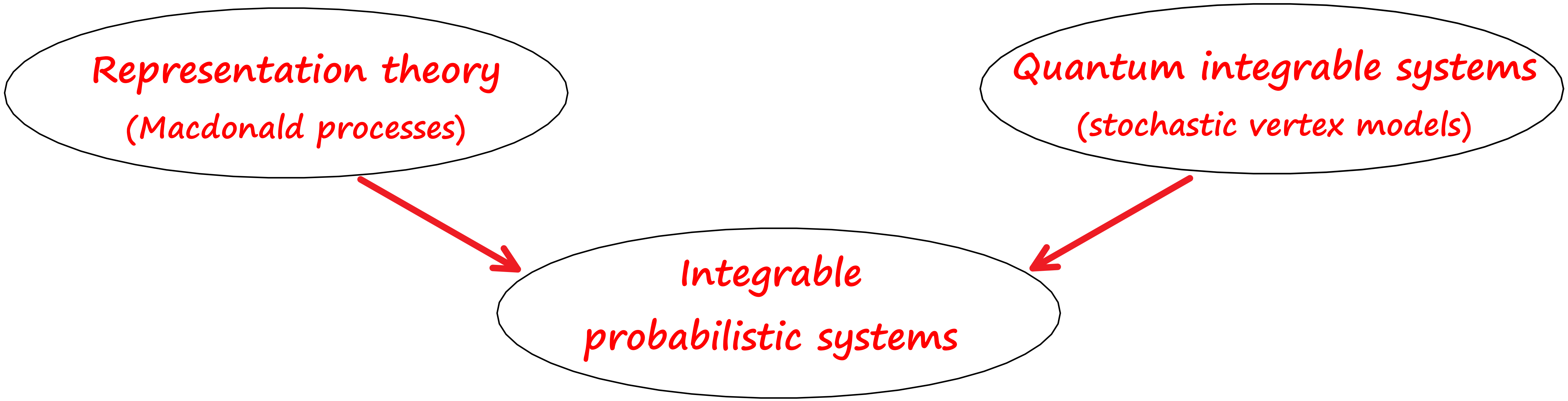
Integrable probabilistic systems

- Admit exact and concise formulas for expectations of a variety of observables of interest.
- Asymptotics of systems, observables and formulas lead to detailed descriptions of wide universality classes and limiting phenomena.

These special systems come from algebraic structures:



Representation theory

(Macdonald processes)

Quantum integrable systems

(stochastic vertex models)

Integrable

probabilistic systems

Universality of asymptotics requires different tools – mostly open!

Coin flipping

- The number of heads in N fair coin flips is given exactly by the Binomial distribution:

$$\text{Probability (heads = h)} = 2^{-N}\binom{N}{h}$$

- Law of large numbers [Bernoulli 1713]: as N grows, heads/N -> 1/2
- Central limit theorem [de Moivre 1733], [Laplace 1812]: as N grows

$$\text{Probability}\left(\text{heads} < \frac{N}{2} + \frac{1}{2}\sqrt{N}\,x\right) \to \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

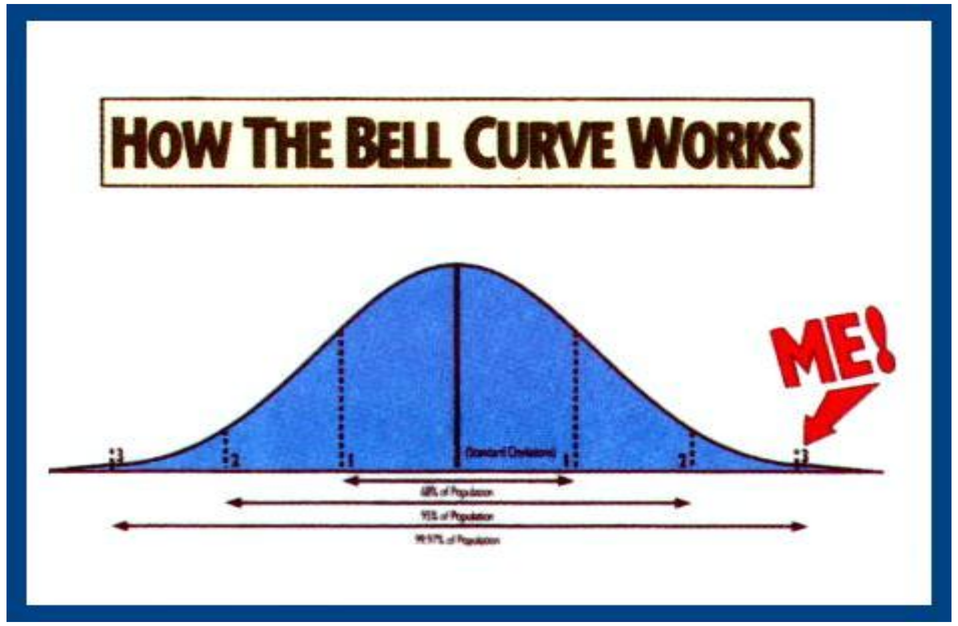
- Proved using asymptotics for N! [de Moivre 1721], [Stirling 1729]:

$$N! = \Gamma(N+1) = \int_0^\infty e^{-t} t^N dt = N^{N+1} \int_0^\infty e^{N(\log z - z)} dz \qquad \approx -1 - \frac{(z-1)^2}{2}$$

$$\approx N^{N+1} e^{-N} \sqrt{2\pi N}$$

The Gaussian central limit theorem

The universality of the Gaussian distribution was not demonstrated until [Lyapunov 1901]. Polya called this the 'central limit theorem' due to its importance in probability theory.



Theorem: Let $X_1, X_2, \ldots$ be independent identically distributed (iid) random variables of finite mean m and variance V. Then for all S, as N grows

$$\text{Probability}\,(X_1+\cdots+X_N < mN + V\sqrt{N}\,S) \to \int_{-\infty}^{S} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

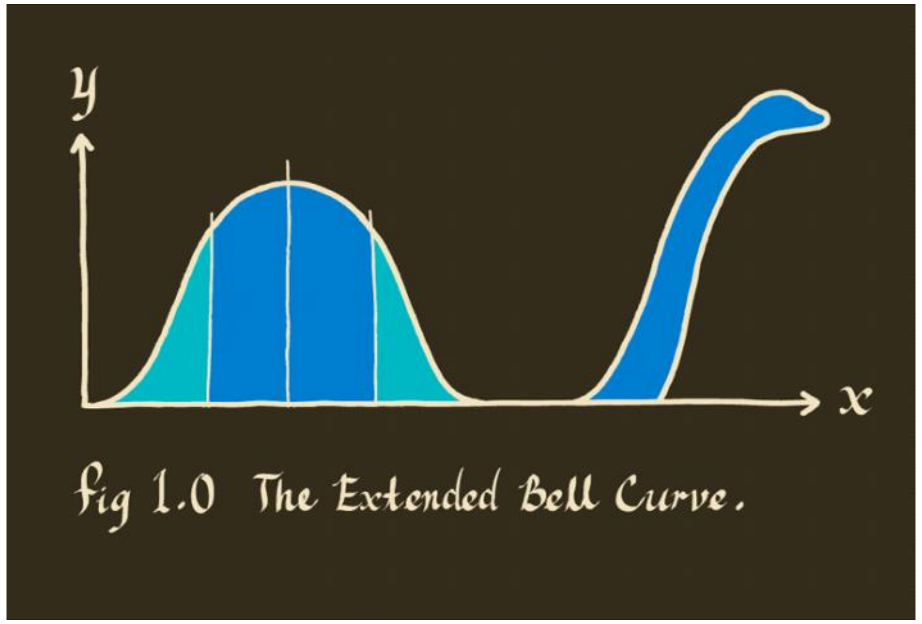
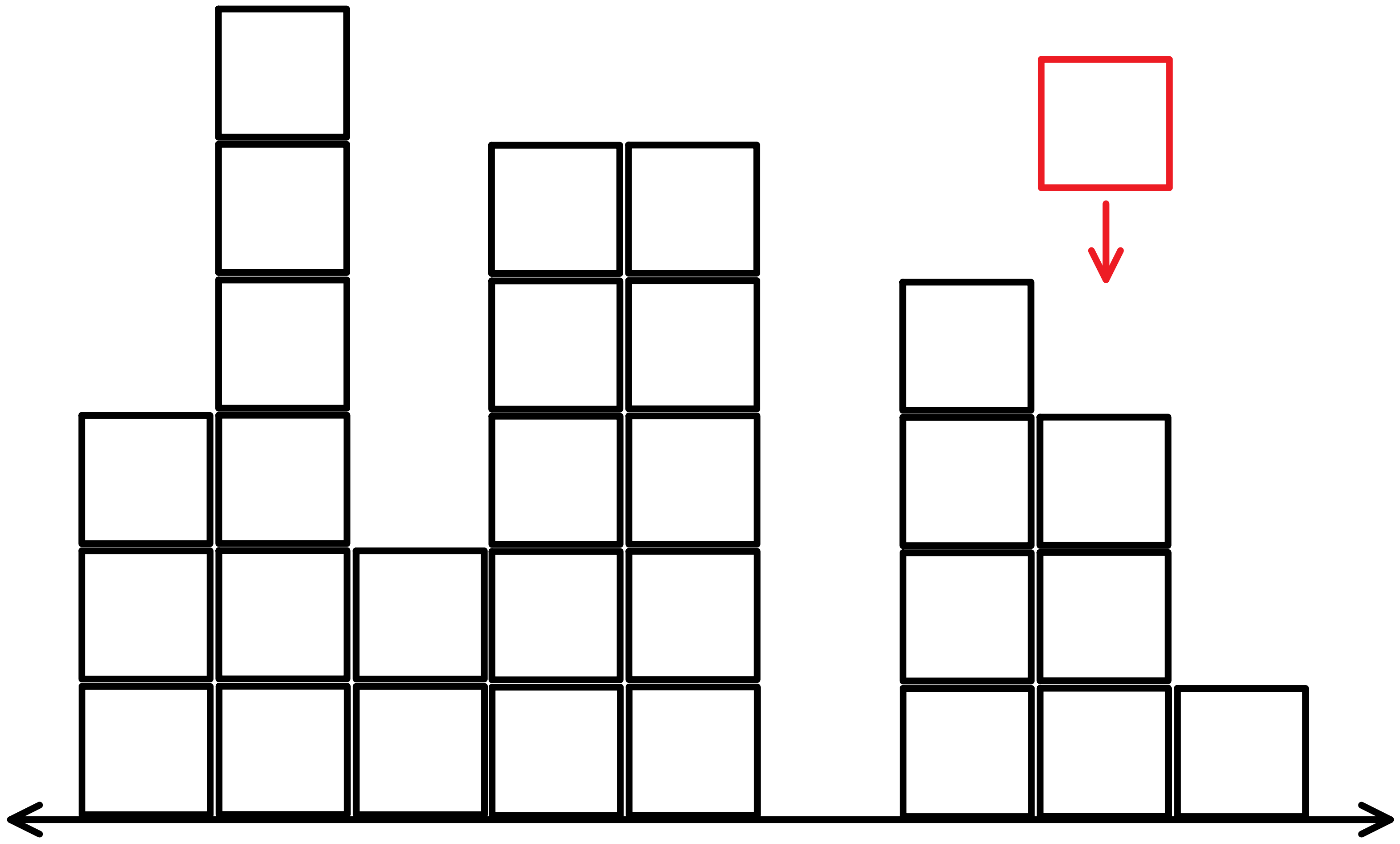


fig 1.0 The Extended Bell Curve.

- Extensions exist for this result, and much of probability deals with Gaussian processes.
- The 'bell curve' is ubiquitous and is the basis for much of classical statistics.

Random deposition model



Blocks fall independent and in parallel above each site according to exponentially distributed waiting times.

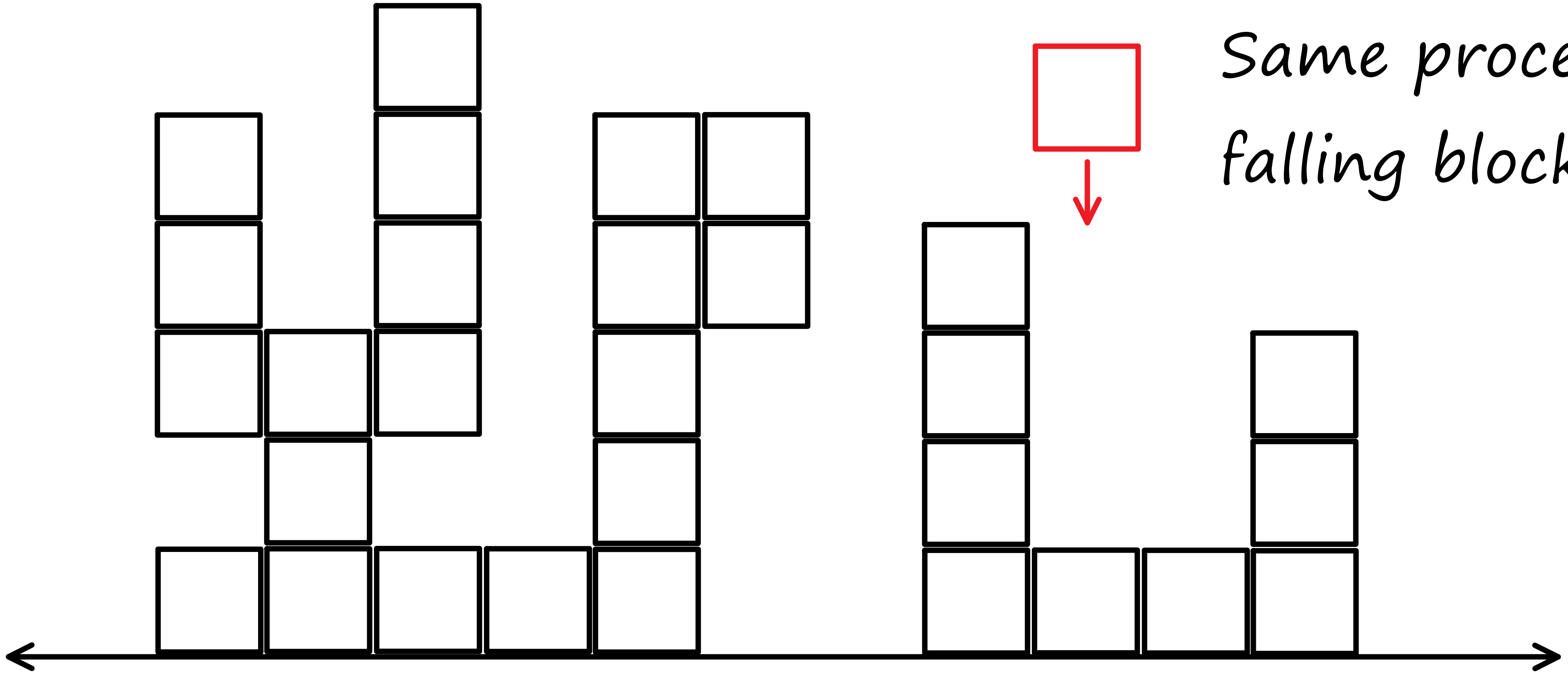
Exponential distribution of rate λ (mean 1/λ):

$$\text{Probability}\ (X>s) = e^{-\lambda s}.$$

Memoryless (Markov), so growth depends only the present state.

