

Knots and 3-manifolds: Problem Set 3

due Tuesday, July 23

1. Find a sequence of Reidemeister moves connecting the diagrams M_0 and M_1 of the trefoil (on page 1 of the handout).

2. A knot diagram is said to be **three-colorable** if each arc of the diagram (from one underpass to the next) can be colored red, blue or green so that all three colors are used and at each crossing either one color or all three colors appear. Show that three-colorability is unchanged by Reidemeister moves. Deduce that the knots 3_1 and 4_1 are distinct.

3. Let L be a link with two components J and K . Prove that $\text{lk}(J, K) = \text{lk}(K, J)$.

Hint: Use induction over the number of crossings of J under K in a link diagram. Investigate what happens if you change an overcrossing into an undercrossing.

4. Show that $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$, for any two knots K_1 and K_2 .

5. Here is a definition of the Alexander polynomial different from the one given in class. Let G be the group of a knot K and let $\langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$ be a presentation of G . Each relation r_i is a product of some x_j 's and x_j^{-1} 's. "Differentiate" each relation r_i with respect to each generator x_j according to the following rules:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}, \quad \frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1},$$

$$\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} + u \frac{\partial v}{\partial x_j}.$$

Here $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise. The end result of this differentiation is a formal sum of elements of the group G . For example,

$$\frac{\partial(xy x^{-1} z)}{\partial x} = 1 - xy x^{-1}.$$

Consider the abelianization map $\phi : G \rightarrow \text{Ab}(G) = H_1(S^3 \setminus K) \cong \mathbb{Z}$. This takes each x_i into a multiple of the generator of \mathbb{Z} . We write this as $\phi(x_i) = t^k$ for some integer k . (For example, in the Wirtinger presentation all generators are mapped to t .) Extend the map ϕ linearly to formal sums of elements of G , such that these are mapped to polynomials in t . Now consider the m by n matrix M with elements

$$m_{ij} = \phi\left(\frac{\partial r_i}{\partial x_j}\right).$$

Take the determinant of any $(n-1) \times (n-1)$ minor of the matrix M . Up to sign and multiplication by a power of t , this is the Alexander polynomial of K . (You are not being asked to prove this.)

Find the Alexander polynomial of the trefoil in this fashion, starting with a Wirtinger presentation.