Version B

Calculus III sec. 003 - First Midterm

You have 1 hour 15 minutes. No books, notes or calculators are allowed.

Do not open the exam until told to do so.

Part I

No justifications are needed in this part.

1. True/False: Circle the correct answer. (1 point each)

(a) Let $\vec{v}, \vec{w}$ be two vectors in space perpendicular to a given plane Σ. Then $\vec{v} - \vec{w}$ is also perpendicular to Σ.

T

(b) If two vectors $\vec{v}$ and $\vec{w}$ are perpendicular, then $\vec{v} \times \vec{w} = 0$.

F

(c) For any nonzero vectors $\vec{v}, \vec{w}$ in space, the vector $(\vec{v} \times \vec{w}) \times \vec{v}$ is parallel to $\vec{w}$.

F

(d) Let $\vec{v}, \vec{w}$ be two vectors in space such that $\vec{v} \cdot \vec{w} = 0$ and $\vec{v} \times \vec{w} = 0$. Then at least one of $\vec{v}$ and $\vec{w}$ is the zero vector.

T

(e) Let Z be the set of points in the plane equidistant from the point $(2, 5)$ and the line $x - y = 1$. Then Z is a hyperbola.

F

2. (1 point each)
(a) Give an example of a quadratic equation in x, y and z describing a cylinder.

$$x^2 + y^2 - 1 = 0$$

(b) Give an example of a quadratic equation in x, y and z describing the union of two parallel planes.

$$x(x - 1) = 0$$

(c) Give an example of a quadratic equation in x, y and z describing a single line in space.

$$x^2 + y^2 = 0$$

Part II

For the rest of the problems, to get full credit you need to fully justify your answers.

3. The point P has Cartesian coordinates $(\sqrt{3}, -1)$. Find its polar coordinates. (3 points)

We have

$$r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$$

and

$$\tan\theta = \frac{y}{x} = -\frac{\sqrt{3}}{3}.$$

The latter equation gives $\theta = 5\pi/6$ or $\theta = 11\pi/6$. Since the point is in the fourth quadrant, we have $\theta = 11\pi/6$.

4. Find the length of the curve given in polar coordinates by $r = e^{2\theta},\ 0 \leq \theta \leq 2\pi$. (4 points)

$$\int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta = \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}} d\theta =$$

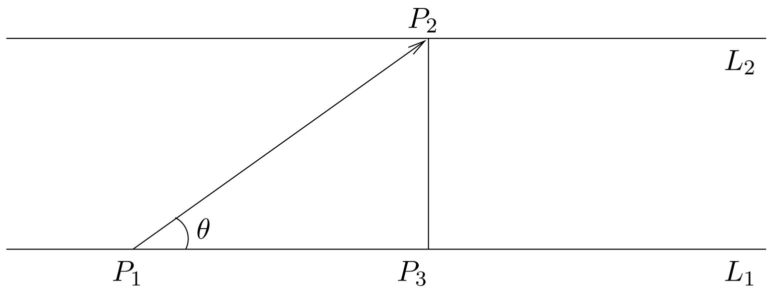
$$= \int_0^{2\pi} \sqrt{5} e^{2\theta} d\theta = \sqrt{5} \frac{e^{2\theta}}{2} \Big|_0^{2\pi} = \frac{\sqrt{5}}{2}(e^{4\pi} - 1).$$

5. Consider the following four points in space: $P_1(1,1,1), P_2(1,1,-1), P_3(0,2,0)$ and $P_4(5,-3,1)$. Do they lie in the same plane? (5 points)

We have $\overrightarrow{P_1P_2} = \langle 0,0,-2\rangle,\ \overrightarrow{P_1P_3} = \langle -1,1,-1\rangle,\ \overrightarrow{P_1P_4} = \langle 4,-4,0\rangle$.
The triple scalar product $\overrightarrow{P_1P_2} \cdot (\overrightarrow{P_1P_3} \times \overrightarrow{P_1P_4})$ evaluates to

$$\begin{vmatrix} 0 & 0 & -2 \\ -1 & 1 & -1 \\ 4 & -4 & 0 \end{vmatrix} = (-2)\begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} = (-2)\cdot((-1)\cdot(-4) - 1\cdot 4) = 0.$$

Since this is zero, the points are coplanar.



6. Consider the following two lines in space, represented by the parametric equations:

$$L_1: \quad x(t) = -1 + t,\ y(t) = 2 - 2t,\ z(t) = 1 - t$$

$$L_2: \quad x(s) = 1 - 2s,\ y(s) = 1 + 4s,\ z(s) = 2s.$$

(a) Determine whether L_1 and L_2 are parallel, intersecting, or skew. (2 points)

Their direction vectors are $\vec{v_1} = \langle 1, -2, -1 \rangle$ and $\vec{v_2} = \langle -2, 4, 2 \rangle$. Since $\vec{v_2} = -2\vec{v_1}$, the lines are parallel.