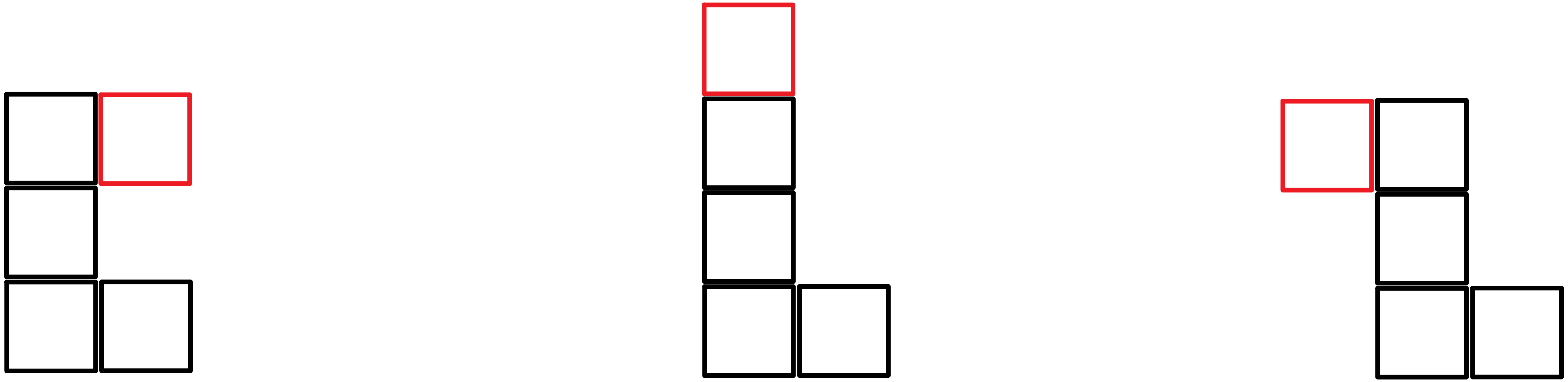
Gaussian behavior since each column is a sum of iid random variables

Ballistic deposition model (sticky blocks)



Same process of falling blocks

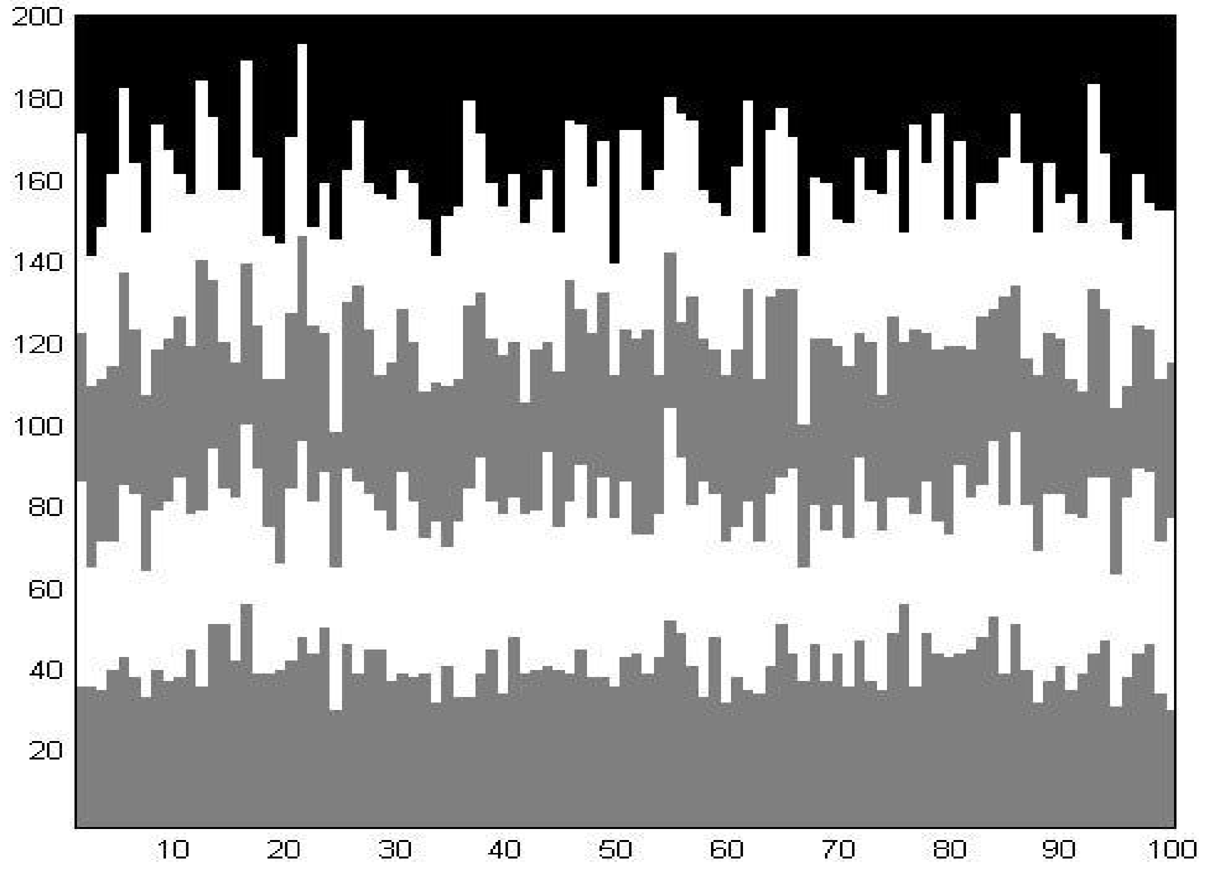
Sticky blocks introduce partial correlation [Vold 1959]



Random vs. ballistic deposition

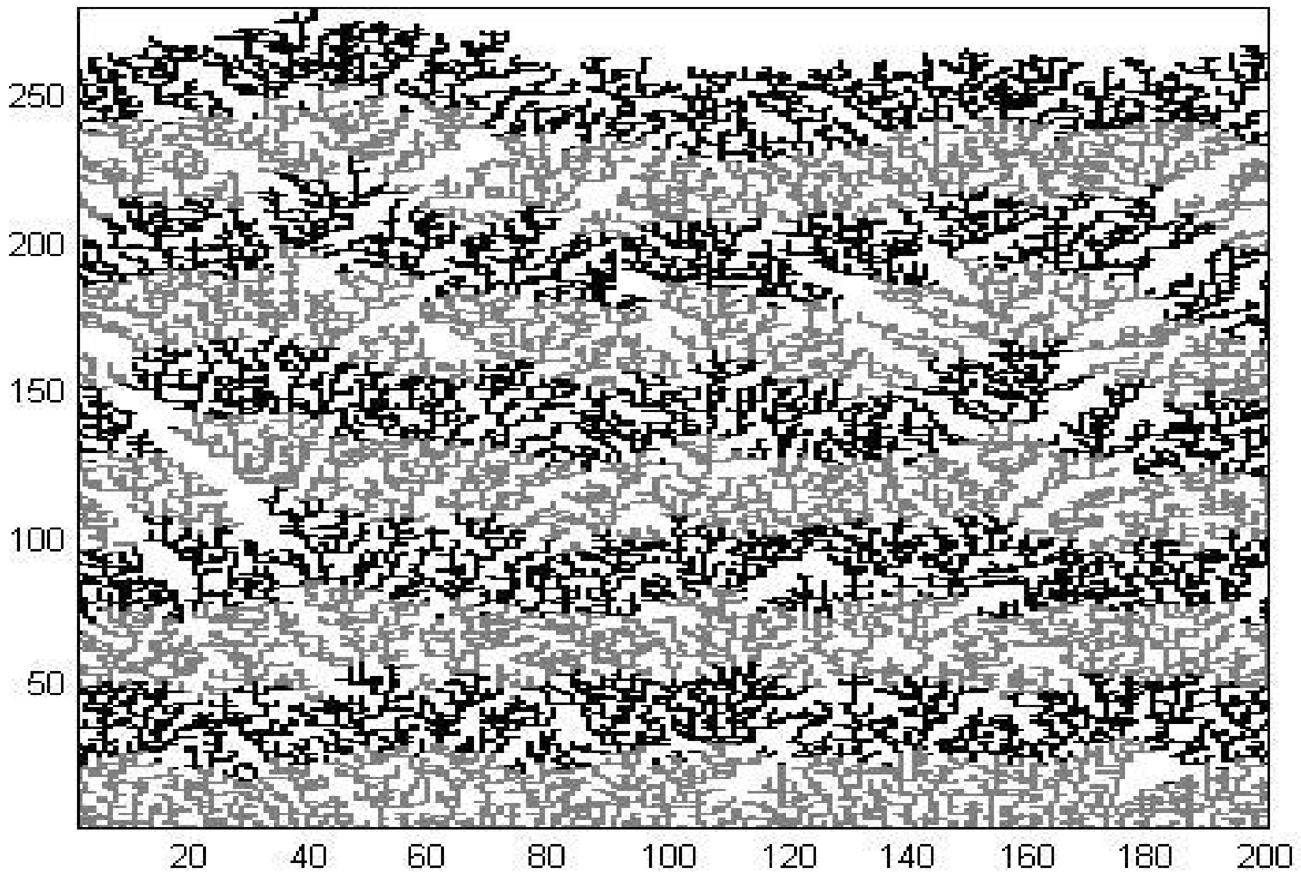
Gaussian universal class

- Linear growth (known speed)
- $t^{1/2}$ fluctuations with Gaussian limit (CLT)
- No spatial correlation

