SPECTRAL MOMENT FORMULAE FOR $GL(3) \times GL(2)$ L-FUNCTIONS

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ABSTRACT. Spectral moment formulae of various shapes have proven to be very successful in studying the statistics of central L-values. In this article, we establish, in a completely explicit fashion, such formulae for the family of $GL(3) \times GL(2)$ Rankin-Selberg L-functions using the period integral method. The Kuznetsov and the Voronoi formulae are not needed in our argument. We also prove the essential analytic properties and explicit formulae for the integral transform of our moment formulae. It is hoped that our method will provide insights into moments of L-functions for higher-rank groups.

1. INTRODUCTION

1.1. **Background.** The study of L-values at the central point $s = 1/2$ has been the center stage for many branches of number theory in the past decades because of the great arithmetic significance behind them. There have been a variety of interesting perspectives furnishing our understanding of the nature of central L-values. As an example, one may wish to take a statistical look at them. Fundamental questions in this direction include the determination of (non-)vanishing and sizes of these L-values. A particularly effective way to approach problems of this sort is via *moments of L-functions*. Techniques from analytic number theory have proven to be very successful in estimating the sizes of moments of all kinds. Moreover, spectacular results can be obtained when moment estimates join forces with arithmetic geometry and automorphic representations.

This line of investigation is nicely exemplified by the landmark result of Conrey-Iwaniec [CI00]. Let χ be a real primitive Dirichlet character $(\bmod q)$ with q odd and square-free. The main object of [CI00] is the cubic moment of $GL(2)$ automorphic L-functions of the congruence subgroup $\Gamma_0(q)$ twisted by χ. An upper bound of Lindelöf strength in the q-aspect was established therein. When combining this upper bound with the celebrated results of [Wa81, KoZa81, KS93, Gu96], the famous Burgess 3/16-bound for Dirichlet L-functions was improved for the first time since the 1960's. In fact, [CI00] proved the bound

$$L\left(\frac{1}{2}, \chi\right) \ll_\epsilon q^{\frac{1}{6}+\epsilon}. \tag{1.1}$$

Understanding the effects of a sequence of intricate transformations, which can be of arithmetic or analytic nature, constitutes a significant part of moment calculations. In the context of [CI00], such a sequence ends up in a beautiful, unexpected *exact identity* which showcases the duality behind this particular moment problem, see Petrow [Pe15], Frolenkov [Fr20], as well as the earlier work of Ivić [Iv01, Iv02]. The cubic average over a basis of $GL(2)$ automorphic forms (Maass or holomorphic) of level one dualizes to the fourth moment of a $GL(1)$ L-function. Also, it transforms a moment problem about the *spectral aspect* to a different one about the *t-aspect*. Roughly speaking, it takes the shape

$$\sum_f L\left(\frac{1}{2}, f\right)^3 = \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt + (***), \tag{1.2}$$

where the weight functions for the moments are suppressed and $(***)$ represents certain polar contributions.

Besides its structural elegance, the identity (1.2) comes with immediate applications. It leads to sharp moment estimates as a consequence of exact evaluation. As an extra benefit, it cleans up the analysis in the traditional but approximate approach. In [Pe15], such an identity was termed a 'Motohashi-type identity'. Indeed, Motohashi [Mo93, Mo97] discovered an identity of this sort but with the choice of test function made

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on the fourth moment side instead. It greatly enhances our understanding of the fourth moment of the ζ-function. There are also the recent works of Blomer-Humphries-Khan-Milinovich [BHKM20], Topacogullari [To21] and Kaneko [Ka21+] extending Motohashi's work to Dirichlet L-functions.

Very recently, this type of Motohashi phenomenon is also nicely accounted under various insightful frameworks in automorphic representations. See the work of Nelson [Ne20+], Wu [Wu21+] and Balkanova-Frolenkov-Wu [BFW21+]. In [Ne20+], the regularized period method proposed by Michel-Venkatesh [MV10] was fully and rigorously developed. In [Wu21+, BFW21+], the authors developed a relative trace formula of Godement-Jacquet type.

Ten years after [CI00], Xiaoqing Li [Li11] made a breakthrough by applying the techniques of [CI00] together with new inputs of her own to settings of higher-rank groups. She studied the first moment of $GL(3)\times GL(2)$ Rankin-Selberg L-functions in the $GL(2)$ spectral aspect. This gave the first instance of subconvexity for $GL(3)$ automorphic L-functions.

1.2. **Main Results.** The main purpose of this article is to illustrate a different strategy towards moment problems and identities of Motohashi type, and more importantly, with a view towards generalizations to higher-rank groups. As a concrete example, our method is implemented in the context of [Li11] and the main result of this article is stated as follows.

Theorem 1.1. *Let*

- Φ *be a fixed, Hecke-normalized Maass cusp form of* $SL_3(\mathbb{Z})$ *with the Langlands parameters* $(\alpha_1,\alpha_2,\alpha_3)\in(i\mathbb{R})^3$, *and* $\widetilde{\Phi}$ *be the dual form of* Φ;
- $(\phi_j)_{j=1}^{\infty}$ *be an orthogonal basis of even, Hecke-normalized Maass cusp forms of* $SL_2(\mathbb{Z})$ *which satisfy* $\Delta\phi_j=\left(\frac{1}{4}-\beta_j^2\right)\phi_j$;
- $L\left(s,\phi_j\otimes\Phi\right)$ *and* $L\left(s,\Phi\right)$ *be the Rankin-Selberg L-function of the pair* (ϕ_j,Φ) *and the standard L-function of* Φ *respectively, where* L^* *denotes the corresponding complete L-functions;*
- $\mathcal{C}_\eta$ $(\eta>40)$ *be the class of holomorphic functions* H *defined on the vertical strip* $|\operatorname{Re}\beta|<2\eta$ *such that* $H(\beta)=H(-\beta)$ *and has rapid decay:*
$$H(\beta)\ \ll\ e^{-10|\beta|}\qquad\qquad(|\operatorname{Re}\beta|\ <\ 2\eta).$$
- *For* $H\in\mathcal{C}_\eta$, $\left(\mathcal{F}_\Phi H\right)(s_0,s)$ *is the integral transform defined in (4.5).*

Then on the domain $\frac{1}{4}+\frac{1}{200}<\sigma<\frac{3}{4}$, *we have the following moment identity:*

$$\frac{1}{2}\sum_{j=1}^{\infty}H\left(\beta_j\right)\ \frac{L^*\left(s,\phi_j\otimes\widetilde{\Phi}\right)}{\langle\phi_j,\phi_j\rangle}\ +\ \frac{1}{8\pi}\int_{\mathbb{R}}H\left(i\mu\right)\frac{L^*\left(s+i\mu,\widetilde{\Phi}\right)L^*\left(1-s+i\mu,\Phi\right)}{\left|\zeta^*(1+2i\mu)\right|^2}\,d\mu$$
$$=\ \frac{\pi^{-3s}}{4}\ L(2s,\Phi)\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\prod_{i=1}^{3}\Gamma\left(\frac{s+\beta-\alpha_i}{2}\right)\Gamma\left(\frac{s-\beta-\alpha_i}{2}\right)\frac{d\beta}{2\pi i}$$
$$+\ \frac{1}{4}\ L(2s-1,\Phi)\ \left(\mathcal{F}_\Phi H\right)(2s-1,s)$$
$$+\ \frac{1}{4}\int_{(1/2)}\zeta\left(2s-s_0\right)L\left(s_0,\Phi\right)\cdot\left(\mathcal{F}_\Phi H\right)(s_0,s)\ \frac{ds_0}{2\pi i}. \tag{1.3}$$

In Section 7, we provide more expressions for $\left(\mathcal{F}_\Phi H\right)(s_0,s)$. In particular,

Proposition 1.2. *For* $\frac{1}{2}+\frac{1}{100}<\sigma<1$, *we have*

$$\left(\mathcal{F}_\Phi H\right)(2s-1,s)\ =\ \pi^{\frac{1}{2}-s}\prod_{i=1}^{3}\frac{\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)}{\Gamma\left(1-s-\frac{\alpha_i}{2}\right)}\cdot\int_{(0)}\frac{H(\beta)}{\left|\Gamma(\beta)\right|^2}\cdot\prod_{i=1}^{3}\prod_{\pm}\Gamma\left(\frac{1-s+\alpha_i\pm\beta}{2}\right)\frac{d\beta}{2\pi i}. \tag{1.4}$$

More generally, the transform can be expressed in terms of *hypergeometric functions* of special types. Recently, the articles [BF18, BF21], [BBFR20], [BFW21+] have brought in the powerful asymptotic analysis of hypergeometric functions into the study of moments and obtain sharp estimates in the spectral aspect.

Also, our class of admissible test functions in Theorem 1.1 is large enough for such prospects for the family of $GL(3)\times GL(2)$ L-functions, see Remark 3.6 for a further discussion.

Remark 1.3. *If the fixed cusp form Φ in Theorem 1.1 is replaced by the minimal parabolic Eisenstein series of $SL_3(\mathbb{Z})$, the proof of Theorem 1.1 gives a new proof of the cubic moment identity (1.2). In this case, $L(s,\Phi)$ becomes a product of three ζ-functions and there will be extra polar contributions from the degenerate part of the Fourier expansion of the Eisenstein series.*

Moreover, if one replaces Φ by a maximal parabolic Eisenstein series twisted by a Maass cusp form ϕ of $SL_2(\mathbb{Z})$, one obtains an identity of the form

$$\sum_{j=1}^{\infty} L\left(\frac{1}{2},\phi\otimes\phi_j\right)L\left(\frac{1}{2},\phi_j\right) \;=\; \int_{\mathbb{R}} L\left(\frac{1}{2}+it,\phi\right)\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \;+\; (***). \tag{1.5}$$

Remark 1.4. *It is possible to incorporate twists of Hecke eigenvalues into the spectral average of Theorem 1.1 for broader applications, see Appendix A.*

1.3. **Features of Our Method.** An important feature of our method is that we are able to uncover the dual moment, i.e., the right side of (1.3), quickly and naturally thanks to the structural advantages provided by the period integral:

$$\left\langle P(*;h)\ ,\ (\mathbb{P}_2^3\Phi)\cdot|\det *|^{\bar{s}-\frac{1}{2}}\right\rangle_{L^2(SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}))}, \tag{1.6}$$

where $P(*;h)$ is a Poincaré series of $SL_2(\mathbb{Z})$ (with a test function h), Φ is a fixed Maass cusp form of $SL_3(\mathbb{Z})$, $(\mathbb{P}_2^3\Phi)(g) := \Phi\begin{pmatrix} g & \\ & 1\end{pmatrix}$, and $\langle\cdot\,,\,\cdot\rangle_{L^2(SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}))}$ is the Petersson inner product.

