Solution Set 10

1. (a) -1 ;
(b) $(1+4i)(2+4i) = 2+4i+8i+16i^2 = -14+12i$;
(c) $(2-3i)(2+3i) = 4+6i-6i-9i^2 = 13$.

2.
(a)

$$\frac{2+i}{3+i} = \frac{(2+i)(3-i)}{(3+i)(3-i)} = \frac{7+i}{10} = \frac{7}{10} + \frac{1}{10}i$$

(b)

$$\frac{1+4i}{2+8i} = \frac{(2+i)}{2(1+4i)} = \frac{1}{2}$$

(c)

$$\frac{2-3i}{3+2i} = \frac{(2-3i)(3-2i)}{(3+2i)(3-2i)} = \frac{-13i}{13} = -i.$$

3. (a) We have $r = \sqrt{1^2+(-\sqrt{3})^2} = 2$ and $\theta = \tan^{-1}(-\sqrt{3}) = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. We pick the latter because $(1, -\sqrt{3})$ is in the fourth quadrant. Hence:

$$1-\sqrt{3}i = 2(\cos(5\pi/3) + i\sin(5\pi/3)).$$

Similar reasoning gives:
(b)

$$-5+5i = 5\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4)).$$

(c)

$$\pi i = \pi(\cos(\pi/2) + i\sin(\pi/2)).$$

4.

$$3-2i = \sqrt{13}\Big(\cos(\tan^{-1}(-\frac{2}{3})) + i\sin(\tan^{-1}(-\frac{2}{3}))\Big).$$

5.
(a)

$$e^{-\pi i/4} = \cos(-\pi/4) + i\sin(-pi/4) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$$

(b)

$$e^{1+\pi i} = e(\cos\pi + i\sin\pi) = e(-1+0\cdot i) = -e.$$

(c)

$$e^{3+i} = e^3\cos(1) + ie^3\sin(1).$$

6.
In polar form we have

$$\sqrt{3}-i = 2(\cos(-\pi/6) + i\sin(-\pi/6)),$$

so, by DeMoivre's Theorem,

$$(\sqrt{3}-i)^7 = 2^7(\cos(-7\pi/6) + i\sin(-7\pi/6)) = 128(-\frac{\sqrt{3}}{2} + i\frac{1}{2}) = -64\sqrt{3} + 64i.$$

Similarly, in polar form

$$1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)),$$

hence

$$(1+i)^9 = \sqrt{2}^9(\cos(9\pi/4) + i\sin(9\pi/4)) = 16 + 16i.$$

7. We divide both sides of the equation by z^5 we get

$$(\frac{z+1}{z})^5 = 1.$$

Thus, $w = \frac{z+1}{z}$ is a fifth root of unity. By the formula on p.5 of the handout, we have

$$w = \cos(\frac{2k\pi}{5}) + i\sin(\frac{2k\pi}{5}),$$

where k can be $0, 1, 2, 3$, or 4.

We can express z in terms of w:

$$w = \frac{z+1}{z} \text{ implies } z + 1 = wz \text{ implies } z = \frac{1}{w-1}.$$

This doesn't work for $k = 1$, because in that case $w = 1$ and we cannot divide by $w-1 = 0$. We are left with four solutions, for $k = 1, 2, 3, 4$:

$$z = \frac{1}{\cos(\frac{2k\pi}{5}) + i\sin(\frac{2k\pi}{5}) - 1}.$$

8. The polar coordinates of $(1, 1)$ are $r = \sqrt{2}, \theta = \pi/4$, so in polar form

$$1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(pi/4)) = \sqrt{2}e^{i\pi/4}.$$

The formula for the complex logarithm is

$$\log(re^{i\theta}) = \ln r + (\theta + 2n\pi)i,$$

where n is an arbitrary integer.

Therefore,

$$\log(1+i) = \ln\sqrt{2} + (\pi/4 + 2n\pi)i = \frac{ln(2)}{2} + (2n + \frac{1}{4})\pi i,$$

where n is an arbitrary integer.

To find i^i, we first find $\log(i)$. The polar coordinates for i are $r = 1$ and $\theta = \pi/2$, so

$$\log(i) = \ln(1) + (\pi/2 + 2n\pi)i = (2n + \frac{1}{2})\pi i.$$

From here we obtain

$$i^i = e^{i\log(i)} = e^{(2n+\frac{1}{2})\pi i^2} = e^{-(2n+\frac{1}{2})\pi},$$

where n can be any integer.

9. Following the hint, let $z = e^{it}$, so that

$$x = \cos(t) = \frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}(z + z^{-1}).$$

The idea is to first express z in terms of x and then find $t = \cos^{-1}(x)$.

To express z in terms of x, note that $x = \frac{1}{2}(z + z^{-1})$ implies that z is a solution of the quadratic equation

$$z^2 - 2xz + 1 = 0.$$

By completing the squares, we get

$$(z - x)^2 = x^2 - 1,$$

so

$$z = x + \sqrt{x^2 - 1}. \tag{1}$$

Note that this is the complex square root, which can take two different values. If $x^2 - 1$ were a positive real number, there would be a preferred square root (the positive one). However, in general $x^2 - 1$ is a complex number, so its square root is not unique; there are two square roots, one being the negative of the other, but there is no preferred square root, so $\sqrt{x^2 - 1}$ denotes either of them.

Going back to $z = e^{it}$, this gives $it = \log(z)$ so

$$\cos^{-1}(x) = t = \frac{1}{i} \cdot \log(z) = -i\log(z).$$

Substituting the formula (1) for z gives

$$\cos^{-1}(x) = -i\log(x + \sqrt{x^2 - 1}).$$

Here $\sqrt{\cdot}$ is the complex square root (with two possible values), and log is the complex logarithm (with infinitely many possible values).