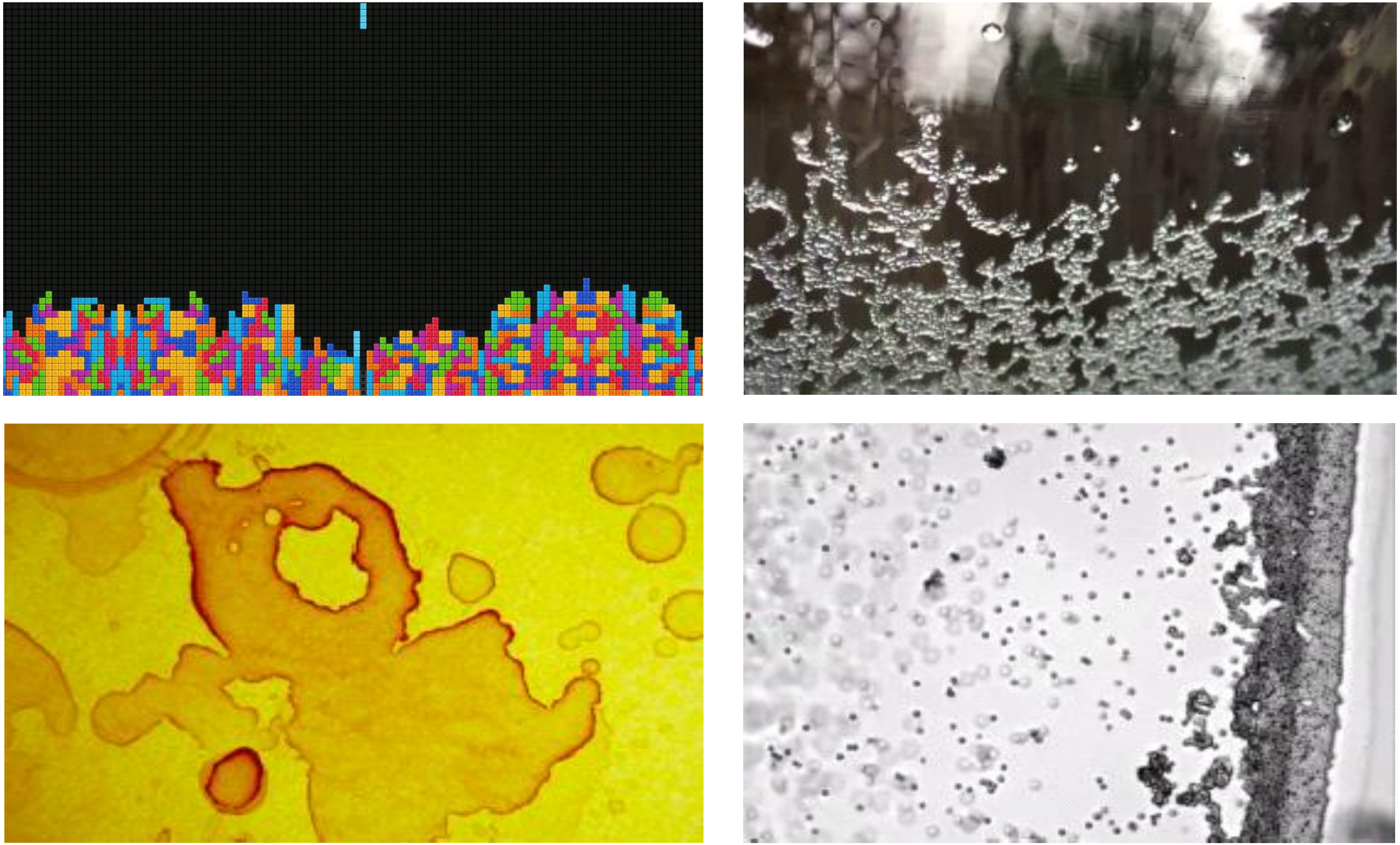


KPZ universality class

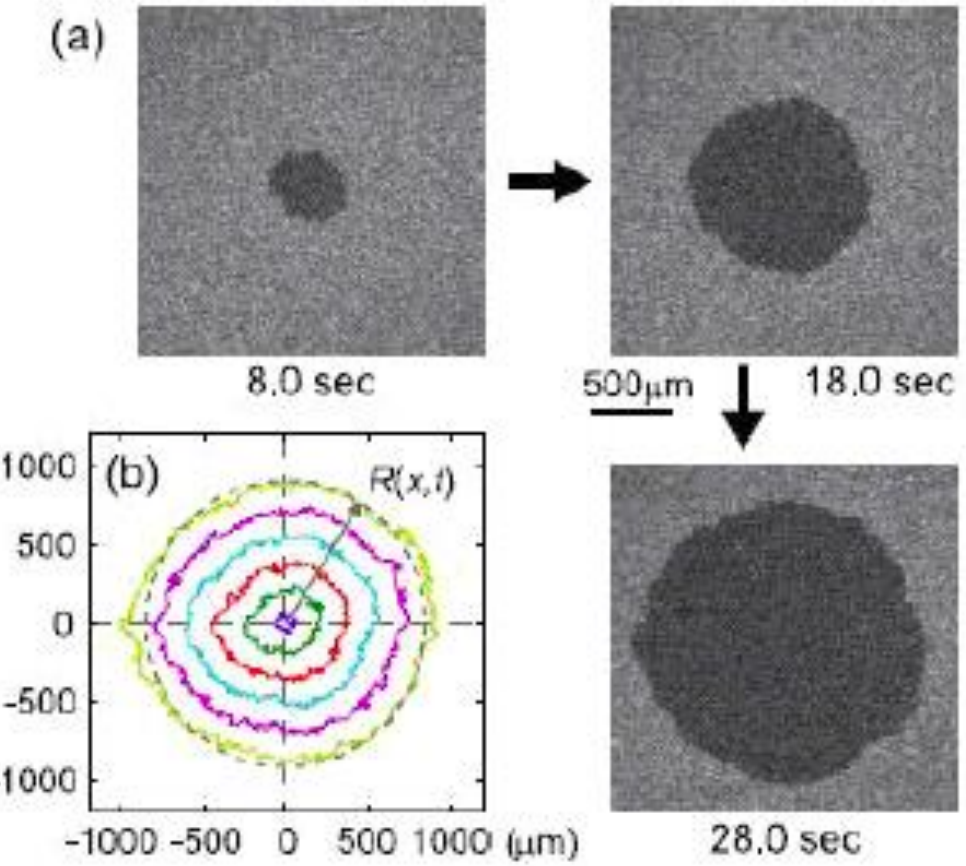
- Linear growth (unknown speed)
- Conjectural $t^{1/3}$ fluctuations with GOE Tracy-Widom limit.
- Conjectural $t^{2/3}$ spatial correlation

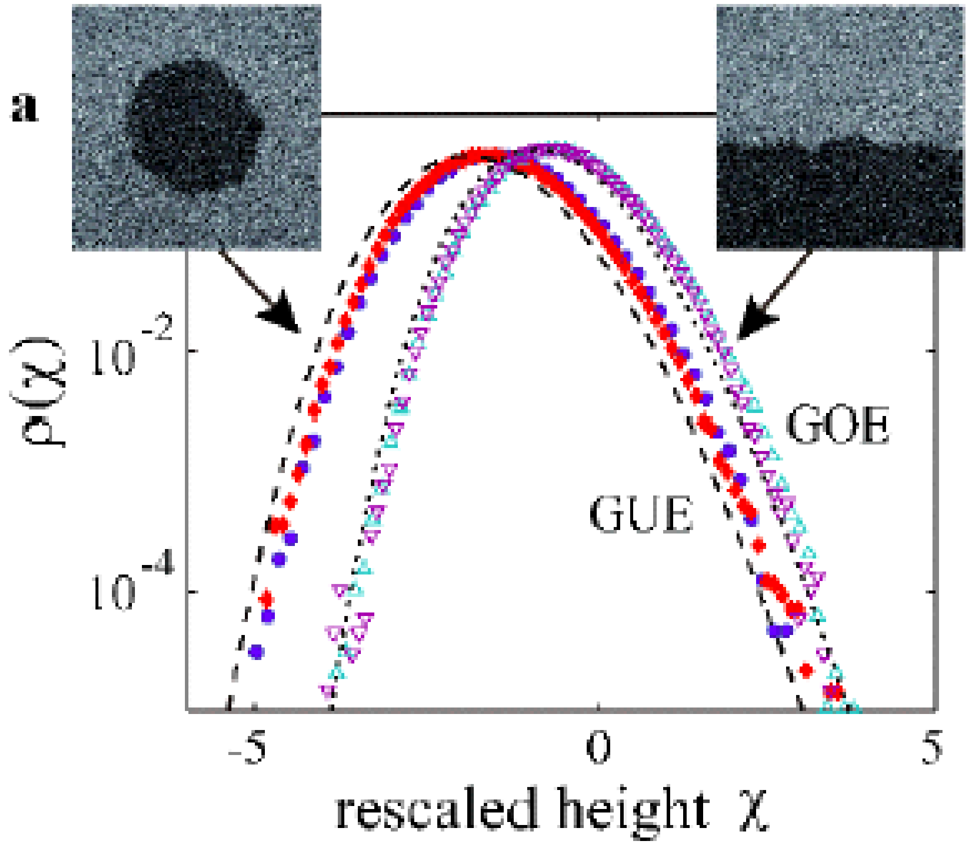


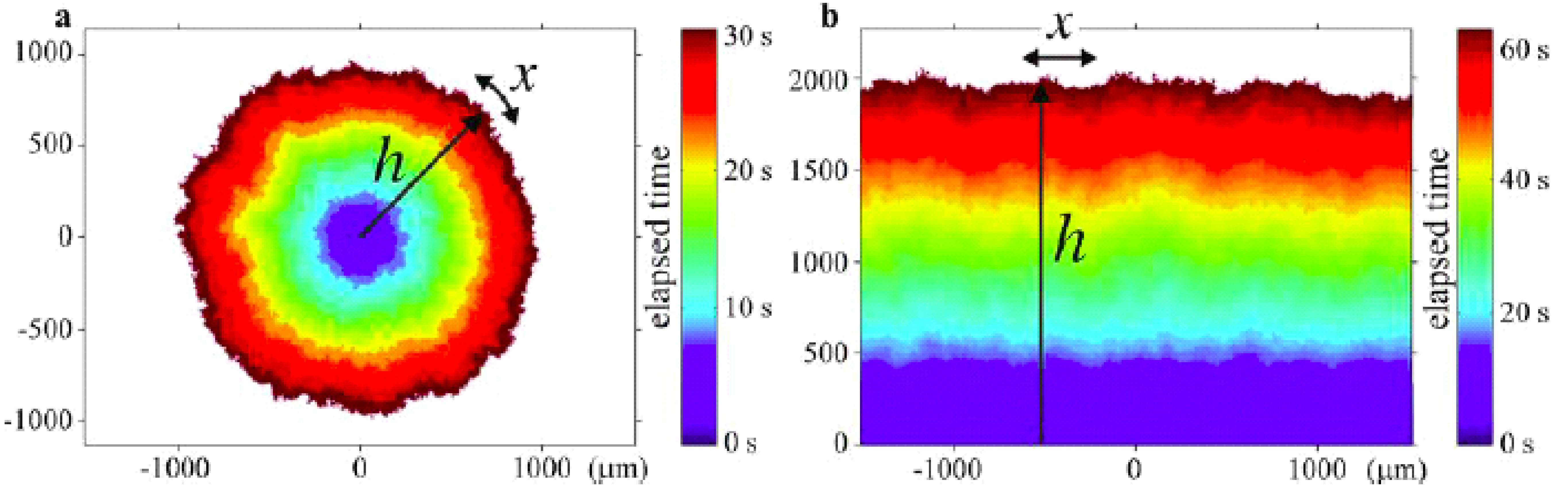
Ballistic deposition in 'nature'