Actually, Theorem 1.1 is an equality of two Rankin-Selberg unfoldings of $GL(2)$ in two distinct directions. This is possible because of the 'disparity of ranks' present in our case. In comparison, [CI00, Pe15, Fr20, Ne20+, Wu21+, BFW21+] stayed in the $GL(2)$ setting only. The first unfolding uses the Fourier expansion of the form Φ, which has the effect of 'projecting' Φ from $GL(3)$ to $GL(2)$, and gives the desired spectral average for $GL(3)\times GL(2)$ L-functions. This is a $GL(2)$ calculation in essence. The second unfolding is performed using the definition of the Poincaré series $P(*;h)$. We are then led to a $GL(3)$ calculation regarding an *incomplete unipotent integration* of the Maass form Φ. This makes the dual structures completely visible for linear algebra reasons. Not only is our method more direct, but generalizations to higher-rank situations are much more likely when compared to the traditional 'Kuznetsov-Voronoi' approach.

The Kuznetsov trace formula, or more generally the relative trace formula, has been a cornerstone in the analytic theory of L-functions for $GL(2)$ during the past few decades. It is an equality between the spectral average of Fourier coefficients over a basis of automorphic forms and the geometric expansion consisting of exponential sums (known as the *Kloosterman sums*) and oscillatory integrals. Unfortunately, the geometric expansion becomes substantially more complicated once we reach $GL(3)$ and it surely presents huge obstacles for the traditional approach. Therefore, this prompts us to re-think our strategies towards moment problems.

Upon further reflection, the author of this paper believes that the *Bruhat decomposition* is a source of complications. Such a decomposition is the main reason why Kloosterman sums and certain oscillatory integrals appear on the geometric side.

Remark 1.5 (Archimedean Oscillatory Integrals). *In $GL(2)$, a couple of coincidences allow us to identify the oscillatory integrals with some well-studied special functions, see* [Mo97], [I02]. *However, such a phenomenon does not exist in $GL(3)$ and there turn out to be many unexpected analytic difficulties, see Buttcane* [Bu13, Bu16]. *Furthermore, the complicated formulae for the oscillatory integrals make the Kuznetsov trace formula for $GL(3)$ challenging to apply, see Blomer-Buttcane* [BlBu20].

Remark 1.6 (Non-archimedean Kloosterman Sums). *The Kloosterman sums encode the arithmetic of moments of L-functions and are subject to delicate transformations. On one hand, the Kloosterman sums of*

$GL(3)$ are complicated, for example, [1]

$$S(m_1, m_2, n_1, n_2;\, D_1, D_2) := \sum_{B_1 \pmod{D_1}} \sum_{C_1 \pmod{D_1}} \sum_{B_2 \pmod{D_2}} \sum_{C_2 \pmod{D_2}} e\left(\frac{m_1 B_1 + n_1(Y_1 D_2 - Z_1 B_2)}{D_1}\right) \cdot e\left(\frac{m_2 B_2 + n_2(Y_2 D_1 - Z_2 B_1)}{D_2}\right). \quad (1.7)$$

On the other hand, the changes of structures brought by the summation formulae of $GL(n)$, or the Voronoi formulae (see [GoLi06, IT13] *for instance), can be quite subtle when $n \geqslant 3$. In fact, it was essential to apply the $GL(3)$ Voronoi formula twice in* [Li11]*, which were remarkably non-involutary, when studying the spectral moment of the left side of (1.3) using the traditional approach. Attempts beyond $GL(3)$ are very limited, see* [BLM19] *and* [CL20].

Thus, we are motivated to look for an approach that is *free from the Bruhat decomposition* and thus any involvement of geometric sums and integrals. The period integral method, along the line of Blomer [Bl12] for example, has provided some instances in which such a goal is indeed attainable. This is the approach adopted in this article. Altogether, the Kuznetsov formula, the Voronoi formula, and the approximate functional equation, which belong to the standard toolbox in analytic number theory (see [CI00, Pe15, Li11] for instance), are completely avoided in our proof of Theorem 1.1.

In view of Remark 1.5, although our calculation on the dual side is about $GL(3)$, the integrals under consideration (see (3.6)) are relatively simple, say when compared to those in the $GL(3)$ Kuznetsov of [BFG88] (In fact, $GL(2)$ Kuznetsov to some extent) or the $GL(3)$ Voronoi formulae (see [GoLi06]). Moreover, the crucial archimedean ingredient in our proof generalizes to $GL(n)$. It is known as *Stade's formula* (see [St01]), which allows us to rewrite the archimedean part completely in terms of integrals Γ-functions and it possesses remarkable recursive structures. In our case, it turns out to be sufficient to work with such representation and obtain the needed analytic continuation for the identity (1.3).

In view of Remark 1.6, it would be favourable if one can detect the structures of moments more directly without having exponential sums as intermediate objects. This is achieved in the context of Theorem 1.1, which is a direct consequence of no Bruhat decomposition is ever involved in our argument.

1.4. **Work In-progress.** The present article constitutes part of the author's thesis. We have restricted ourselves to the simplest possible setting to illustrate the main ideas with focus on the archimedean aspect. In our upcoming work, we plan to

(1) Develop the moment formula behind the more general version of the period integral (1.6), i.e.,
$$\left\langle P^a,\ \mathbb{P}_n^N \Phi \cdot |\det *|^{\bar{s}-\frac{1}{2}} \right\rangle_{L^2(GL_n(F)\backslash GL_n(\mathbb{A}_F))}, \quad (1.8)$$
where $\mathbb{P}_n^N$ is defined in [Cog04], P^a is a Poincaré series of $GL(n)$ and Φ is an automorphic form of $GL(N)$. Our method uses integral representation, which will allow local treatment and handling of ramifications.

(2) Refine our investigation of the archimedean aspect of the problem, say along the line of the series of papers [Bu13, Bu16, BlBu20, Bu20, Bu21]. The main goal is to provide good estimations for the integral transforms.

Remark 1.7. *Peter Humphries has kindly informed the author that the moment of Theorem 1.1 can also be investigated under the framework of* [BK19a, BK19b, BLM19]*. It arises naturally from the context of the L^4-norm problem of Maass forms of $GL(2)$. This is his on-going work with Rizwanur Khan.*

2. Outline, Notations & Preliminary

2.1. **Outline of this Paper.** In Section 2, we collect some essential notions and results for later parts of the article. In particular, we strive to maintain all notations to be with convenient normalizations for the

[1] where the definitions of Y_i, Z_i's along with some congruence and coprimality conditions are suppressed, see [Bu13] for detail.

analytic theory of $GL(3)$ automorphic forms. Also, we want to ensure the compatibility of the conventions between the $GL(2)$ and the $GL(3)$ theories.

The proof of Theorem 1.1 is divided into five sections. In Section 3, we prove the key identity of this article (see Corollary 3.2). In Section 4, we develop such an identity into moments of L-functions on the region of absolute convergence. In particular, the intrinsic structure of the problem allows one to easily see the shape of the dual moment (see Proposition 4.2).

In Section 5, we obtain the region of holomorphy and growth of the archimedean transform. In Section 6, a step-by-step analytic continuation argument is performed based on the analytic information obtained in Section 5. In Section 7, we provide several explicit formulae of the transforms. In Appendix A, we provide a 'twisted' version of Theorem 1.1.

Throughout this article, we use the following notations: $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ $(s \in \mathbb{C})$; $e(x) := e^{2\pi i x}$ $(x \in \mathbb{R})$; $\Gamma_n := SL_n(\mathbb{Z})$ $(n \geqslant 2)$.

2.2. **(Spherical) Whittaker Functions & Transforms.** The Whittaker function of $GL_2(\mathbb{R})$ is more familiar and is given by $W_\beta(y) := 2\sqrt{y}K_\beta(2\pi y)$ for $\beta \in \mathbb{C}$ and $y > 0$. For the group $GL_3(\mathbb{R})$, we first introduce the function [2]

$$I_\alpha(y_0, y_1) \;=\; I_\alpha\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix} \;:=\; y_0^{1-\alpha_3}y_1^{1+\alpha_1}$$

for $y_0, y_1 > 0$ and $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)} := \left\{(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{C}^3 : \alpha_1+\alpha_2+\alpha_3 = 0\right\}$. Then the Whittaker function for $GL_3(\mathbb{R})$, denoted by $W_\alpha(y_0,y_1) = W_\alpha\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}$, can be defined using *Jacquet's integral*:

$$\prod_{1\leqslant j<k\leqslant 3}\Gamma_{\mathbb{R}}(1+\alpha_j-\alpha_k)\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} I_\alpha\left[\begin{pmatrix} & & 1\\ & -1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & u_{1,2} & u_{1,3}\\ & 1 & u_{2,3}\\ & & 1\end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right]\cdot e(-u_{1,2}-u_{2,3})\,du_{1,2}\,du_{1,3}\,du_{2,3}$$

for $y_0, y_1 > 0$ and $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$. See Chapter 5.5 of [Gold] for details. Moreover, it admits the following useful Mellin-Barnes representation:

Proposition 2.1. *Assume $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ is tempered, i.e., $\operatorname{Re}\alpha_i = 0$ $(i = 1,2,3)$. Then for any $\sigma_0, \sigma_1 > 0$,*

$$W_{-\alpha}(y_0,y_1) \;=\; \frac{1}{4}\int_{(\sigma_0)}\int_{(\sigma_1)} G_\alpha(s_0,s_1)y_0^{1-s_0}y_1^{1-s_1}\,\frac{ds_0}{2\pi i}\frac{ds_1}{2\pi i}, \qquad y_0,y_1 \;>\; 0, \tag{2.1}$$

where

$$G_\alpha(s_0,s_1) \;:=\; \frac{\prod_{i=1}^{3}\Gamma_{\mathbb{R}}(s_0+\alpha_i)\,\Gamma_{\mathbb{R}}(s_1-\alpha_i)}{\Gamma_{\mathbb{R}}(s_0+s_1)}. \tag{2.2}$$

Proof. See (6.1.4)−(6.1.5) of [Gold]. For a proof, see Chapter X of [Bump84]. □

Corollary 2.2. *For any $-\infty < A_0, A_1 < 1$, we have*

$$|W_{-\alpha}(y_0,y_1)| \;\ll\; y_0^{A_0}y_1^{A_1}, \qquad y_0,y_1 \;>\; 0, \tag{2.3}$$

where the implicit constant depends only on α, A_0, A_1.