(b) Consider the points $P_1(-1, 2, 1)$ on L_1 (corresponding to $t = 0$) and $P_2(1, 1, 0)$ on L_2 (corresponding to $s = 0$). Let θ be the angle between the vector $\overrightarrow{P_1P_2}$ and the line L_1. Find $\cos\theta$ and $\sin\theta$. (4 points)

We have $\overrightarrow{P_1P_2} = \langle 2, -1, -1 \rangle$ of length $|\overrightarrow{P_1P_2}| = \sqrt{4+1+1} = \sqrt{6}$. We are interested in its angle with the vector $\vec{v_1} = \langle 1, -2, -1 \rangle$ of length $|\vec{v_1}| = \sqrt{1+4+1} = \sqrt{6}$. Since

$$\overrightarrow{P_1P_2} \cdot \vec{v_1} = 2 \cdot 1 + (-1) \cdot (-2) + (-1) \cdot (-1) = 5,$$

we obtain

$$\cos\theta = \frac{\overrightarrow{P_1P_2} \cdot \vec{v_1}}{|\overrightarrow{P_1P_2}| \cdot |\vec{v_1}|} = \frac{5}{\sqrt{6} \cdot \sqrt{6}} = \frac{5}{6}$$

and

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{25}{36}} = \frac{\sqrt{11}}{6}$$

(c) Find the distance between L_1 and L_2. (4 points)

Since the lines are parallel, the distance between them is just the distance from any point on L_2 (for example P_2) to the line L_1. In the figure above, the distance is the magnitude of the segment P_2P_3, where P_3 is the projection of P_2 to L_1. We get

$$|P_2P_3| = |P_1P_2| \cdot \sin\theta = \sqrt{6} \cdot \frac{\sqrt{11}}{6} = \frac{\sqrt{66}}{6}$$

7. Let Σ be the surface consisting of all points in space whose distance to the origin is $\sqrt{2}$ times their distance to the plane $z = 1$.

(a) Find an equation for Σ. (4 points)

Let $P(x, y, z)$ be the coordinates of a point on Σ. The distance between P and $(0, 0, 0)$ is

$$\sqrt{x^2 + y^2 + z^2}$$

while the distance from P to the plane $0 \cdot x + 0 \cdot y + 1 \cdot z - 1 = 0$ is

$$\frac{0 \cdot x + 0 \cdot y + 1 \cdot z - 1}{\sqrt{0 + 0 + 1}} = |z - 1|$$

Thus the equation is

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{2}|z - 1|,$$
$$x^2 + y^2 + z^2 = 2(z - 1)^2,$$

or

$$x^2 + y^2 = (z - 2)^2 - 2.$$

(b) Sketch Σ, by drawing at least three traces. (As a suggestion, look at the intersections of Σ with various horizontal planes, given by equations $z =$ constant.) (4 points)

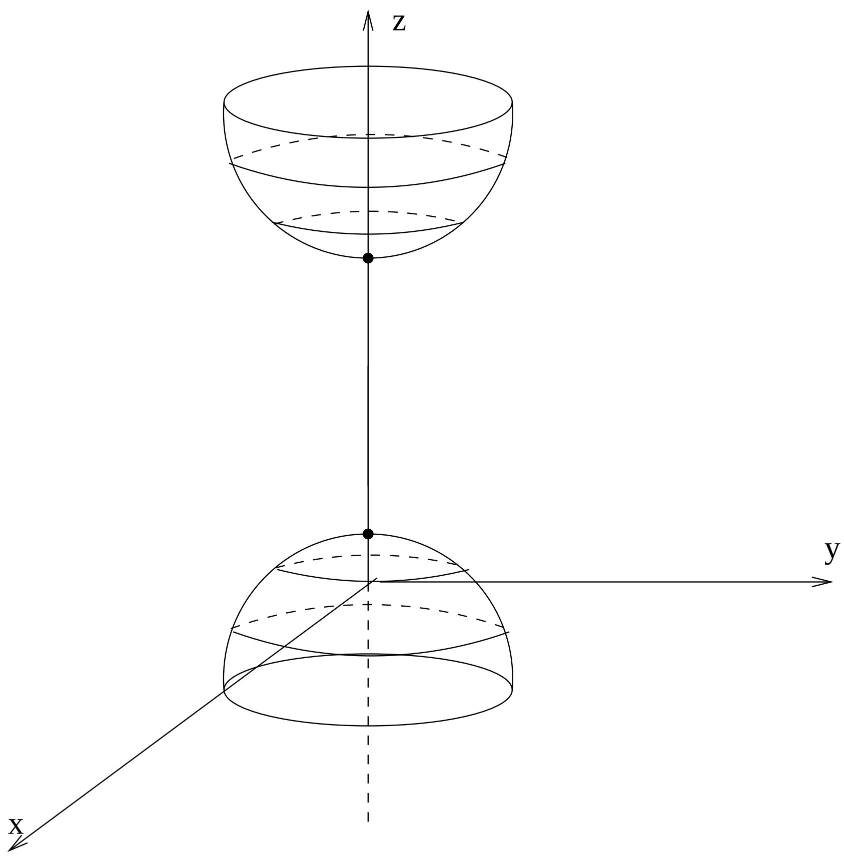
The horizontal traces are given by

$$z = z_0, \; x^2 + y^2 = (z_0 - 2)^2 - 2.$$

For z_0 between $-\sqrt{2} + 2$ and $\sqrt{2} + 2$, there are no solutions to $x^2 + y^2 = (z_0 - 2)^2 - 2$.
For $z_0 > \sqrt{2} + 2$, the traces are circles, getting bigger and bigger as z_0 increases.
For $z_0 < -\sqrt{2} + 2$, the traces are circles, getting bigger and bigger as z_0 decreases.
The graph looks like the picture above.

(c) What kind of quadric is Σ? (2 points)

A hyperboloid of two sheets.



z
y
x