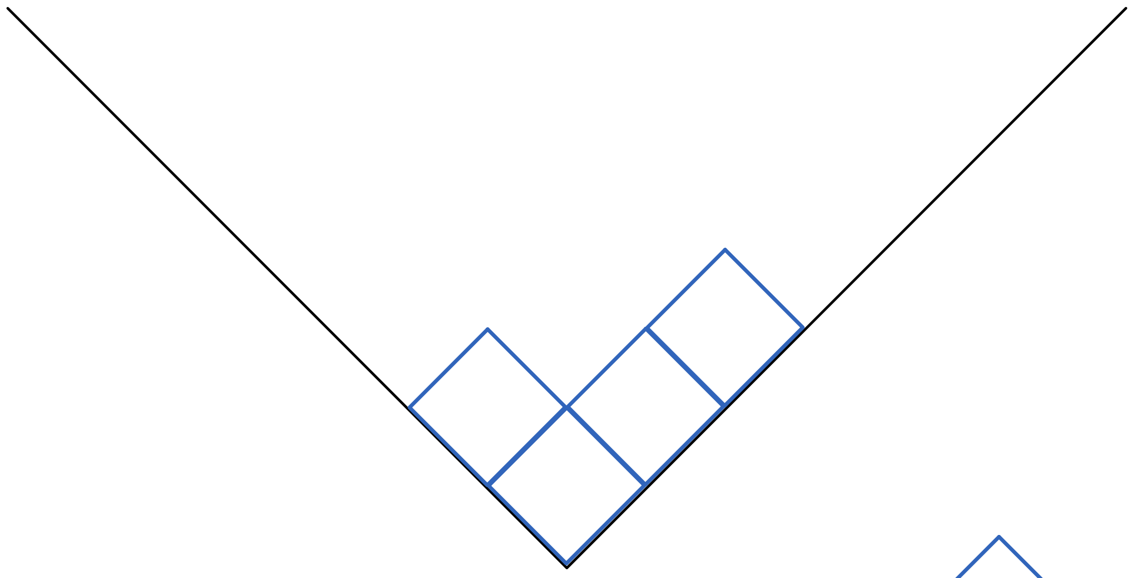
Disordered liquid crystal growth

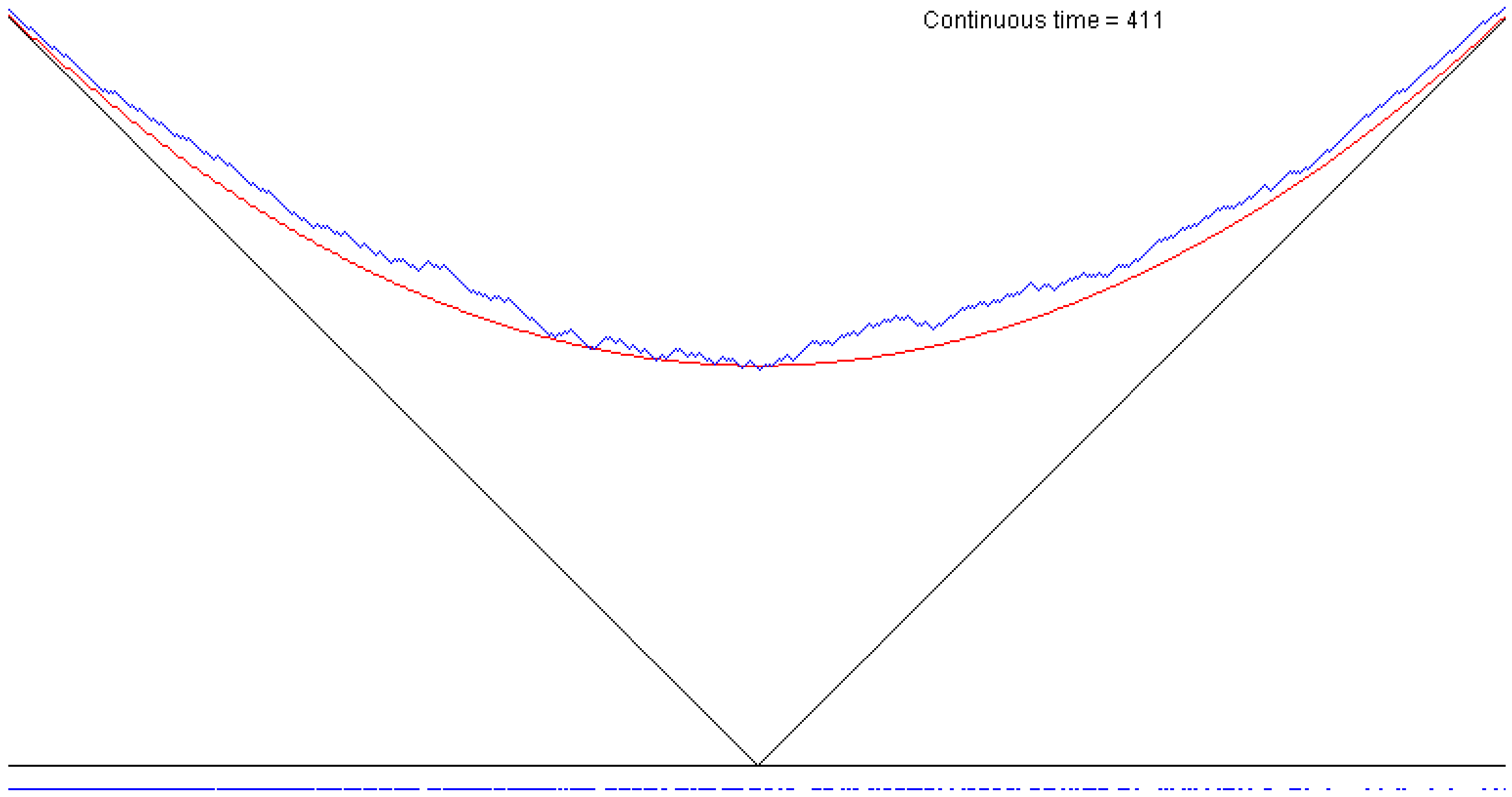


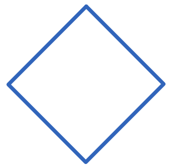




Corner growth model – an integrable example





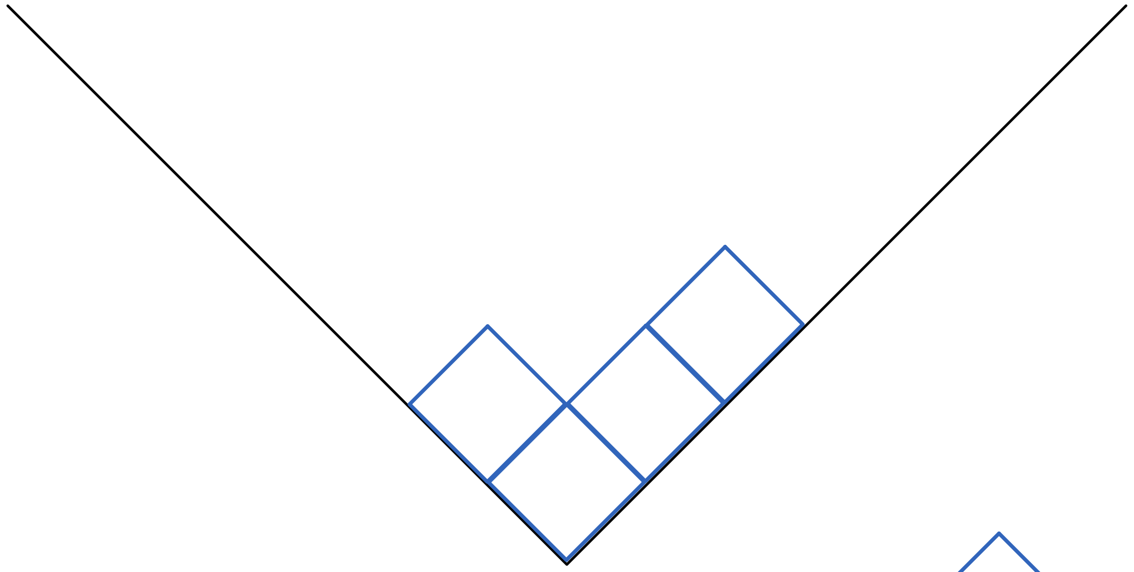
Each  turns into  after an exponential rate 1 waiting time.

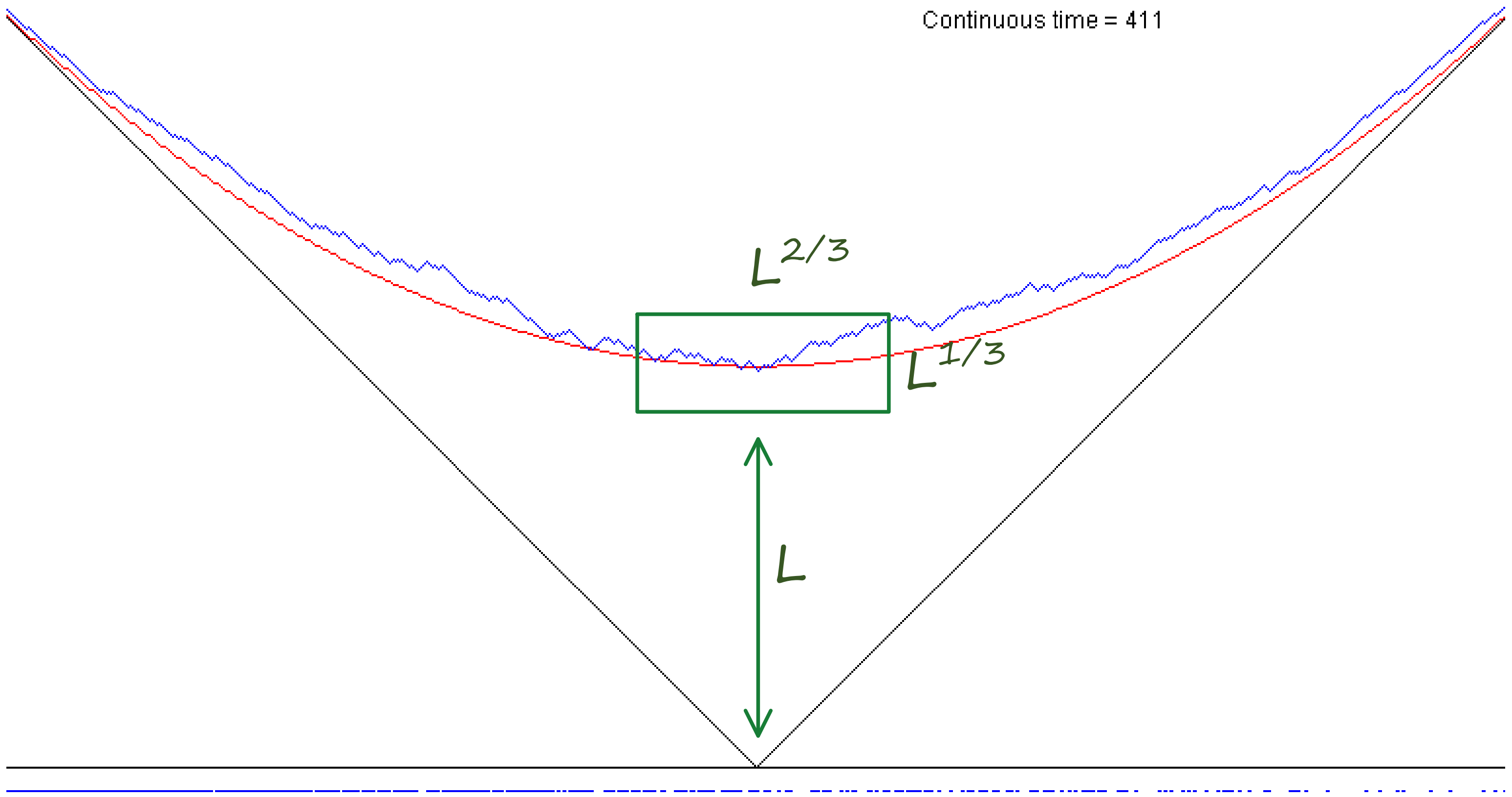
$h(t,x)$ = height above x at time t. Wedge initial data is $h(0,x)=|x|$.

Theorem [Rost 1981]: For wedge initial data as t grows,

$$\frac{h(t,tx)}{t} \longrightarrow \begin{cases} \frac{1-x^2}{2}, & |x|<1 \\ |x|, & |x|\geq 1 \end{cases}$$

Corner growth model – an integrable example





Each ∨ turns into ◇ after an exponential rate 1 waiting time.

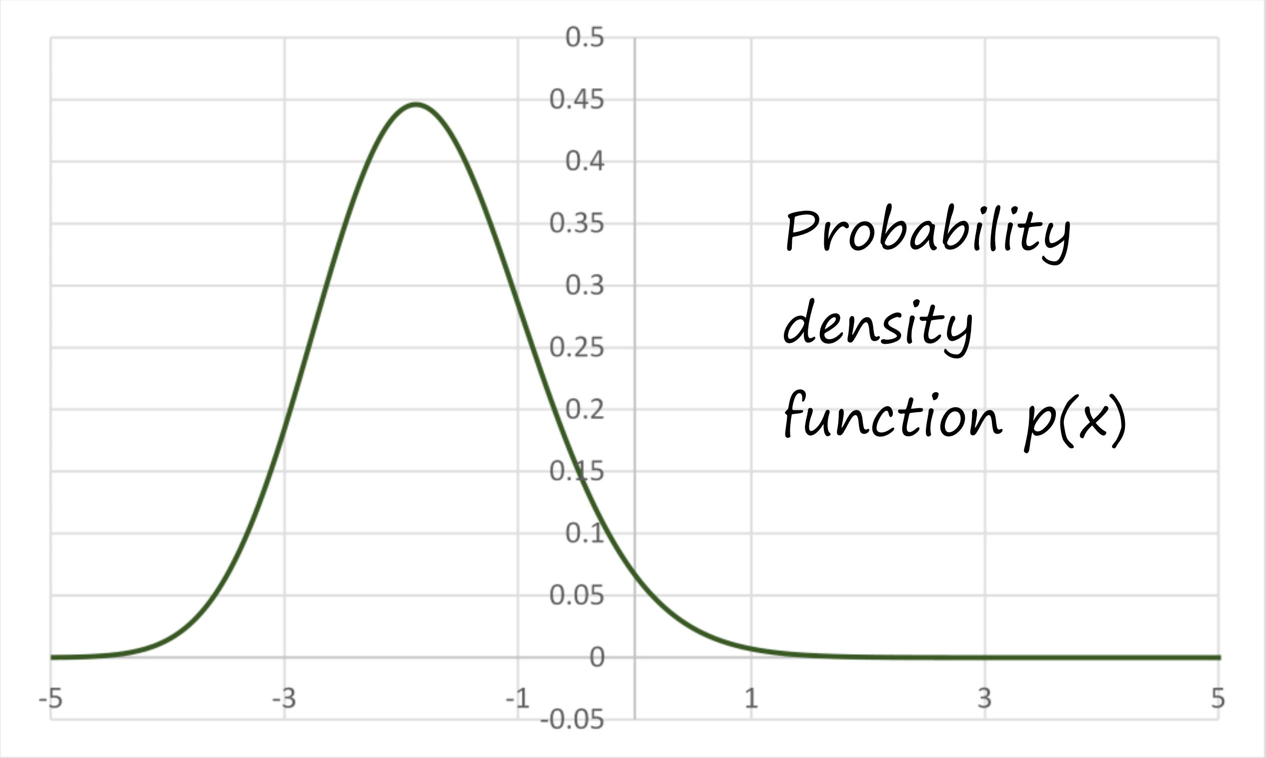
Define the rescaled height function $h_L(t,x) = L^{-1/3}\left[h(Lt, L^{2/3}x) - \frac{Lt}{2}\right]$

Theorem [Johansson 1999]: For wedge initial data as L grows

$$\text{Probability}\left(h_L(t,0) > -s\right) \to F_{GUE}(s).$$

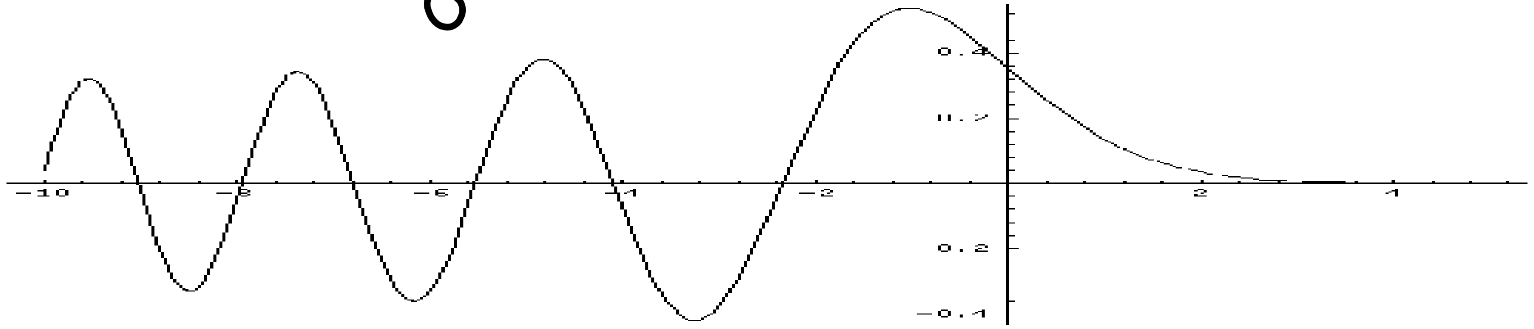
GUE Tracy-Widom distribution (F_{GUE} or F_2)