Proof. Follows directly from Proposition 2.1. □

We will need the evaluation of the $GL_3(\mathbb{R})\times GL_2(\mathbb{R})$ Rankin-Selberg integral:

Proposition 2.3. *Let W_β and W_α be the Whittaker functions of $GL_2(\mathbb{R})$ and $GL_3(\mathbb{R})$ respectively. For $\operatorname{Re}s \gg 0$, we have*

$$\int_0^\infty\int_0^\infty W_\beta(y_1)\cdot\overline{W_\alpha}(y_0,y_1)\cdot(y_0^2y_1)^{s-\frac{1}{2}}\,\frac{dy_0dy_1}{y_0y_1^2} \;=\; \frac{1}{4}\cdot\prod_{j=1}^{2}\prod_{K=1}^{3}\Gamma_{\mathbb{R}}\left(s+\beta_j+\overline{\alpha_K}\right). \tag{2.4}$$

Proof. See [Bump88]. □

[2] The normalization here for the I_α-function is more convenient than that of eq. 5.1.1 in [Gold].

The following pair of integral transforms plays an important role in the archimedean aspect of this article.

Definition 2.4. Let $h:(0,\infty)\to\mathbb{C}$ and $H: i\mathbb{R}\to\mathbb{C}$ be measurable functions with $H(\beta)=H(-\beta)$. Let $W_\beta(y):=2\sqrt{y}K_\beta(2\pi y)$. Then the Kontorovich-Lebedev transform of h is defined by

$$h^{\#}(\beta) \;:=\; \int_0^\infty h(y)\cdot W_\beta(y)\,\frac{dy}{y^2}, \tag{2.5}$$

whereas its inverse transform is defined by [3]

$$H^\flat(y) \;=\; \frac{1}{4\pi i}\int_{(0)} H(\beta)\cdot W_\beta(y)\,\frac{d\beta}{|\Gamma(\beta)|^2}, \tag{2.6}$$

provided the integrals converge absolutely.

Definition 2.5. Let $\mathcal{C}_\eta$ be the class of holomorphic functions H on the vertical strip $|\operatorname{Re}\beta|<2\eta$ such that

(1) $H(\beta)=H(-\beta)$,
(2) H has rapid decay in the sense that

$$H(\beta) \;\ll\; e^{-10|\beta|} \qquad\qquad (|\operatorname{Re}\beta| \;<\; 2\eta). \tag{2.7}$$

In this article, we take $\eta>40$ without otherwise specified. By contour-shifting and Stirling's formula, we have

Proposition 2.6. *For any $H\in\mathcal{C}_\eta$, the integral (2.6) defining $H^\flat$ converges absolutely. Moreover, we have*

$$H^\flat(y) \;\ll\; \min\{y,y^{-1}\}^\eta \qquad (y \;>\; 0). \tag{2.8}$$

Proof. See Lemma 3.2 of [GK12]. □

Proposition 2.7. *Under the same assumptions of Proposition 2.6, we have*

$$(h^{\#})^\flat(g) \;=\; h(g) \qquad \textit{and} \qquad (H^\flat)^{\#}(\beta) \;=\; H(\beta). \tag{2.9}$$

Proof. See [GK12]. □

2.3. **Automorphic Forms of $GL(2)$ and $GL(3)$.** Let $\mathfrak{h}^2:=\{\left(\begin{smallmatrix}1&u\\&1\end{smallmatrix}\right)\left(\begin{smallmatrix}y&\\&1\end{smallmatrix}\right): u\in\mathbb{R},\ y>0\}$ with its invariant measure given by $dudy/y^2$. An automorphic form $\phi:\mathfrak{h}^2\to\mathbb{C}$ of Γ_2 satisfies $\Delta\phi=\left(\frac{1}{4}-\beta^2\right)\phi$ for some $\beta=\beta(\phi)\in\mathbb{C}$, where $\Delta:=-y^2\left(\partial_x^2+\partial_y^2\right)$. It is convenient to identify β with the pair $(\beta,-\beta)\in\mathfrak{a}_{\mathbb{C}}^{(2)}$.

For $a\in\mathbb{Z}-\{0\}$, the a-th Fourier coefficient of ϕ, denoted by $\mathcal{B}_\phi(a)$, is defined by

$$(\widehat{\phi})_a(y) \;:=\; \int_0^1 \phi\left[\left(\begin{smallmatrix}1&u\\&1\end{smallmatrix}\right)\left(\begin{smallmatrix}y&\\&1\end{smallmatrix}\right)\right]e(-au)\,du \;=\; \frac{\mathcal{B}_\phi(a)}{\sqrt{|a|}}\cdot W_{\beta(\phi)}(|a|y). \tag{2.10}$$

In the case of the Eisenstein series of Γ_2, i.e.,

$$\phi \;=\; E(z;\mu) \;:=\; \frac{1}{2}\sum_{\gamma\in U_2(\mathbb{Z})\backslash\Gamma_2} I_\mu(\operatorname{Im}\gamma z) \qquad (z \;\in\; \mathfrak{h}^2), \tag{2.11}$$

where $I_\mu(y):=y^{\mu+\frac{1}{2}}$, it is well-known that

$$\mathcal{B}(a;\mu) \;=\; \frac{|a|^\mu\sigma_{-2\mu}(|a|)}{\zeta^*(1+2\mu)} \qquad \text{and} \qquad \Delta E(*;\mu) \;=\; \left(\frac{1}{4}-\mu^2\right)E(*;\mu), \tag{2.12}$$

where $\zeta^*(s):=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\sigma_{-2\mu}(|a|):=\sum_{d|a}d^{-2\mu}$. Moreover, the series (2.11) converges absolutely when $\operatorname{Re}\mu>1/2$ and admits a meromorphic continuation to $\mathbb{C}$.

Next, let $\mathfrak{h}^3:=\left\{\left(\begin{smallmatrix}1&u_{1,2}&u_{1,3}\\&1&u_{2,3}\\&&1\end{smallmatrix}\right)\left(\begin{smallmatrix}y_0y_1&&\\&y_0&\\&&1\end{smallmatrix}\right): u_{i,j}\in\mathbb{R},\ y_k>0\right\}$. Let $\Phi:\mathfrak{h}^3\to\mathbb{C}$ be a Maass cusp form of Γ_3 as defined in Definition 5.1.3 of [Gold]. In particular, there exists $\alpha=\alpha(\Phi)\in\mathfrak{a}_{\mathbb{C}}^{(3)}$ such that for any $D\in Z(U\mathfrak{gl}_3(\mathbb{C}))$, [4] we have

$$D\Phi \;=\; \lambda_D\Phi \qquad \text{and} \qquad DI_\alpha \;=\; \lambda_D I_\alpha$$

[3] The normalization constant $1/4\pi i$ in (2.6) is consistent with that in [Mo97], [I02].
[4] The center of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$.

for some $\lambda_D \in \mathbb{C}$. The triple $\alpha(\Phi)$ is said to be the Langlands parameters of Φ. There is also the notion of Fourier coefficients for Φ thanks to the multiplicity-one theorem of Shalika. For a proof, see Theorem 6.1.6 of [Gold]. Once again, we follow the convention of [Gold] (pp. 261).

Definition 2.8. Let $m = (m_1, m_2) \in (\mathbb{Z} - \{0\})^2$ and $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form. For any $y_0, y_1 > 0$, define

$$(\widehat{\Phi})_{(m_1,m_2)}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix} := \int_0^1\int_0^1\int_0^1 \Phi\left[\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1\end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right]\cdot e\left(-m_1u_{2,3} - m_2u_{1,2}\right)\, du_{1,2}\, du_{1,3}\, du_{2,3}. \tag{2.13}$$

Then the (m_1, m_2)-th **Fourier coefficient** of Φ is the complex number $\mathcal{B}_\Phi(m_1, m_2)$ for which

$$(\widehat{\Phi})_{(m_1,m_2)}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix} = \frac{\mathcal{B}_\Phi(m_1,m_2)}{|m_1m_2|}\, W_{\alpha(\Phi)}^{\operatorname{sgn}(m_2)}\begin{pmatrix} (|m_1|y_0)(|m_2|y_1) & & \\ & |m_1|y_0 & \\ & & 1\end{pmatrix} \tag{2.14}$$

holds for any $y_0, y_1 > 0$.

Convention 2.9. *For the rest of the article,*

1. *All Maass cusp forms will be simultaneous eigenfunctions of the Hecke operators and will be either even or odd. Also, their first Fourier coefficients are equal to* 1*. In this case, the forms are said to be* ***Hecke-normalized****. Note that there are no odd form for $SL_3(\mathbb{Z})$, see Proposition 9.2.5 of* [Gold].
2. *Our fixed Maass cusp form Φ of $SL_3(\mathbb{Z})$ is assumed to be* ***tempered at*** ∞*, i.e., its Langlands parameters are purely imaginary. This merely serves as a simplification of our exposition. In fact, all Maass cusp forms of $SL_3(\mathbb{Z})$ are conjectured to be tempered at ∞. The non-tempered forms constitute a density zero set (see* [Mil01]*).*
3. *Denote by θ the best progress towards the Ramanujan conjecture for the Maass cusp forms of $SL_3(\mathbb{Z})$. We have $\theta \leqslant \frac{1}{2} - \frac{1}{10}$, see Theorem 12.5.1 of* [Gold].

2.4. **Automorphic L-functions.** The Maass cusp forms Φ and ϕ below are Hecke-normalized and their Langlands parameters are $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ and $\beta \in \mathfrak{a}_{\mathbb{C}}^{(2)}$ respectively.

Definition 2.10. Let $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form of Γ_3. We define the standard L-function of Φ by the Dirichlet series

$$L(s,\Phi) := \sum_{n=1}^{\infty} \frac{\mathcal{B}_\Phi(1,n)}{n^s} \qquad (\operatorname{Re} s \gg 1). \tag{2.15}$$

Let $\widetilde{\Phi}(g) := \Phi\left({}^tg^{-1}\right)$ be the dual form of Φ. The L-function $L(s,\Phi)$ admits an entire continuation and satisfies the functional equation:

$$L^*(s,\Phi) := \prod_{K=1}^{3}\Gamma_{\mathbb{R}}(s+\alpha_K)\cdot L(s,\Phi) = \prod_{K=1}^{3}\Gamma_{\mathbb{R}}(1-s-\alpha_K)\cdot L\left(1-s,\widetilde{\Phi}\right) := L^*\left(1-s,\widetilde{\Phi}\right). \tag{2.16}$$

For a proof, see Section 6.5 of [Gold] or [JPSS].

Definition 2.11. Let ϕ (resp. Φ) be a Maass cusp form of Γ_2 (resp. Γ_3). We define the Rankin-Selberg L-function of ϕ and Φ by the Dirichlet series

$$L(s,\phi\otimes\Phi) := \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} \frac{\mathcal{B}_\phi(m_2)\mathcal{B}_\Phi(m_1,m_2)}{\left(m_1^2m_2\right)^s} \qquad (\operatorname{Re} s \gg 1). \tag{2.17}$$

The corresponding complete L-function is given by

$$L^*(s,\phi\otimes\Phi) := L_\infty(s,\phi\otimes\Phi)\cdot L(s,\phi\otimes\Phi) := \prod_{k=1}^{2}\prod_{K=1}^{3}\Gamma_{\mathbb{R}}(s+\beta_k+\alpha_K)\cdot L(s,\phi\otimes\Phi). \tag{2.18}$$

Proposition 2.12. *Suppose Φ (resp. ϕ) is a Maass cusp form of Γ_3 (resp. Γ_2). For $\operatorname{Re} s \gg 1$, if ϕ is even, then*

$$\left\langle \phi,\ (\mathbb{P}_2^3\Phi)\cdot|\det *|^{\overline{s}-\frac{1}{2}}\right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))} = \frac{1}{2}\, L^*\left(s,\phi\otimes\widetilde{\Phi}\right) \tag{2.19}$$

where $(\mathbb{P}_2^3\Phi)(g) := \Phi\begin{pmatrix} g & \\ & 1\end{pmatrix}$ $(g \in GL_2(\mathbb{R}))$; whereas (2.19) is 0 if ϕ is odd.

Proof. We take this opportunity to correct some minor inaccuracies in Section 12.2 of [Gold]. Indeed, the assumptions on the parities are missing in [Gold] and the inner product should be taken over the quotient $\Gamma_2\backslash GL_2(\mathbb{R})$ instead of $\Gamma_2\backslash\mathfrak{h}^2$.

As a brief sketch, we replace $\mathbb{P}_2^3\Phi$ by its Fourier-Whittaker expansion (Theorem 5.3.2 of [Gold]) on the left side of (2.19) and unfold. Then one may extract the Dirichlet series (2.17) by using (2.10) and (2.13). The integral of Whittaker functions can be computed by Proposition 2.3. □

When ϕ is even, the involution $g \mapsto {}^tg^{-1}$ and the above Proposition give the functional equation

$$L^*\left(s, \phi\otimes\widetilde{\Phi}\right) \ = \ L^*\left(1-s, \phi\otimes\Phi\right).$$

Proposition 2.13. *For* $\operatorname{Re}(s\pm\mu) \gg 1$, *we have*

$$\left\langle E(*;\mu),\ \left(\mathbb{P}_2^3\Phi\right)\cdot|\det *|^{\bar{s}-\frac{1}{2}}\right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))} \ = \ \frac{1}{2}\ \frac{L^*\left(s+\mu,\ \widetilde{\Phi}\right)L^*\left(s-\mu,\ \widetilde{\Phi}\right)}{\zeta^*(1+2\mu)}. \tag{2.20}$$

Proof. Parallel to Proposition 2.12. Meanwhile, we make use of (2.12). □

Remark 2.14. *By analytic continuation, (2.19) and (2.20) hold for* $s\in\mathbb{C}$ *and away from the poles of* $E(*;\mu)$. *In fact, the rapid decay of* Φ *at* ∞ *guarantees the inner products converge absolutely.*

2.5. **Calculation on the Spectral Side.** As indicated in the introduction, our approach differs from the 'Kuznetsov-Voronoi' one right from the start — we will not make use of the Dirichlet series (2.17). Instead, the moment of $GL(3)\times GL(2)$ L-functions is first interpreted in terms of the period integral of Proposition 2.12 using a Poincaré series.

Definition 2.15. Let $a \geqslant 1$ be an integer and $h\in C^\infty(0,\infty)$. The Poincaré series of Γ_2 is defined by

$$P^a(z;h) \ := \sum_{\gamma\in U_2(\mathbb{Z})\backslash\Gamma_2} h(a\operatorname{Im}\gamma z)\cdot e\left(a\operatorname{Re}\gamma z\right) \qquad \left(z\ \in\ \mathfrak{h}^2\right) \tag{2.21}$$

provided it converges absolutely.

It is not hard to see that if the bounds

$$h(y) \ \ll\ y^{1+\epsilon} \quad (\text{as}\ \ y\to 0) \quad \text{ and } \quad h(y)\ \ll\ y^{\frac{1}{2}-\epsilon} \quad (\text{as}\ \ y\to\infty) \tag{2.22}$$

are satisfied, then the Poincaré series $P^a(z;h)$ converges absolutely and represents an L^2-function. In this article, we take $h := H^\flat$ with $H\in\mathcal{C}_\eta$ and $\eta > 40$. By Proposition 2.6, conditions (2.22) clearly holds. We often use the shorthand $P^a := P^a(*;h)$.

Lemma 2.16. *Let* ϕ *be a Maass cusp form of* Γ_2, $\Delta\phi = \left(\frac{1}{4}-\beta^2\right)\phi$, *and* $\mathcal{B}_\phi(a)$ *be the* a*-th Fourier coefficient of* ϕ. *Then*

$$\left\langle P^a,\ \phi\right\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)} \ = \ |a|^{1/2}\cdot\overline{\mathcal{B}_\phi(a)}\cdot h^{\#}\left(\overline{\beta}\right).$$

Proof. Replace P^a in $\left\langle P^a,\ \phi\right\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)}$ by its definition and unfold, we find that

$$\left\langle P^a,\ \phi\right\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)} \ = \ \int_0^\infty h(ay)\cdot\overline{\left(\widehat{\phi}\right)_a(y)}\ \frac{dy}{y^2}.$$

The result follows at once upon plugging-in (2.10) and making the change of variable $y\to|a|^{-1}y$. □

Similarly, the following holds away from the poles of $E(*;\mu)$:

Lemma 2.17.

$$\left\langle P^a,\ E(*;\mu)\right\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)} \ = \ |a|^{1/2}\cdot\frac{|a|^{\overline{\mu}}\sigma_{-2\overline{\mu}}(|a|)}{\zeta^*(1+2\overline{\mu})}\cdot h^{\#}\left(\overline{\mu}\right). \tag{2.23}$$

Proposition 2.18 (Spectral Expansion)**.** *Suppose $f\in L^2(\Gamma_2\backslash\mathfrak{h}^2)$ and $\langle f,1\rangle=0$. Then*

$$f(z) \;=\; \sum_{j=1}^{\infty}\frac{\langle f,\phi_j\rangle}{\langle\phi_j,\phi_j\rangle}\cdot\phi_j(z)+\frac{1}{4\pi}\int_{-\infty}^{\infty}\Big\langle f,\;E(*;i\mu)\Big\rangle\cdot E(z;i\mu)\,d\mu\qquad (z\;\in\;\mathfrak{h}^2) \tag{2.24}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle:=\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Gamma_2\backslash\mathfrak{h}^2)}$ and $(\phi_j)_{j\geqslant1}$ is any orthogonal basis of Maass cusp forms for Γ_2.