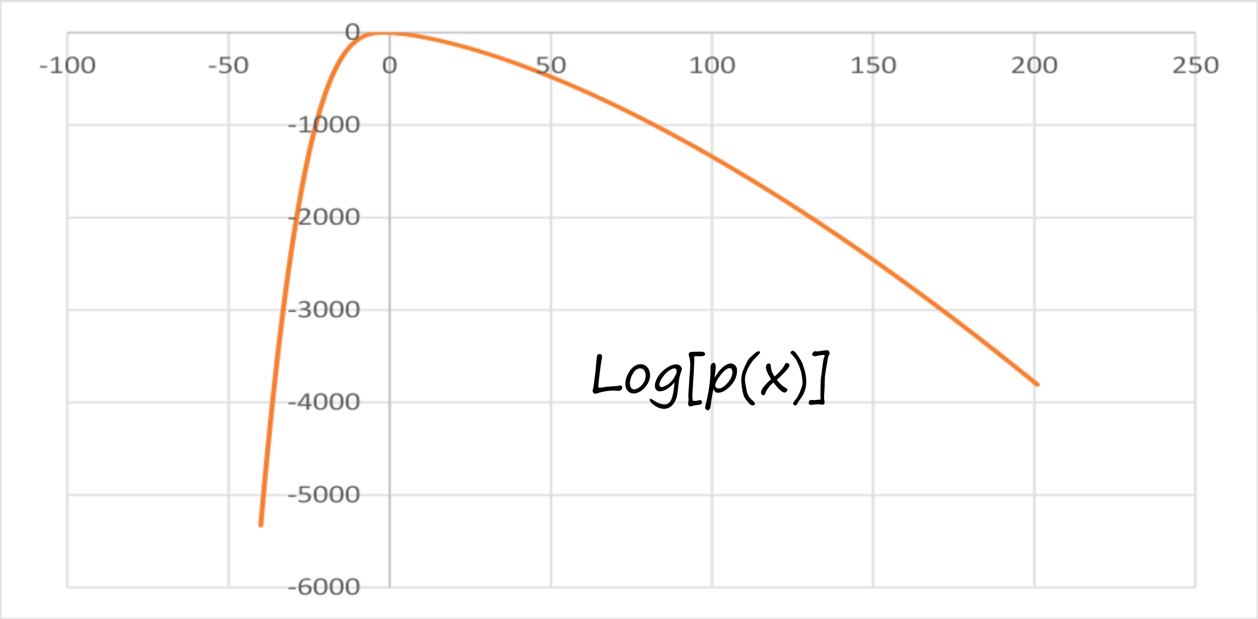
- First arose in the study of random matrices [Tracy-Widom 1993]
- Negative mean, lower tail like e^{-cy^3} and upper tail like $e^{-\tilde{c}\, y^{3/2}}$
- Defined via a Fredhold determinant:



$$F_{GUE}(s) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_s^{\infty} dx_1 \cdots \int_s^{\infty} dx_n \det\left[K(x_i, x_j)\right]_{i,j=1}^{n}$$

$$K(x,y) = \int_0^{\infty} dr\, Ai(x+r)\, Ai(y+r)$$



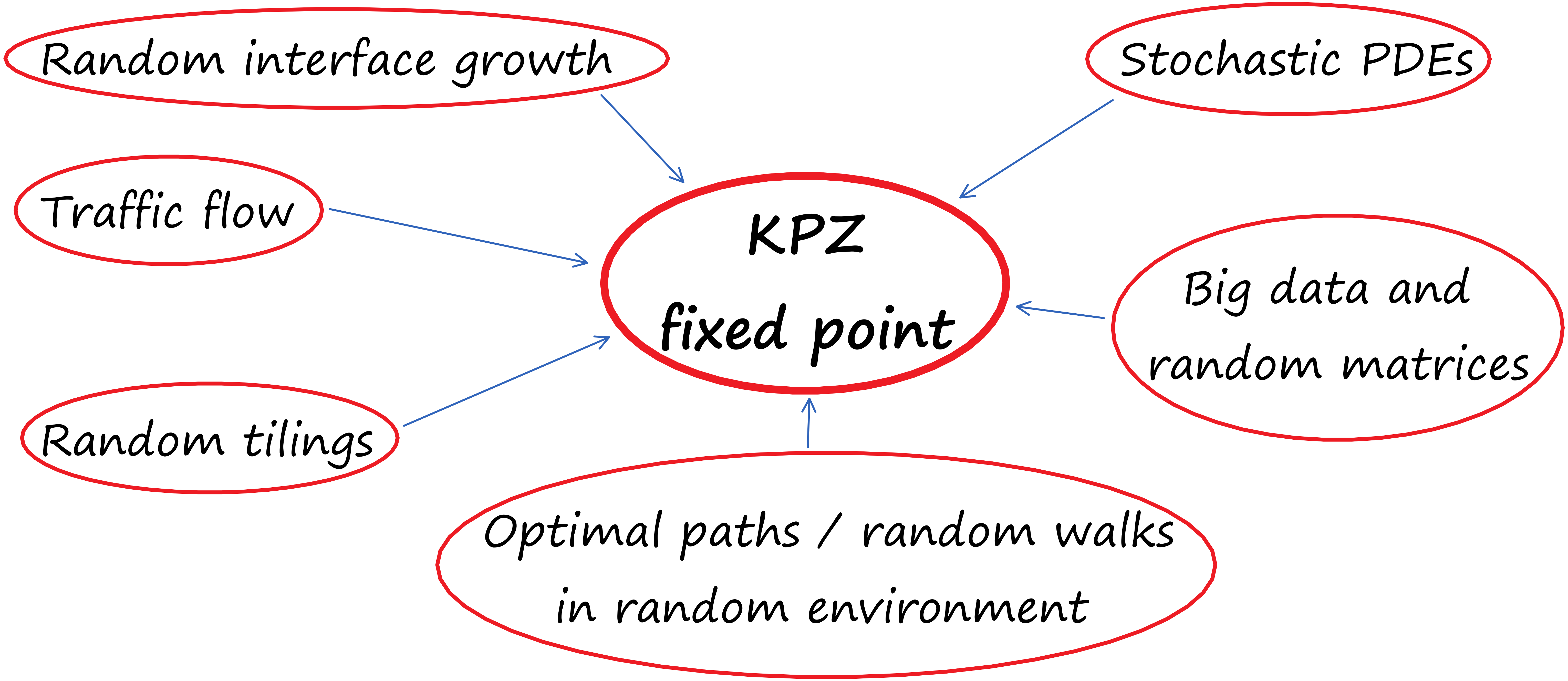


1+1 dimensional Kardar-Parisi-Zhang universality class

- Entire growth processes has a limit - the KPZ fixed point.
- 3 : 2 : 1 scaling of time : space : fluctuation is called 'KPZ scaling'.
- Believed to arise in 1+1 dimensional growth processes which enjoy
    - Local dynamics
    - Smoothing
    - Slope dependent (or lateral) growth rate
    - Space-time random driving forces

- There are a number of other types of systems which can (at least in special cases or approximations) be maps into growth processes. Hence these become included into the universality class too.

Filling in the KPZ universality class



Random interface growth

Stochastic PDEs

Traffic flow

KPZ

fixed point

Big data and random matrices

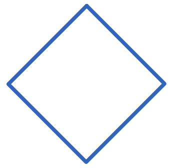
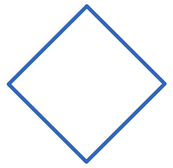
Random tilings

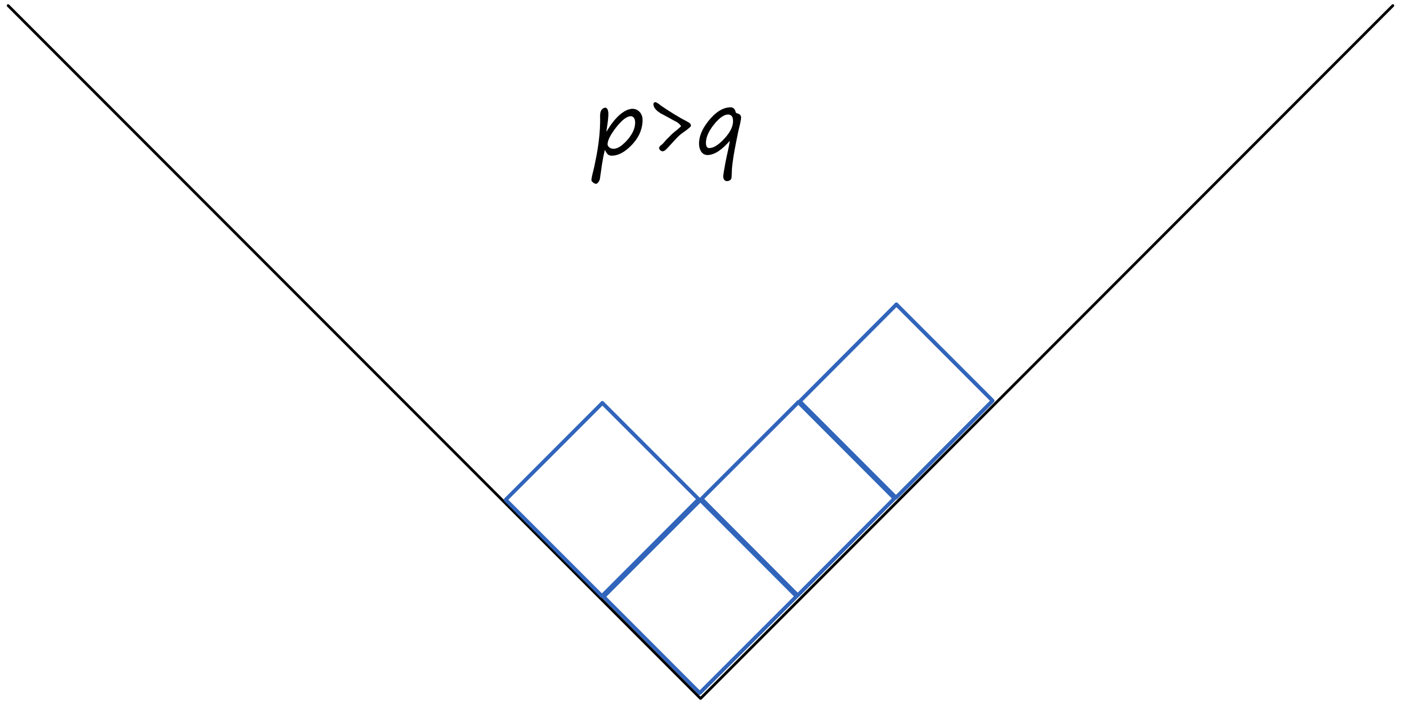
Optimal paths / random walks in random environment

KPZ fixed point should be the universal limit under 3:2:1 scaling.

This is mainly conjectural and only proved for integrable models.

Random interface growth

- Partially asymmetric corner growth model:
  - ✧ Each  turns into  after an exponential rate p waiting time.
  - ✧ Each  turns into  after an exponential rate q waiting time.



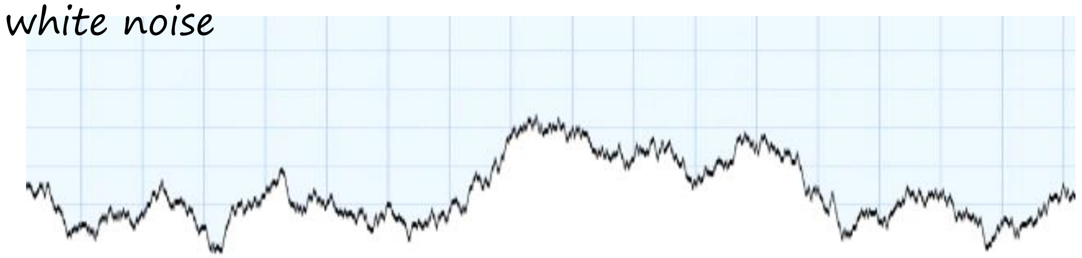
<u>Theorem [Tracy-Widom '09]:</u> Same law of large numbers and fluctuation limit theorems hold with t -> t/(p-q).

When p=q the law of large numbers and fluctuations change nature. This corresponds with the Edwards-Wilkinson universality class which has 4:2:1 scaling and Gaussian limiting behavior.

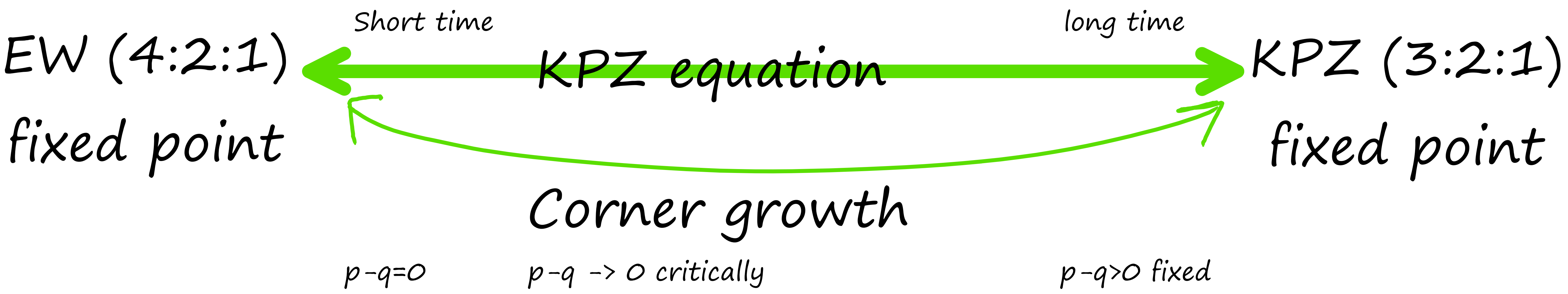
Stochastic partial differential equations: KPZ equation

space-time white noise

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x)$$



- Continuum growth model studied by [Kardar-Parisi-Zhang '86] using work of [Forster-Nelson-Stephen '77] to predict 3:2:1 scaling.
- [Bertini-Cancrini '95], [Bertini-Giacomin '97] make sense of this.
- **KPZ equation is in the KPZ universality class** proved recently:
    - 3:2:1 scaling [Balazs-Quastel-Seppalainen '09]
    - F_{GUE} limit [Amir-C-Quastel '10]



EW (4:2:1) fixed point

Short time

KPZ equation

long time

KPZ (3:2:1) fixed point

Corner growth

p-q=0

p-q -> 0 critically

p-q>0 fixed

Rescaling the KPZ equation

The rescaled solution $h_\varepsilon(x,t) := \varepsilon^{b} h(\varepsilon^{-z} t, \varepsilon^{-1} x)$ satisfies

$$\partial_t h_\varepsilon = \frac{1}{2} \varepsilon^{2-z} \partial_{xx} h_\varepsilon + \frac{1}{2} \varepsilon^{2-z-b} (\partial_x h_\varepsilon)^2 + \varepsilon^{b-z/2+1/2} \xi$$

KPZ scaling: **b=1/2, z=3/2** [Forster-Nelson-Stephen '77], [KPZ '86]

All growth processes with key features (locality, smoothing, lateral growth, noise) should renormalize to **KPZ fixed point** [C-Quastel '11] (e.g. GUE Tracy-Widom law). Unclear exactly what this limit is!