Proof. See Theorem 3.16.1 of [Gold]. □

Proposition 2.19. *Let Φ be a Maass cusp form of Γ_3 and P^a be a Poincaré series of Γ_2. Then for any $s\in\mathbb{C}$,*

$$\begin{aligned}
&2\cdot|a|^{-1/2}\cdot\Big\langle P^a,\;(\mathbb{P}_2^3\Phi)\cdot|\det *|^{\bar{s}-\frac12}\Big\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))}\\
&\quad=\;\sum_{j=1}^{\infty}{}' \;h^{\#}\left(\overline{\beta_j}\right)\cdot\frac{\overline{\mathcal{B}_j(a)}\cdot L^*\left(s,\phi_j\otimes\widetilde{\Phi}\right)}{\langle\phi_j,\phi_j\rangle}\\
&\qquad\qquad+\frac{1}{4\pi}\int_{\mathbb{R}}h^{\#}(i\mu)\,\frac{\sigma_{-2i\mu}(|a|)|a|^{-i\mu}L^*\left(s+i\mu,\widetilde{\Phi}\right)L^*\left(1-s+i\mu,\Phi\right)}{|\zeta^*(1+2i\mu)|^2}\,d\mu,
\end{aligned}\tag{2.25}$$

where the sum is restricted to an orthogonal basis (ϕ_j) of even Hecke-normalized Maass cusp forms for Γ_2 with $\Delta\phi_j=\left(\frac14-\beta_j^2\right)\phi_j$ and $\mathcal{B}_j(a):=\mathcal{B}_{\phi_j}(a)$. [5]

Proof. Substitute the spectral expansion of P^a as in (2.24) into $\langle P^a,\;(\mathbb{P}_2^3\Phi)\cdot|\det *|^{\bar{s}-\frac12}\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))}$. The inner products involved have been computed in Lemma 2.16–2.17 and Proposition 2.12–2.13. □

Remark 2.20. *In many applications, it is important to have refined control in the spectral component and study the analytic properties of the relevant integral transforms. As a result, the ability of making flexible choices of test functions on the spectral side is crucial. This is one of the strengths of the Kuznetsov formula over the period integral methods (say [Bl12, Nu20+, Za21, Za20+]) and might partly explain why the former seems to be more ubiquitous in the current literature.*

Although our method is period-integral based (see (1.6)), we are able to put a large class of test functions on the spectral side as in the Kuznetsov approach, by using the pair of transforms introduced in Definition 2.4. Such transforms have been generalized to $GL(n)$ in a simple and explicit fashion in [GK12]. They have played important roles in the recent development of the Kuznetsov formulae of higher-rank (see [GK13], [GSW21], [Bu20, Bu21]).

Our method preserves the advantages of both the Kuznetsov and the period integral approaches — the former being the precision in the archimedean aspect whereas the latter being the structural insights in the nonarchimedean aspect.

Remark 2.21. *Readers may wonder about the possiblity of using an automorphic kernel in place of a Poincaré series in studying the moment of L-functions in Theorem 1.1. Although this offers extra flexibility in incorporating new structures, the analysis behind the integral transforms (the spherical transforms) becomes quite complicated, see [Bu13] for the case of $GL(3)$. The approach using Poincaré series seems to be more adapted to the analytic number theory of higher-rank groups.*

3. Basic Identity for Dual Moment

3.1. **Unipotent Integration.** We are ready to work on the dual side of our moment formula. As a simplification of our argument, we shall only consider $P=P^a(*;h)$ with $a=1$ in the following. For discussions of

[5] also the a-th Hecke eigenvalue of ϕ_j.

the general case, see Appendix A. Suppose $\mathrm{Re}\, s > 1 + \frac{\theta}{2}$. We begin by replacing P by its definition in the inner product $\left\langle P,\ \mathbb{P}_2^3\Phi \cdot |\det *|^{\overline{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))}$. We find upon unfolding:

$$\left\langle P,\ \mathbb{P}_2^3\Phi \cdot |\det *|^{\overline{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))} = \int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-\frac{1}{2}} \cdot \int_0^1 \overline{\Phi} \left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e(u_{1,2})\, du_{1,2}\, \frac{dy_0 dy_1}{y_0 y_1^2}. \tag{3.1}$$

The main task of this section is to compute the inner, 'incomplete' unipotent integral in (3.1). We wish to evaluate it in terms of the Fourier coefficients of Φ (see Definition 2.8) as they are relevant in the constructions of various L-functions associated to Φ, say those discussed in Section 2.4.

Certainly, this can be obtained by plugging in the *full* Fourier expansion of [JPSS] (see [Gold] Theorem 5.3.2) and look for possible simplifications. This is in fact not necessary. We prefer a self-contained and conceptual treatement. It simply follows from two one-dimensional Fourier expansions and the automorphy of Φ. In essence, this is where 'summation formulae' take place in our approach, and they are nicely packaged in an elementary, clean, and global fashion.

Proposition 3.1. *For any automorphic function Φ of Γ_3, we have, for any $y_0, y_1 > 0$,*

$$\int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e(-u_{1,2})\, du_{1,2} = \sum_{a_0, a_1 = -\infty}^{\infty} (\widehat{\Phi})_{(a_1,1)} \left[\begin{pmatrix} 1 & & \\ & 1 & \\ & -a_0 & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right]. \tag{3.2}$$

Proof. Firstly, we Fourier-expand along the abelian subgroup $\left\{ \left(\begin{smallmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \right\}$:

$$\int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e(-u_{1,2})\, du_{1,2} = \sum_{a_0=-\infty}^{\infty} \int_{\mathbb{Z}^2\backslash\mathbb{R}^2} \Phi \left[\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e(-u_{1,2} - a_0 \cdot u_{1,3})\, du_{1,2}\, du_{1,3}. \tag{3.3}$$

Secondly, for each $a_0 \in \mathbb{Z}$, consider a unimodular change of variables of the form $(u_{1,2}, u_{1,3}) = (u'_{1,2}, u'_{1,3}) \cdot \left(\begin{smallmatrix} 1 & \\ -a_0 & 1 \end{smallmatrix} \right)$. One can readily observe that

$$\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -a_0 & 1 \end{pmatrix}.$$

Together with the automorphy of Φ with respect to Γ_3, we have

$$\int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e(-a_2 \cdot u_{1,2})\, du_{1,2} = \sum_{a_0=-\infty}^{\infty} \int_{\mathbb{Z}^2\backslash\mathbb{R}^2} \Phi \left[\begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -a_0 & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e(-u'_{1,2})\, du'_{1,2}\, du'_{1,3}. \tag{3.4}$$

The result follows from the third and final Fourier expansion along the abelian subgroup $\left\{\begin{pmatrix}1 & & \\ & 1 & * \\ & & 1\end{pmatrix}\right\}$:

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2})\, du_{1,2}$$

$$= \sum_{a_0,a_1=-\infty}^{\infty}\int_0^1\int_0^1\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & -a_0 & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right]$$

$$\cdot\, e(-u_{1,2} - a_1\cdot u_{2,3})\, du_{1,2}\, du_{1,3}\, du_{2,3}.$$

□

We then explicate Proposition 3.1 when Φ is a Maass cusp form of Γ_3. This constitutes the *basic identity* of the present article. Theorem 1.1 is a natural consequence of this identity and the diagonal/ off-diagonal structures on the dual side become apparent (see Proposition 4.2).

Corollary 3.2. *Suppose Φ is a Maass cusp form of Γ_3. Then*

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2})\, du_{1,2}$$

$$= \sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(a_1,1)}{|a_1|}\cdot W_{\alpha(\Phi)}\left(|a_1|y_0, y_1\right)$$

$$+\sum_{a_0\neq 0}\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(a_1,1)}{|a_1|}\cdot W_{\alpha(\Phi)}\left(\frac{|a_1|y_0}{1+(a_0y_0)^2},\ y_1\sqrt{1+(a_0y_0)^2}\right)$$

$$\cdot\, e\left(-\frac{a_0a_1y_0^2}{1+(a_0y_0)^2}\right). \tag{3.5}$$

Proof. By cuspidality, $(\widehat{\Phi})_{(0,1)}\equiv 0$. The result follows from a straight-forward linear algebra calculation:

$$\begin{pmatrix}1 & & \\ & 1 & \\ & -a_0 & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix} \equiv \begin{pmatrix}1 & & \\ & 1 & -\frac{a_0y_0^2}{1+(a_0y_0)^2} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{y_0}{1+(a_0y_0)^2}\cdot y_1\sqrt{1+(a_0y_0)^2} & & \\ & \frac{y_0}{1+(a_0y_0)^2} & \\ & & 1\end{pmatrix}$$

under the right quotient by $O_3(\mathbb{R})\cdot\mathbb{R}^\times$. This can be verified by the mathematica command *IwasawaForm[]* in the *GL(n)pack* (*gln.m*). [6] □

Remark 3.3. *Readers might be interested in comparing the approach of* [Li11] *with ours. In* [Li11]*, the $GL(3)$ Voronoi formula (*[GoLi06]*) was used and the unipotent subgroup of interest was $\left\{\begin{pmatrix}1 & & * \\ & 1 & * \\ & & 1\end{pmatrix}\right\}$. In our case, the natural unipotent subgroup (upon unfolding) turns out to be the complementary one: $\left\{\begin{pmatrix}1 & * & \\ & 1 & \\ & & 1\end{pmatrix}\right\}$. From (3.5), observe that the diagonal and off-diagonal terms are encapsulated simultaneously.*

3.2. **Initial Simplification and Absolute Convergence.** We temporarily restrict ourselves to the vertical strip $1+\frac{\theta}{2}<\sigma:=\operatorname{Re}s<4$. As we shall see, this guarantees the absolute convergence of all sums and integrals.

Suppose $H\in\mathcal{C}_\eta$ with $\eta>40$ (see Proposition 2.6). Then the bound (2.8) for $h:=H^\flat$ implies its Mellin transform $\widetilde{h}(w):=\int_0^\infty h(y)y^w\, d^\times y$ is holomorphic on the strip $|\operatorname{Re}w|<\eta$. Substituting (3.5) into (3.1), and

[6] The user manual and the package can be downloaded from Kevin A. Broughan's website: https://www.math.waikato.ac.nz/~kab/glnpack.html.

apply the changes of variables $y_0 \to |a_1|^{-1}y_0$, $y_0 \to |a_0|^{-1}y_0$ to the first, second piece of the resultant,

$$\begin{aligned} &\left\langle P,\ \mathbb{P}_2^3\Phi \cdot |\det *|^{\bar{s}-\frac{1}{2}} \right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))} \\ &= 2\cdot L(2s,\Phi)\cdot \int_0^\infty\int_0^\infty h(y_1)\cdot (y_0^2y_1)^{s-\frac{1}{2}}\cdot W_{-\alpha(\Phi)}(y_0,y_1)\ \frac{dy_0dy_1}{y_0y_1^2} \\ &\quad + \sum_{a_0\neq 0}\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(1,a_1)}{|a_0|^{2s-1}|a_1|}\cdot \int_0^\infty\int_0^\infty h(y_1)\cdot (y_0^2y_1)^{s-\frac{1}{2}}\cdot e\left(\frac{a_1}{a_0}\cdot\frac{y_0^2}{1+y_0^2}\right) \\ &\qquad \cdot W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right|\cdot\frac{y_0}{1+y_0^2},\ y_1\sqrt{1+y_0^2}\right)\frac{dy_0dy_1}{y_0y_1^2}. \end{aligned} \tag{3.6}$$

Definition 3.4.

$$\begin{aligned} OD_\Phi(s) := \sum_{a_0\neq 0}\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(1,a_1)}{|a_0|^{2s-1}|a_1|}\cdot \int_0^\infty\int_0^\infty h(y_1)\cdot (y_0^2y_1)^{s-\frac{1}{2}}\cdot e\left(\frac{a_1}{a_0}\cdot\frac{y_0^2}{1+y_0^2}\right) \\ \cdot W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right|\cdot\frac{y_0}{1+y_0^2},\ y_1\sqrt{1+y_0^2}\right)\frac{dy_0dy_1}{y_0y_1^2}. \end{aligned} \tag{3.7}$$

Proposition 3.5. *When $H\in\mathcal{C}_\eta$ and $4>\sigma>\frac{1+\theta}{2}$, we have*

$$\begin{aligned} &\int_0^\infty\int_0^\infty h(y_1)\cdot (y_0^2y_1)^{s-\frac{1}{2}}\cdot W_{-\alpha(\Phi)}(y_0,y_1)\ \frac{dy_0dy_1}{y_0y_1^2} \\ &\qquad = \frac{\pi^{-3s}}{8}\cdot\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\prod_{i=1}^3\Gamma\left(\frac{s+\beta-\alpha_i}{2}\right)\Gamma\left(\frac{s-\beta-\alpha_i}{2}\right)\ \frac{d\beta}{2\pi i}. \end{aligned} \tag{3.8}$$

Proof. From Proposition 2.7, we have

$$\begin{aligned} &\int_0^\infty\int_0^\infty h(y_1)\cdot (y_0^2y_1)^{s-\frac{1}{2}}\cdot W_{-\alpha(\Phi)}(y_0,y_1)\ \frac{dy_0dy_1}{y_0y_1^2} \\ &\qquad = \frac{1}{2}\cdot\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\int_0^\infty\int_0^\infty W_\beta(y_1)W_{-\alpha(\Phi)}(y_0,y_1)(y_0^2y_1)^{s-\frac{1}{2}}\ \frac{dy_0dy_1}{y_0y_1^2}\ \frac{d\beta}{2\pi i}. \end{aligned}$$

The y_0, y_1-integrals can be evaluated by Proposition 2.3 and (3.8) follows. Moreover, the right side of (3.8) is holomorphic on $\sigma>0$. ☐

Remark 3.6. *If one wishes to prove a smooth Weyl law for the L-values $L\big(1/2,\phi_j\otimes\widetilde{\Phi}\big)$ in the $GL(2)$ spectral aspect, one may pick*

$$H(\beta) = H_{R,\eta}(\beta;\Phi) := e^{(\beta/R)^2}\cdot\frac{\Gamma(2\eta+\beta)\Gamma(2\eta-\beta)}{\prod_{i=1}^3\Gamma\left(\frac{\frac{1}{2}+\beta-\alpha_i}{2}\right)\Gamma\left(\frac{\frac{1}{2}-\beta-\alpha_i}{2}\right)}, \tag{3.9}$$

where η is a large, absolute constant and $R\gg 1$. In (3.9),

- *the factor $e^{(\beta/R)^2}$ serves as a smooth cut-off for $|\beta_j|<R$ and gives the needed decay in Proposition 2.6;*
- *the factors $\prod_{i=1}^3\Gamma\left(\frac{\frac{1}{2}+\beta-\alpha_i}{2}\right)\Gamma\left(\frac{\frac{1}{2}-\beta-\alpha_i}{2}\right)$ cancels out the archimedean factors of $L^*\big(1/2,\phi_j\otimes\widetilde{\Phi}\big)$ on the spectral expansion (2.25) and in the diagonal contribution (3.8);*
- *the factors $\Gamma(2\eta+\beta)\Gamma(2\eta-\beta)$ balance off the exponential growth from $d\beta/|\Gamma(\beta)|^2$, $||\phi_j||^{-2}$ and $|\zeta^*(1+2i\mu)|^{-2}$. Also, a large enough region of holomorphy of (3.9) is maintained so that $h(y):=H^\flat(y)$ has sufficient decay at 0 and ∞.*

In Theorem 1.1, the choice (3.9) give a main term $\asymp_\eta L(1,\Phi)\cdot R^{4\eta+1}$ as $R\to\infty$.

Proposition 3.7. *The off-diagonal $OD_\Phi(s)$ converges absolutely when $4>\sigma>1+\frac{\theta}{2}$ and $H\in\mathcal{C}_\eta$ $(\eta>40)$.*

Proof. Upon inserting absolute values, breaking up the y_0-integral into $\int_0^1 + \int_1^\infty$, and applying the bounds (2.3) [7] and $|\mathcal{B}_\Phi(1, a_1)| \ll |a_1|^\theta$, observe that

$$OD_\Phi(s) \ll \sum_{a_0=1}^{\infty} \sum_{a_1=1}^{\infty} \frac{1}{a_0^{2\sigma-1} a_1^{1-\theta}} \left(\int_{y_0=1}^{\infty} + \int_{y_0=0}^{1} \right) \int_{y_1=0}^{\infty} |h(y_1)| (y_0^2 y_1)^{\sigma - \frac{1}{2}} \left(\frac{a_1 a_0^{-1} y_0}{1+y_0^2} \right)^{A_0} \left(y_1 \sqrt{1+y_0^2} \right)^{A_1} \frac{dy_0 dy_1}{y_0 y_1^2},$$

where the implicit constant depends only on Φ, A_0, A_1 with $-\infty < A_0, A_1 < 1$.

The convergence of both of the series is guaranteed if

$$A_0 \; < \; -\theta \quad \text{and} \quad \sigma \; > \; 1 - \frac{A_0}{2}. \tag{3.10}$$

Claim 3.8. *If we have (3.10) and*

$$A_1 \; < \; A_0 - 2\sigma + 1, \tag{3.11}$$

then the y_0-integrals converge.

Indeed, observe that $2\sigma + A_0 - 2 > -1$ by (3.10), and

$$\int_{y_0=0}^{1} y_0^{2\sigma + A_0 - 2} \left(1 + y_0^2\right)^{\frac{A_1}{2} - A_0} dy_0 \; \asymp_{A_0, A_1} \; \int_{y_0=0}^{1} y_0^{2\sigma + A_0 - 2} \, dy_0.$$

The last integral converges. Also, (3.10)-(3.11) imply $A_1 < \min\{1, 2A_0\}$ and thus,

$$\int_{y_0=1}^{\infty} y_0^{2\sigma + A_0 - 2} \left(1 + y_0^2\right)^{\frac{A_1}{2} - A_0} dy_0 \; \leqslant \; \int_{y_0=1}^{\infty} y_0^{2\sigma + A_1 - A_0 - 2} \, dy_0.$$

Now, the last integral converges because of (3.11).

For the y_1-intgeral, the integrals

$$\int_{y_1=1}^{\infty} |h(y_1)| y_1^{\sigma + A_1 - \frac{5}{2}} \, dy_1 \quad \text{and} \quad \int_{y_1=0}^{1} |h(y_1)| y_1^{\sigma + A_1 - \frac{5}{2}} \, dy_1$$

converge whenever $H \in \mathcal{C}_\eta$ (we then have (2.8)) and

$$\eta \; > \; \left| \sigma + A_1 - \frac{3}{2} \right|. \tag{3.12}$$

Let $\delta := \sigma - 1 - (\theta/2)$ (> 0). In view of (3.10) and (3.11), we may take $A_0 := -\theta - \delta$ and $A_1 := -2\theta - 1 - 4\delta$. Also, (3.12) trivially holds as $\eta > 40$ and $\sigma < 4$. The result follows. ☐

Remark 3.9. *Readers will have no trouble in realizing the resemblance of (1.6) to the well-known inner product construction for the Kuznetsov formula.* [8] *However, there are some differences. One of them has been mentioned: our moment identity is an equality between two unfoldings instead of that between spectral and geometric expansions.*

The other is on the technical aspect. In the Kuznetsov formula, it is possible to annihilate the oscillatory factors therein to obtain a primitive form of the trace formula with some applications, see [GK13], [Zh14], [GSW21]*. However, such a treatment is far from sufficient in our case — we have not analytically continued into the critical strip in Proposition 3.7! In other words, the oscillatory factor in $OD_\Phi(s)$ is of intrinsic importance to our problem. It arises naturally from the abstract characterization of Whittaker functions.*

[7] We are allowed to choose different A_0, A_1 in different ranges of the y_0, y_1-integrals.

[8] Indeed, $\mathbb{P}_2^3\Phi$ can be regarded as an (infinite sum of) Poincaré series of $SL_2(\mathbb{Z})$ thanks to its Fourier expansion. (We never adopt this approach in this article.) In a sense, this can be considered as a $GL(3) \times GL(2)$ analog of the Kuznetsov formulae.