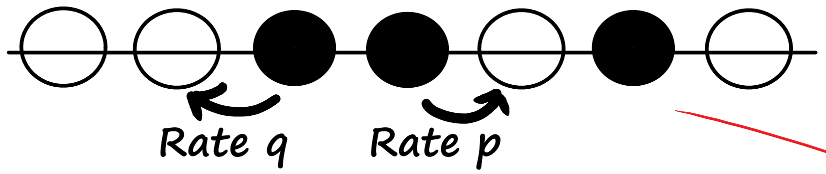
Weak nonlinearity scaling: b=1/2, z=2, scale nonlinearity by $\varepsilon^{1/2}$.

Weak noise scaling: b=0, z=2, scale noise by $\varepsilon^{1/2}$.

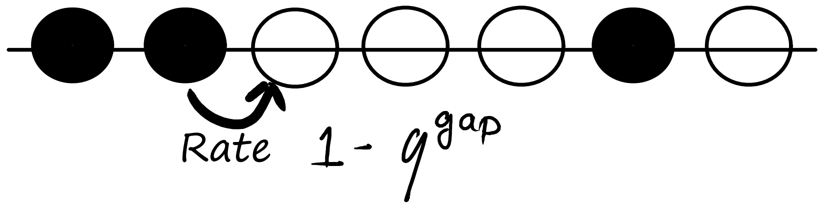
Weak limits are proxies for rescaling discrete models to KPZ equation.

Another big picture

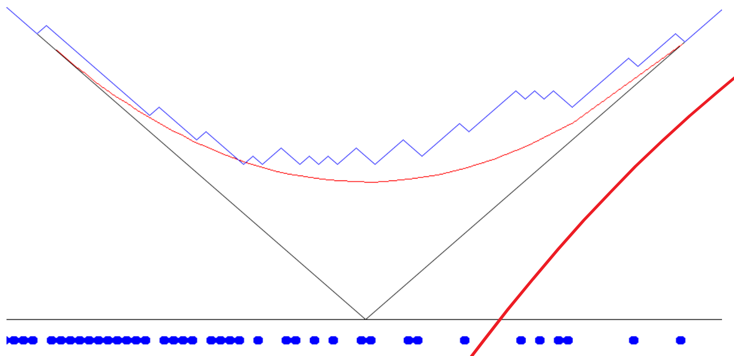
ASEP



q-TASEP



TASEP



Weak nonlinearity scaling

KPZ equation

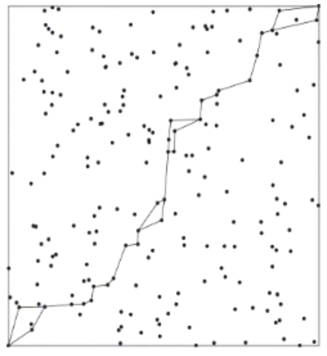
Weak noise scaling

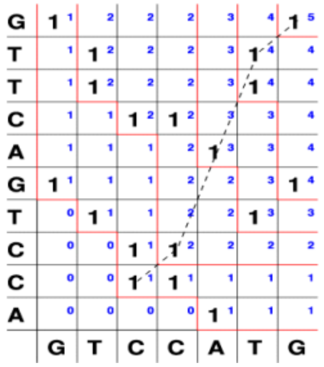
1+1 dimensional semi-discrete and discrete SHE

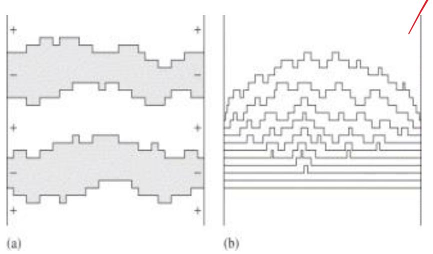
KPZ scaling

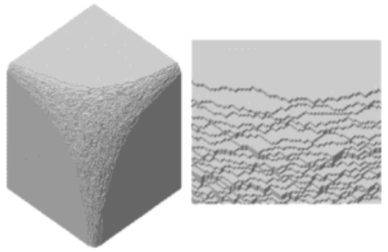
KPZ fixed point

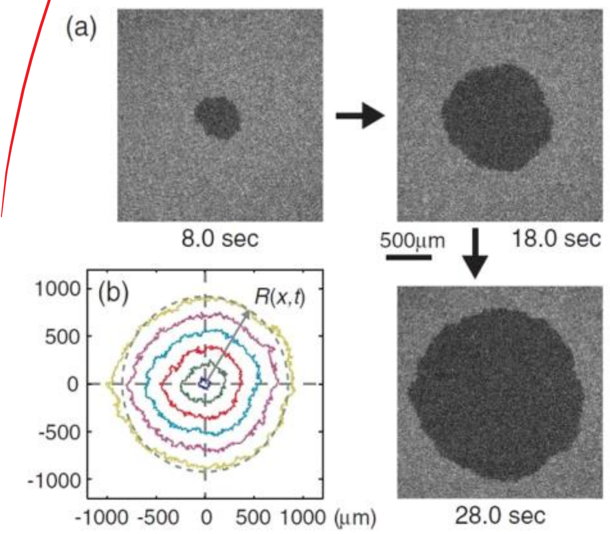
Directed polymers

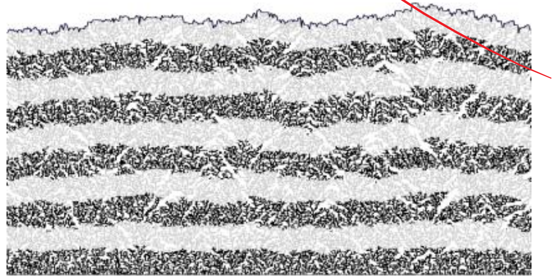


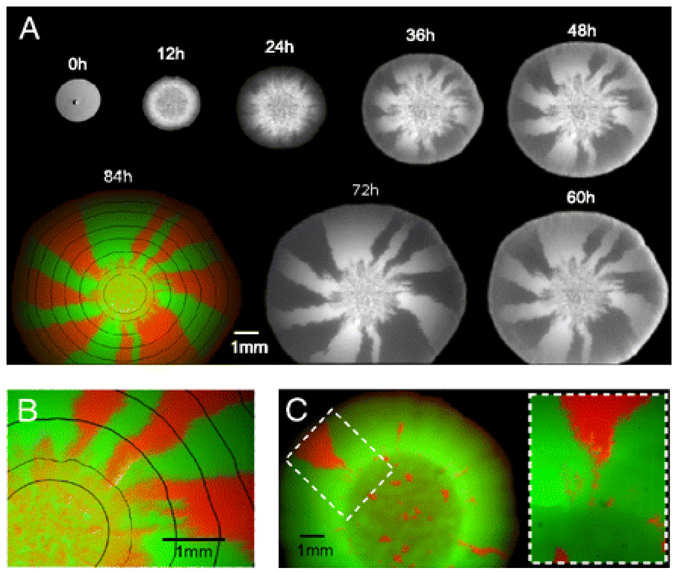




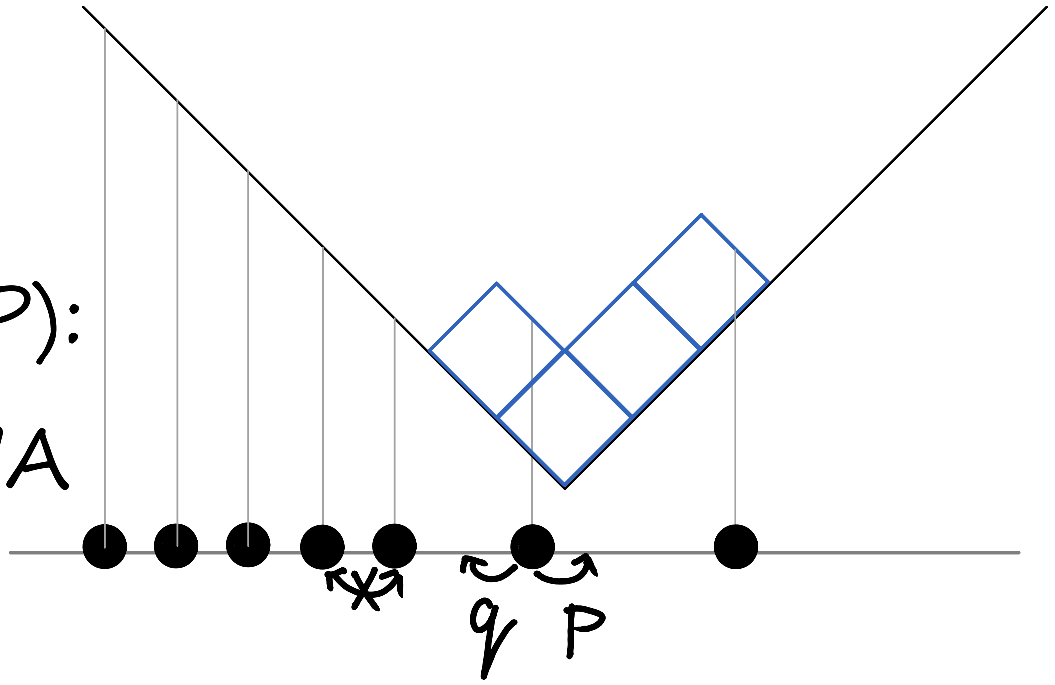




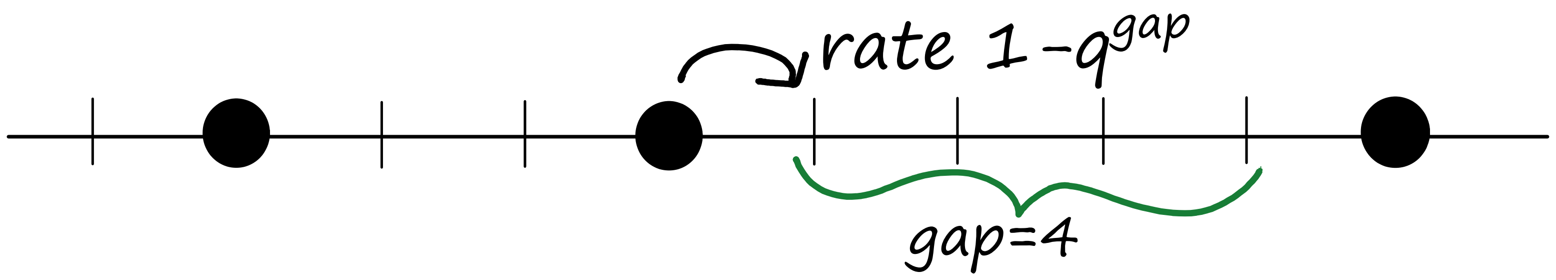




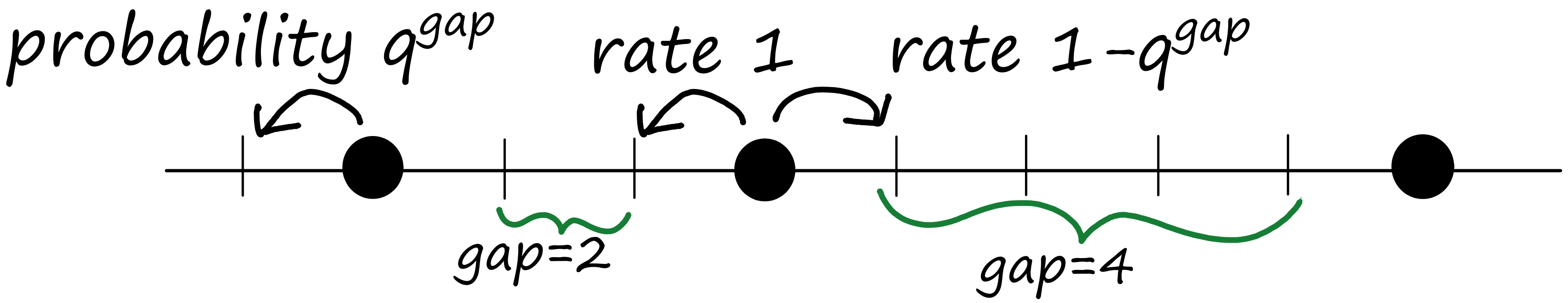
Traffic flow



- Asymmetric simple exclusion process (ASEP): Introduced in biology literature to model RNA transcription [MacDonald-Gibbs-Pipkin '68].
- q-TASEP [Borodin-C '11]: Simple traffic model

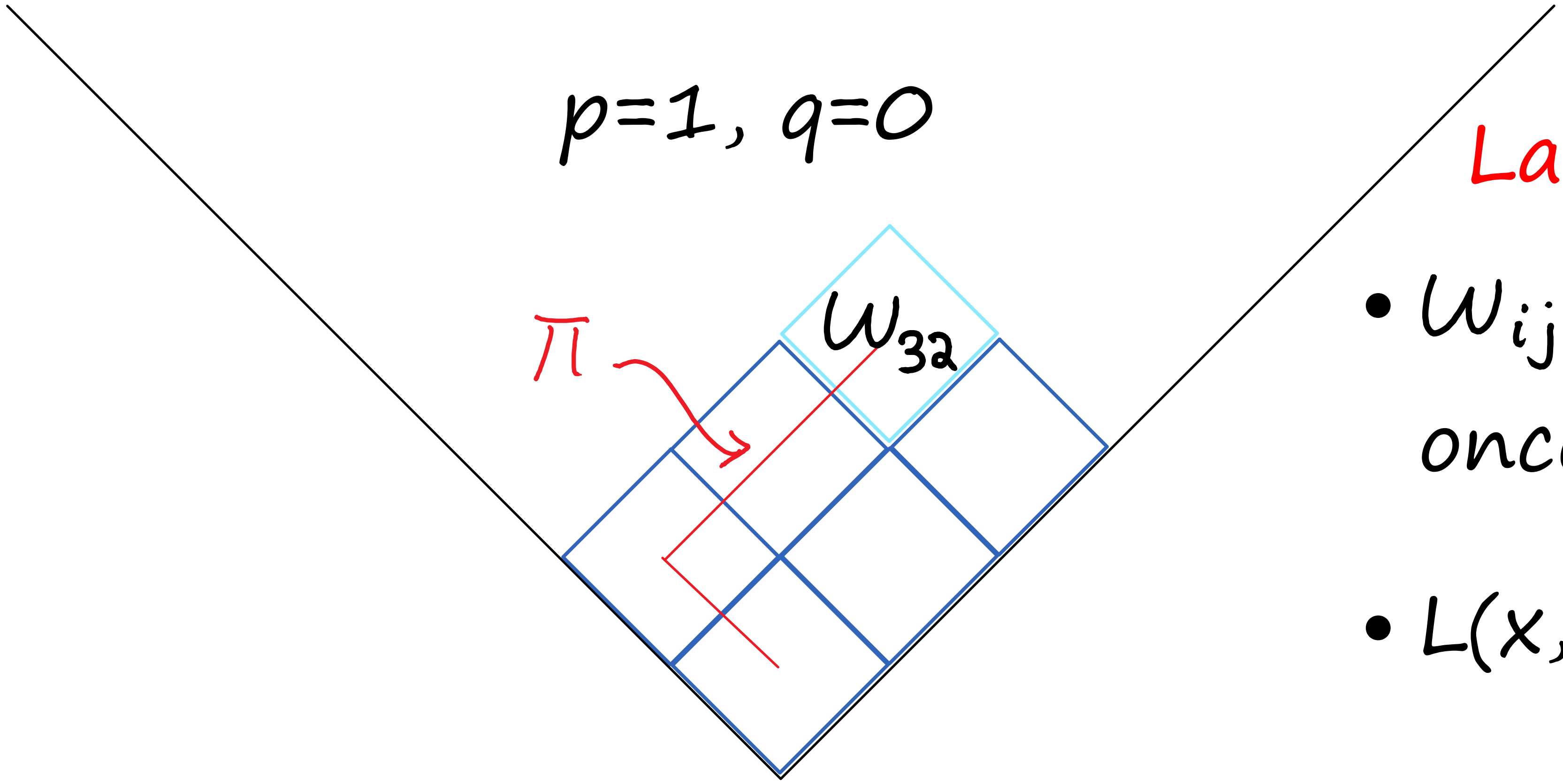


- q-PushASEP [C-Petrov '12]: Includes breaking



<u>KPZ class behavior</u>: For step initial data, the number of particles to cross origin behaves like $ct + c' t^{1/3} \chi$ where χ is F_{GUE} distributed.

Optimal paths in random environment



Last passage percolation [Rost '81]

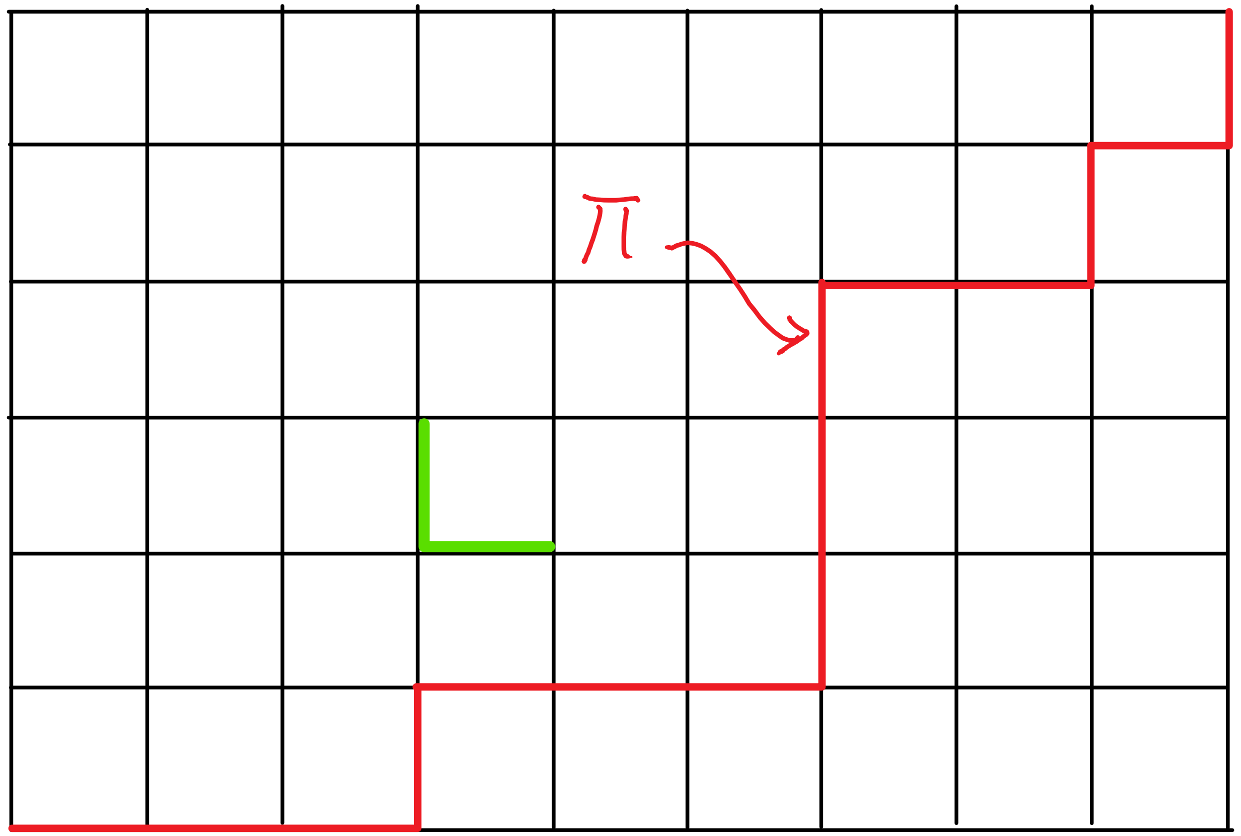
- w_{ij} : time for box (i,j) to grow, once it can (exponential rate 1).
- L(x,y): time when box x,y is grown.

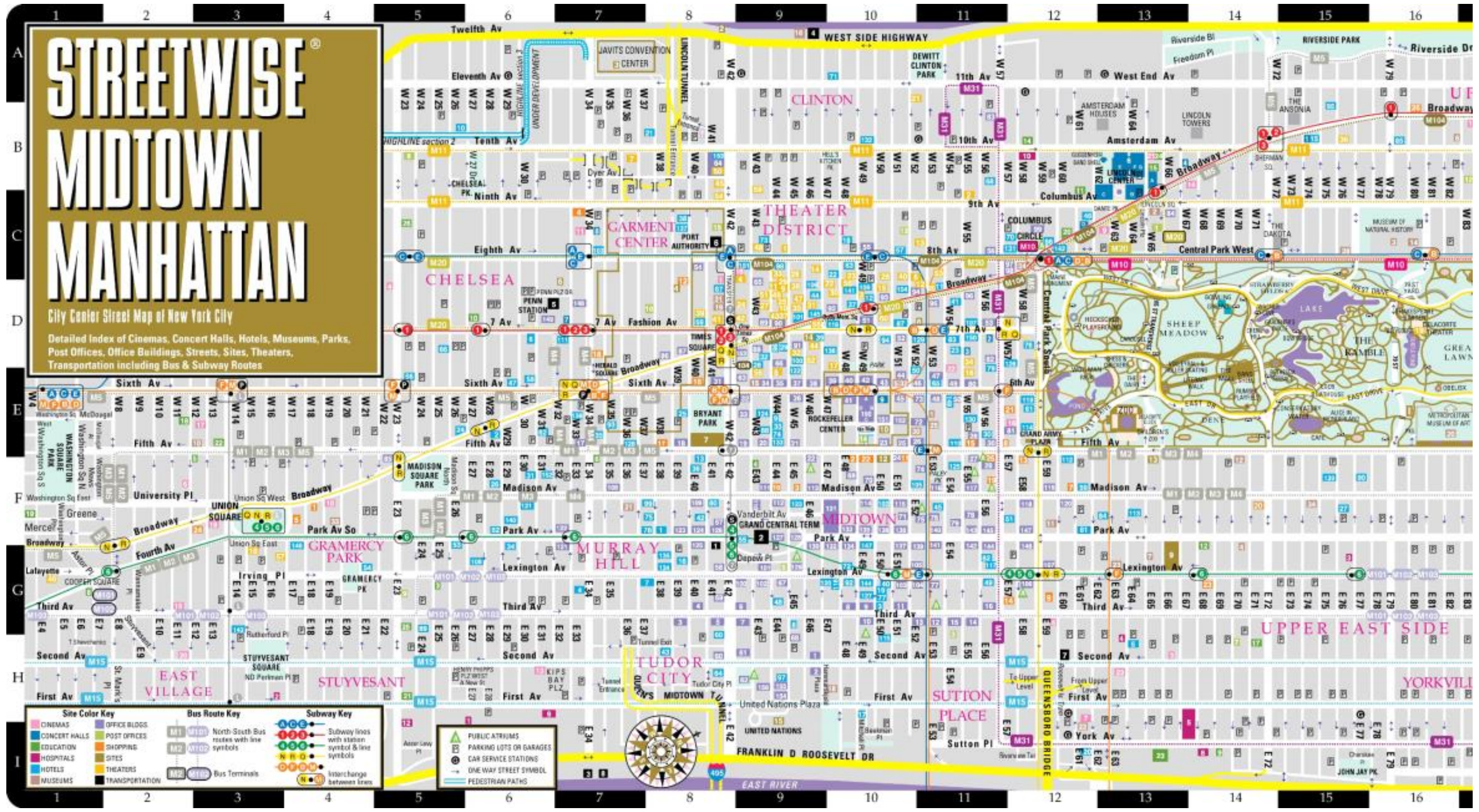
Recursion: $L(x,y) = \max(L(x-1,y),L(x,y-1)) + w_{xy}$

Iterating: $L(x,y) = \max_{\pi:(1,1)\to(x,y)} \sum_{(i,j)\in\pi} w_{ij}$

<u>KPZ class behavior</u>: L(xt,yt) behaves like $ct + c' t^{1/3} \chi$ where χ is F_{GUE} distributed and the constants depend on x,y.

Optimal paths in random environment



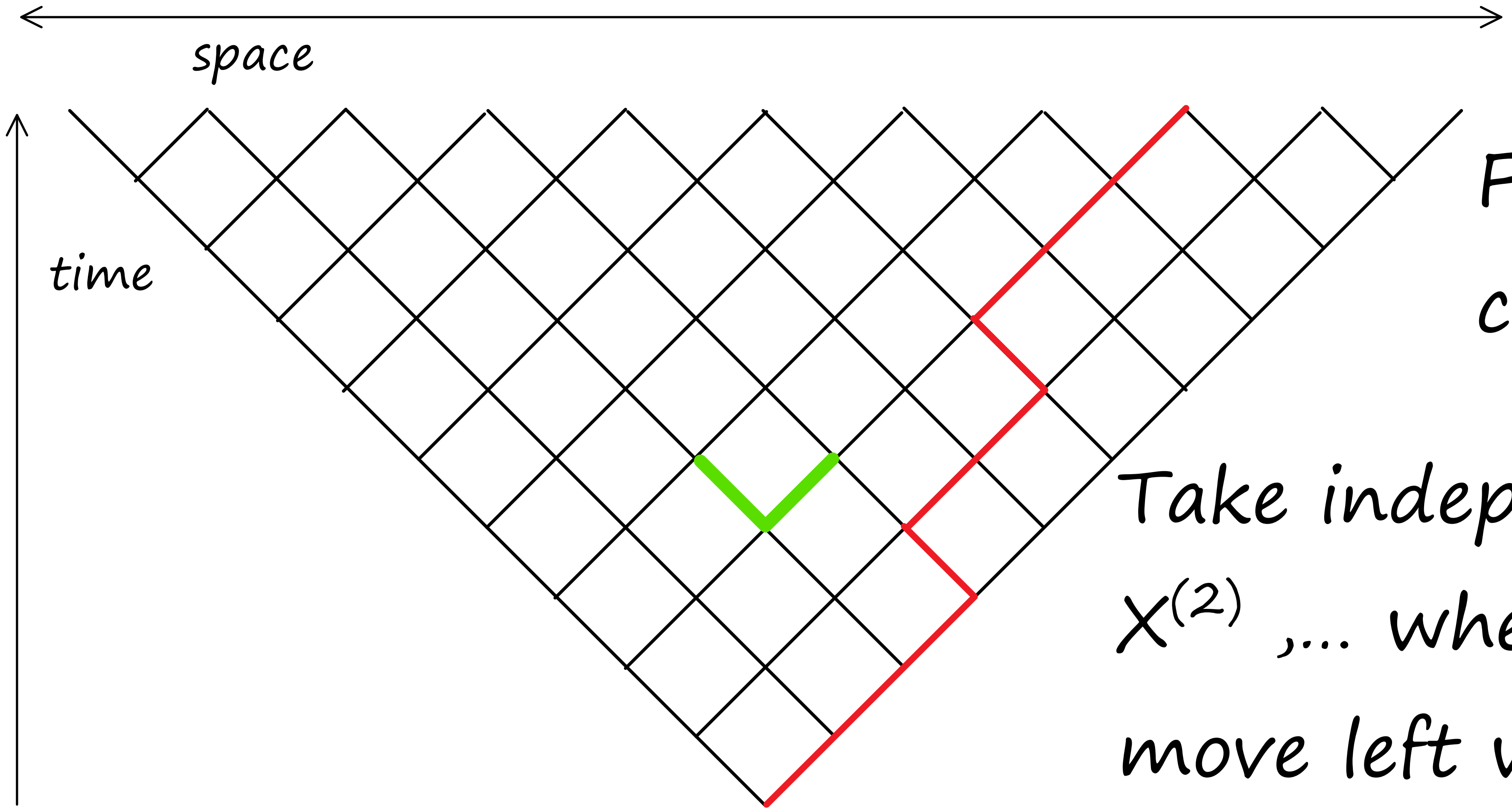


[Barraquand-C '15]: Assign edge weights to each └ so with probability 1/2, horizontal weight is 0 and vertical is exp(1); otherwise reversed.