4. Structures of the Off-diagonal

Fix $\epsilon := 1/100$ (say) [9], $0 < \phi < \pi/2$, [10] and consider the domain $1 + \frac{\theta}{2} + \epsilon < \sigma < 4$ in this section. We define a perturbed version of $OD_\Phi(s)$ as follows:

$$OD_\Phi(s;\phi) \ := \ \sum_{a_0\neq 0}\sum_{a_1\neq 0} \frac{\mathcal{B}_\Phi(1,a_1)}{|a_0|^{2s-1}|a_1|}\int_0^\infty\int_0^\infty h(y_1)\cdot (y_0^2y_1)^{s-\frac{1}{2}} W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right|\frac{y_0}{1+y_0^2},\ y_1\sqrt{1+y_0^2}\right) \cdot e\left(\frac{a_1}{a_0}\frac{y_0^2}{1+y_0^2};\phi\right)\ \frac{dy_0dy_1}{y_0y_1^2}, \tag{4.1}$$

where

$$e\left(x;\phi\right) \ := \ \int_{(\epsilon)} |2\pi x|^{-u} e^{iu\phi\,\mathrm{sgn}(x)}\Gamma(u)\ \frac{du}{2\pi i} \qquad (x\in\mathbb{R}-\{0\}). \tag{4.2}$$

Remark 4.1. *The goals of this section is to obtain an expression of $OD_\Phi(s;\phi)$ that*

- *reveals the structure of the dual moment;*
- *that it can be analytically continued into the critical strip;*
- *and will allow us to pass to the limit $\phi\to\pi/2$.*

In view of these, it is natural to work on the dual side of Mellin transforms. Also, we will be able to separate variables as an added benefit. The main result of this section is as follows:

Proposition 4.2 (Dual Moment)**.** *Let $H\in\mathcal{C}_\eta$ $(\eta>40)$ and $\phi\in(0,\pi/2)$. On the vertical strip*

$$1+\frac{\theta}{2}+\epsilon \ < \ \sigma \ < \ 4, \tag{4.3}$$

we have

$$OD_\Phi(s;\phi) \ = \ \frac{1}{4}\int_{(1+\theta+2\epsilon)} \zeta\left(2s-s_0'\right)L\left(s_0',\Phi\right)\cdot\sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0',\ s;\ \phi\right)\ \frac{ds_0'}{2\pi i}, \tag{4.4}$$

where the transform of H is given by

$$\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0',\ s;\ \phi\right) \ := \ \int_{(15)}\int_{(\epsilon)} \widetilde{h}\left(s-s_1-\frac{1}{2}\right)\cdot\mathcal{G}_\Phi^{(\delta)}\left(s_1,u;\ s_0',s;\ \phi\right)\ \frac{du}{2\pi i}\frac{ds_1}{2\pi i}, \tag{4.5}$$

with $h:=H^\flat$ and the kernel function being

$$\mathcal{G}_\Phi^{(\delta)}\left(s_1,u;\ s_0',s;\ \phi\right) \ := G_\Phi\left(s_0'-u,s_1\right)\cdot(2\pi)^{-u}e^{i\delta\phi u}\Gamma(u)\cdot\frac{\Gamma\left(\frac{u+1-2s+s_1-s_0'}{2}\right)\Gamma\left(\frac{2s-s_0'-u}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0'\right)}. \tag{4.6}$$

Here, $G_\Phi := G_{\alpha(\Phi)}$ was defined in (2.2).

Proof. Plug-in the expression of $W_{-\alpha(\Phi)}$ described in Proposition 2.1 into $OD_\Phi(s;\phi)$ with

$$\sigma_1 \ := \ 15 \quad \text{and} \quad 1+\theta \ < \ \sigma_0 \ < \ 2\sigma-1-\epsilon. \tag{4.7}$$

Inserting absolute values to the resulting expression, the sums and integrals are bounded by

$$\sum_{\delta:=\mathrm{sgn}(a_0a_1)=\pm}\left(\sum_{a_0\neq 0}\frac{1}{|a_0|^{2\sigma-\sigma_0-\epsilon}}\right)\left(\sum_{a_1\neq 0}\frac{|\mathcal{B}_\Phi(1,a_1)|}{|a_1|^{\sigma_0+\epsilon}}\right)\left(\int_{(\sigma_0)}\int_{(\sigma_1)}|G_\Phi(s_0,s_1)|\ \ |ds_0||ds_1|\right) \cdot\left(\int_{(\epsilon)}\left|e^{i\delta\phi u}\Gamma(u)\right|\ |du|\right)\left(\int_0^\infty y_0^{-\sigma_0-2\epsilon+2\sigma}(1+y_0^2)^{\sigma_0+\epsilon-\frac{1+\sigma_1}{2}}\ d^\times y_0\right)\left(\int_0^\infty|h(y_1)|\cdot y_1^{\sigma-\sigma_1-\frac{1}{2}}\ d^\times y_1\right). \tag{4.8}$$

Observe that:

- by Stirling's formula, the s_0, s_1, u-integrals converge as long as

$$\sigma_0,\ \sigma_1,\ \epsilon\ >\ 0, \qquad \phi\ \in\ (0,\pi/2); \tag{4.9}$$

[9] We will stick with this choice of ϵ for the rest of this article.

[10] This should not be pose any confusion with the basis of cusp forms (ϕ_j) of Γ_2.

- the y_0-integral converges as long as
$$\sigma_0+2\epsilon \ < \ 2\sigma \ < \ \sigma_1-\sigma_0+1; \tag{4.10}$$
- by the bound $|\mathcal{B}_\Phi(1,a_1)| \ \ll \ |a_1|^{\theta}$, the a_0-sum and the a_1-sum converge as long as
$$2\sigma-1 \ > \ \sigma_0+\epsilon \ > \ 1+\theta. \tag{4.11}$$

Under (4.7), items (4.9), (4.10), (4.11) hold. Moreover, the y_1-integral converges by (2.8) and $H\in\mathcal{C}_\eta$ ($\eta>40$). Now, upon rearranging sums and integrals, and notice that $\mathcal{B}_\Phi(1,a_1)=\mathcal{B}_\Phi(1,-a_1)$, we have

$$\begin{aligned} OD_\Phi(s;\phi) \ &= \ 2\sum_{\delta=\pm}\int_{(\sigma_0)}\int_{(\sigma_1)}\int_{(\epsilon)} \frac{G_\Phi(s_0,s_1)}{4}\cdot(2\pi)^{-u}e^{i\delta\phi u}\Gamma(u)\left(\int_0^\infty h(y_1)y_1^{s-s_1-\frac{1}{2}}d^\times y_1\right) \\ &\cdot\left(\int_0^\infty y_0^{-s_0-2u+2s}(1+y_0^2)^{s_0+u-\frac{1+s_1}{2}}\ d^\times y_0\right)\left(\sum_{a_0=1}^\infty\sum_{a_1=1}^\infty\frac{\mathcal{B}_\Phi(1,a_1)}{a_0^{2s-1}a_1}\left(\frac{a_1}{a_0}\right)^{1-s_0-u}\right)\frac{ds_0}{2\pi i}\frac{ds_1}{2\pi i}\frac{du}{2\pi i}. \end{aligned} \tag{4.12}$$

Recall the integral identity
$$\int_0^\infty y_0^v(1+y_0^2)^A\ d^\times y_0 \ = \ \frac{1}{2}\ \frac{\Gamma\left(-A-\frac{v}{2}\right)\Gamma\left(\frac{v}{2}\right)}{\Gamma(-A)} \tag{4.13}$$
for $0<\operatorname{Re}v<-2\operatorname{Re}A$. It follows that

$$\begin{aligned} OD_\Phi(s;\phi) \ &= \ 2\sum_{\delta=\pm}\int_{(\sigma_0)}\int_{(\sigma_1)}\int_{(\epsilon)} \zeta\left(2s-s_0-u\right)L\left(s_0+u;\Phi\right)\cdot\widetilde{h}\left(s-s_1-\frac{1}{2}\right) \\ &\cdot\frac{G_\Phi(s_0,s_1)}{4}\cdot(2\pi)^{-u}e^{i\delta\phi u}\Gamma(u)\cdot\frac{1}{2}\ \frac{\Gamma\left(s-\frac{s_0}{2}-u\right)\Gamma\left(\frac{1+s_1-s_0}{2}-s\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0-u\right)}\ \frac{ds_0}{2\pi i}\frac{ds_1}{2\pi i}\frac{du}{2\pi i}. \end{aligned} \tag{4.14}$$

We pick the contour $(\sigma_0):=(1+\theta+\epsilon)$ (we thus impose (4.3)). To isolate the nonarchimedean part of $OD_\Phi(s;\phi)$, we make the change of variable $s_0' \ = \ s_0+u$. Upon plugging-in the expression for $G_\Phi\left(s_0'-u,s_1\right)$ (see (2.2)), we obtain (4.4)-(4.6). By the absolute convergence proven above, we also conclude that the integral transform $\left(\mathcal{F}_\Phi^{(\delta)}h\right)(s_0',\ s;\ \phi)$ is holomorphic on the domain
$$\sigma \ < \ 4 \qquad\text{and}\qquad 1+\theta+\epsilon \ < \ \sigma_0' \ < \ 2\sigma-1. \tag{4.15}$$
This completes the proof. □

Remark 4.3. *In the following, we replace s_0' by s_0 and have (4.15) superseding (4.7) correspondingly.*

Proposition 4.4. *For $4>\sigma>(3+\theta)/2$ and $H\in\mathcal{C}_\eta$, we have*
$$\lim_{\phi\to\pi/2} OD_\Phi(s;\phi) \ = \ OD_\Phi(s). \tag{4.16}$$

Proof. Let $\epsilon:=1/100$, $\sigma_1:=15$, and pick any σ_0 satisfying
$$\frac{3}{2}+\theta+\epsilon \ < \ \sigma_0 \ < \ 2\sigma-1-\epsilon. \tag{4.17}$$

Denote by $\mathcal{C}_\epsilon$ the indented path consisting of the line segments:
$$-\frac{1}{2}-\epsilon-i\infty \ \to \ -\frac{1}{2}-\epsilon-i \ \to \ \epsilon-i \ \to \ \epsilon+i \ \to \ -\frac{1}{2}-\epsilon+i \ \to \ -\frac{1}{2}-\epsilon+i\infty.$$
Replace $e(x;\phi)$ in (4.12) by the expression:
$$e(x;\phi) \ = \ \int_{\mathcal{C}_\epsilon}|2\pi x|^{-u}e^{iu\phi\operatorname{sgn}(x)}\Gamma(u)\ \frac{du}{2\pi i}. \tag{4.18}$$
Note that $\left|e^{iu\phi\operatorname{sgn}(x)}\Gamma(u)\right| \ \ll_\epsilon \ \left(1+|\operatorname{Im}u|\right)^{-1-\epsilon}$ for $u\in\mathcal{C}_\epsilon$ and $\phi\in(0,\pi/2]$. Insert absolute values in (4.12). The resulting sums and integrals converge absolutely when $\phi\in(0,\pi/2]$ and (4.17) holds, which can be seen by the same argument following (4.8). Apply Dominated Convergence and shift the contour of the u-integral to $-\infty$, the residual series obtained is exactly $e\left(\frac{a_1}{a_0}\frac{y_0^2}{1+y_0^2}\right)$. This completes the proof. □

Now, $OD_\Phi(s;\phi)$ is in terms of integrals of Mellin-Barnes type. Note that the Γ-factors from Proposition 2.1 and (4.2) alone are not sufficient for our goals (see Remark 4.1 and (4.9), (4.10), (4.11)). The three extra Γ-factors brought by the y_0-integral, which 'mix' all variables of integrations, will play an important role in Section 5-6.