Minimal passage time $P(x,y) = \min_{\pi:(0,0)\to(x,y)} \sum_{e\in\pi} w_e$.

<u>KPZ class behavior</u>: For $x \neq y$, $P(xt,yt)$ behaves like $ct + c' t^{1/3} \chi$ where χ is F_{GUE} distributed and the constants depend on x,y.

Random walk in random environment



For each (space,time)-vertex choose u_{ys} uniform on [0,1].

Take independent random walks $X^{(1)}$, $X^{(2)}$,... where at time s and position y, move left with probability u_{ys}, right with $1-u_{ys}$. Let $M(t,N) = \max(X^{(1)}, \dots, X^{(N)})$.

<u>KPZ class behavior</u>: For $0<r<1$, $M(t,e^{rt})$ behaves like $ct + c' t^{1/3} \chi$ where χ is F_{GUE} and the constants depend on r [Barraquand-C '15].

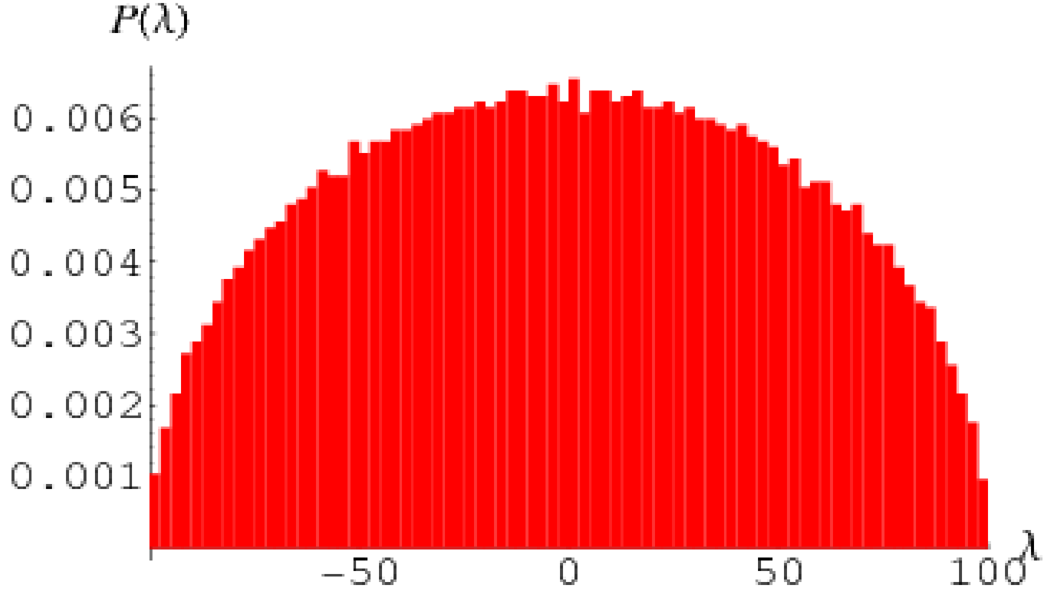
If all $u_{ys}=1/2$ (i.e. simple symmetric random walk), large deviations and extreme value theory implies order one Gumbel fluctuations.

Big data and random matrices

Gaussian Unitary Ensemble (GUE) on N x N complex matrices:

$$H^{(N)} = \left[H_{ij}^{(N)}\right]_{ij=1}^{N} \text{ where } H_{ij} = \overline{H}_{ji} = \begin{cases} \mathcal{N}(0,\sqrt{N}) & i=j \\ \mathcal{N}(0,\sqrt{N/2}) + \sqrt{-1}\,\mathcal{N}(0,\sqrt{N/2}) & i \neq j \end{cases}$$

Introduced by [Wigner '55] to model the energy levels/gaps of atoms too complicated to solve analytically.



Let $\lambda_1^{(N)} \geq \cdots \geq \lambda_N^{(N)}$ denote the (random) real eigenvalues of $H^{(N)}$.

<u>KPZ class behavior</u>: $\lambda_1^{(N)}$ behaves like $2N + N^{1/3}\chi$ where χ is F_{GUE}.

Relationship to growth processes is much less apparent here.

Big data and random matrices

Complex Wishart Ensemble (or sample covariance) on N x M matrices:

$$H^{(N,M)} = \left[H^{(N,M)}_{ij}\right]_{\substack{1\le i\le N\\ 1\le j\le M}} \quad \text{where} \quad H_{ij} = \mathcal{N}(0, \tfrac{1}{2}) + \sqrt{-1}\,\mathcal{N}(0, \tfrac{1}{2})$$

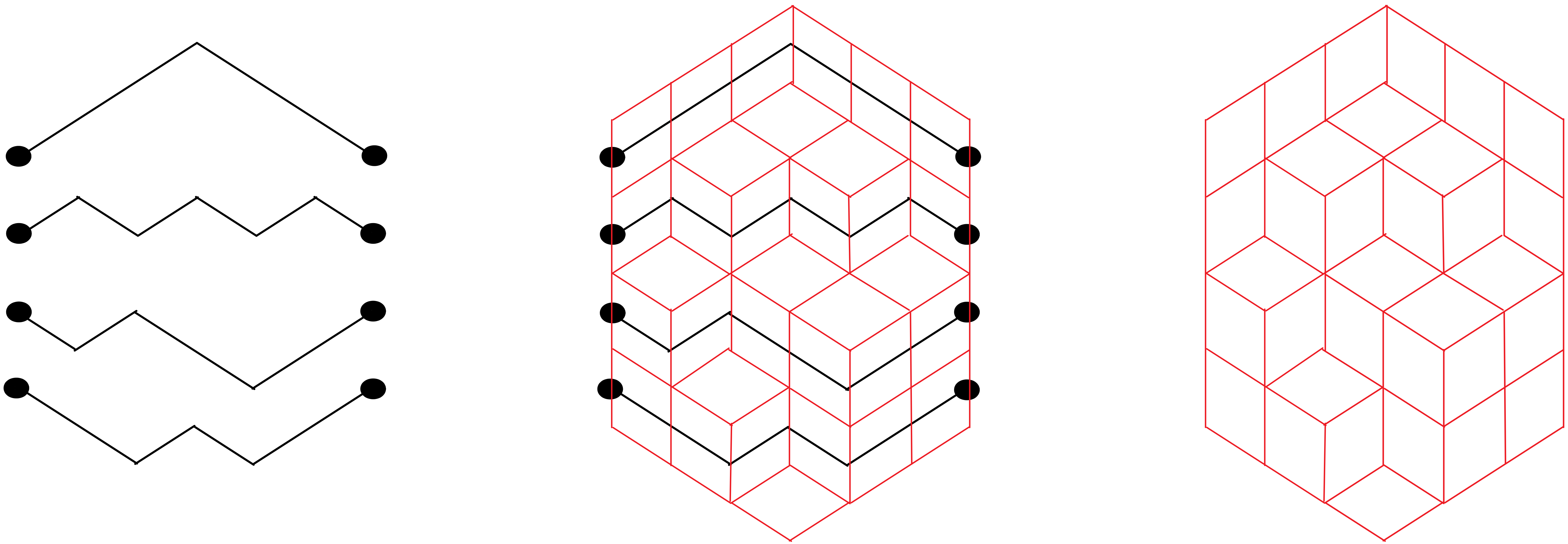
Introduced by [Wishart '28] within statistics. Provides a base-line for noisy data against which to compare Principal Component Analysis

Let $\sigma_1^{(N,M)} \geq \cdots \geq \sigma_N^{(N,M)}$ denote the (random) real positive singular values of $H^{(N,M)}$ (i.e., the square-roots of eigenvalues of $H^{(N,M)}(H^{(N,M)})^*$).

Surprise [Johansson '00]: The distribution of $\sigma_1^{(N,M)}$ equals that of L(N,M).

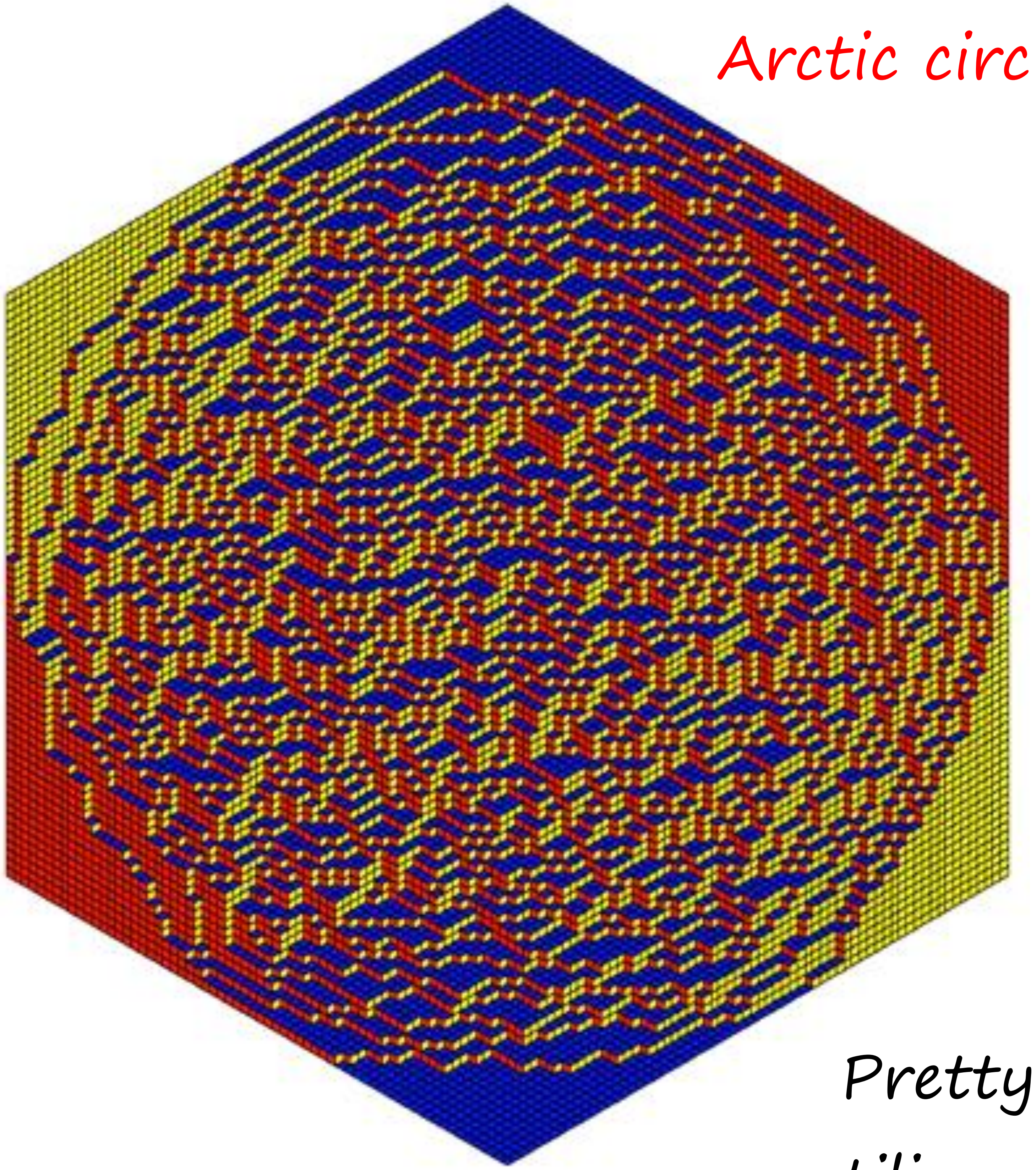
E.G. N=M=1, $\text{Probability}(\sigma_1^{(1,1)} \leq s) = \frac{1}{\pi}\int_{x^2+y^2\leq s^2} e^{-x^2-y^2}\,dx\,dy = \int_0^{s^2} e^{-r^2}\,2r\,dr = \int_0^{s} e^{-t}\,dt$

Vicious walkers and random tilings



Consider N random walks with fixed starting and ending points, conditioned not to touch. This gives rise to a uniform measure on fillings of a box, or tilings of a hexagon by three types of rhombi.

Vicious walkers and random tilings



Arctic circle theorem [Cohn-Larsen-Propp '98]

KPZ class behavior: The top walker (or edge of the arctic circle) has fluctuations of order $N^{1/3}$ and limiting F_{GUE} distribution. [Baik-Kriecherbauer-McLaughlin-Miller '07], [Petrov '12]

Pretty pictures (and math) when tiling various types of domains

Open problems

- **Higher dimension** (e.g. random surface growth)
- **KPZ universality** (scale, distribution, entire space-time limit)
  - Growth processes (e.g. ballistic deposition, Eden model)
  - Interacting particle systems (e.g. non-nearest neighbor exclusion)
  - Last/first passage percolation, RWRE with general weights
- Full description of **KPZ fixed point**
  - Complete space-time multpoint distribution
  - Unique characterization of fixed point
- **Weak universality** of the KPZ equation
  - Under critical weak tuning of the strength of model parameters
- Discover **new integrable examples and tools in their analyses**

Summary

- Integrable probabilistic systems reveal details of large universality classes. They are intimately connected to certain algebraic structure
- Coin flipping and Gaussian universality class is simplest example
- Random interface growth leads to new phenomena such as spatial correlation, smaller fluctuations and new distributions
- KPZ class arises in various growing interfaces, and the analysis of the corner growth model reveals its properties
- KPZ class encompasses many other types of systems, including stochastic PDEs, traffic flow, optimal paths in random environments, random walks in random environments, big data and random matrices, vicious walkers and tilings…