5. Analytic Properties of the Archimedean Transform

In (4.4), the factors $\zeta(2s-s_0)$ and $L(s_0,\Phi)$ are known to admit holomorphic continuation and have polynomial growth in vertical strips (except on the line $2s-s_0=1$). It remains to study the archimedean part of (4.4), i.e., the integral transform

$$\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,\ s;\ \phi) \ := \ \int_{(15)}\int_{(\epsilon)} \widetilde{h}\left(s-s_1-\frac{1}{2}\right)\cdot \mathcal{G}_\Phi^{(\delta)}(s_1,u;\ s_0,s;\ \phi)\ \frac{du}{2\pi i}\frac{ds_1}{2\pi i}, \tag{5.1}$$

where $h:=H^\flat$ and $\mathcal{G}_\Phi^{(\delta)}(\cdots)$ as defined in (4.6). In Section 4, we have shown that when $\phi\in(0,\pi/2)$, the function $(s_0,s)\mapsto\left(\mathcal{F}_\Phi^{(\delta)}h\right)(s_0,\ s;\ \phi)$ is holomorphic on the domain (4.15), i.e.,

$$\sigma \ < \ 4 \qquad \text{and} \qquad 1+\theta+\epsilon \ < \ \sigma_0 \ < \ 2\sigma-1.$$

In this section, we establish a larger region of holomorphy for $(s_0,s)\mapsto\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,s;\phi)$ that holds for $\phi\in(0,\pi/2]$. We write

$$s \ = \ \sigma+it, \qquad s_0=\sigma_0+it_0, \qquad s_1=\sigma_1+it_1, \qquad \text{and} \qquad u=\ \epsilon+iv,$$

with $\epsilon:=1/100$. It is sufficient to consider s inside the rectangular box $\epsilon<\sigma<4$ and $|t|\leqslant T$, for any given $T\geqslant 1000$. Moreover, $\alpha_k:=i\gamma_k\in i\mathbb{R}$ $(k=1,2,3)$ by our assumptions on Φ. The main result of this section can be stated as follows:

Proposition 5.1. *Suppose $H\in\mathcal{C}_\eta$.*

(1) *For any $\phi\in(0,\pi/2]$, the transform $\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,s;\phi)$ is holomorphic on the domain*

$$\sigma_0 \ > \ \epsilon, \qquad \sigma \ < \ 4, \qquad \text{and} \qquad 2\sigma-\sigma_0-\epsilon \ > \ 0. \tag{5.2}$$

(2) *Whenever $(\sigma_0,\sigma)\in$ (5.2), $|t|<T$, and $\phi\in(0,\pi/2)$, the transform $\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,\ s;\ \phi)$ has exponential decay as $|t_0|\to\infty$.* [11]

Remark 5.2. *The domain (5.2) is chosen in a way that the function $(s_0,s)\mapsto\mathcal{G}_\Phi^{(\delta)}(s_1,u;\ s_0,s;\ \phi)$ is holomorphic on (5.2) when $\operatorname{Re}s_1=\sigma_1\geqslant 15$ and $\operatorname{Re}u=\epsilon$. Moreover, if we have $15\leqslant\sigma_1\leqslant\eta-\frac{1}{2}$ and (5.2), then $s-s_1-1/2$ lies inside the region of holomorphy of $\widetilde{h}$.*

Proof. The proof is based on a careful application of the Stirling estimate

$$|\Gamma(a+ib)| \ \asymp_a \ (1+|b|)^{a-\frac{1}{2}}\,e^{-\frac{\pi}{2}|b|} \qquad (a\ \neq\ 0,\ -1,\ -2,\ \ldots,\quad b\ \in\ \mathbb{R}) \tag{5.3}$$

to the kernel function $\mathcal{G}_\Phi^{(\delta)}(s_1,u;\ s_0,s;\ \phi)$. The following set of conditions will be repeated throughout the proof:

$$\begin{cases} \qquad\quad 0 \ < \ \phi \ \leqslant \ \pi/2, \\ \sigma_0 \ > \ \epsilon, \quad \sigma \ < \ 4, \quad 2\sigma-\sigma_0-\epsilon \ > \ 0, \\ \operatorname{Re}s_1 \ = \ \sigma_1 \ \geqslant \ 15, \quad \operatorname{Re}u \ = \ \epsilon. \end{cases} \tag{5.4}$$

[11] The explicit estimate is stated in the proof below and the implicit constant depends only on T and Φ.

Assuming (5.4), apply (5.3) to the kernel function (4.6). It follows that

$$\left|\mathcal{G}^{(\delta)}_{\Phi}(s_1,u;\ s_0,s;\ \phi)\right| \asymp (1+|v|)^{\epsilon-\frac12}\, e^{-\left(\frac{\pi}{2}-\phi\right)|v|}\cdot\prod_{k=1}^{3}\,(1+|t_1-\gamma_k|)^{\frac{\sigma_1-1}{2}}\, e^{-\frac{\pi}{4}|t_1-\gamma_k|}$$
$$\cdot\prod_{k=1}^{3}\,(1+|t_0-v+\gamma_k|)^{\frac{\sigma_0-\epsilon-1}{2}}\, e^{-\frac{\pi}{4}|t_0-v+\gamma_k|}\cdot(1+|2t-t_0-v|)^{\frac{2\sigma-1-\sigma_0-\epsilon}{2}}\, e^{-\frac{\pi}{4}|2t-t_0-v|}$$
$$\cdot(1+|v-2t+t_1-t_0|)^{\frac{\epsilon-2\sigma+\sigma_1-\sigma_0}{2}}\, e^{-\frac{\pi}{4}|v-2t+t_1-t_0|}$$
$$\cdot(1+|t_1-2t_0|)^{-\left(\frac{\sigma_1}{2}-\sigma_0\right)}\, e^{\frac{\pi}{4}|t_1-2t_0|}\ \cdot(1+|t_0+t_1-v|)^{-\frac{\sigma_0+\sigma_1-\epsilon-1}{2}}\, e^{\frac{\pi}{4}|t_0+t_1-v|}, \tag{5.5}$$

where the implicit constant depends at most on σ_1. [12] Let $\mathcal{P}^{\Phi}_s(t_0,t_1,v)$ be the 'polynomial part' of (5.5) and

$$\mathcal{E}^{\Phi}_s(t_0,t_1,v) \ :=\ \sum_{k=1}^{3}\ \{|t_1-\gamma_k|+|t_0-v+\gamma_k|\}+|2t-t_0-v|+|v-2t+t_1-t_0|-|t_1-2t_0|-|t_0+t_1-v|.$$

We first examine the exponential phase $\mathcal{E}^{\Phi}_s(t_0,t_1,v)$ of (5.5) as it determines the effective support of $\big(\mathcal{F}^{(\delta)}_{\Phi}H\big)(s_0,s;\phi)$. By the triangle inequality and the fact $\gamma_1+\gamma_2+\gamma_3=0$, we have

$$\left|\mathcal{G}^{(\delta)}_{\Phi}(s_1,u;\ s_0,s;\ \phi)\right| \ \ll_{\sigma_1}\ e^{\pi T}\cdot\mathcal{P}^{\Phi}_s(t_0,t_1,v)\cdot\exp\left(-\frac{\pi}{4}\mathcal{E}(t_0,t_1,v)\right)\cdot e^{-\left(\frac{\pi}{2}-\phi\right)|v|} \tag{5.6}$$

with

$$\mathcal{E}(t_0,t_1,v)\ :=\ 3|t_1|+3|t_0-v|-|t_1-2t_0|+|v+t_1-t_0|+|t_0+v|-|t_0+t_1-v|, \tag{5.7}$$

whenever we have (5.4) and $|t|\leqslant T$,

Claim 5.3. *For any $t_0,t_1,v\in\mathbb{R}$, we have $\mathcal{E}(t_0,t_1,v)\ \geqslant\ 0$. Equality holds if and only if*

$$t_1\ =\ 0\qquad\text{and}\qquad t_0-v\ =\ 0. \tag{5.8}$$

Proof. Adding up the inequalities $|t_1|+|t_0-v|\geqslant|t_0+t_1-v|$ and $|v+t_1-t_0|+|t_0+v|\geqslant|t_1-2t_0|$, we have

$$\mathcal{E}(t_0,t_1,v)\ \geqslant\ 2\,(|t_1|+|t_0-v|)\ \geqslant\ 0. \tag{5.9}$$

The equality case is apparent. ☐

Claim 5.4. *When (5.4) and $|t|\leqslant T$ hold, the integral*

$$\iint_{\substack{(\mathrm{Re}\,s_1,\mathrm{Re}\,u)=(\sigma_1,\epsilon),\\ (t_1,v):(5.11)\text{ holds}}}\ \widetilde{h}\left(s-s_1-\frac12\right)\cdot\mathcal{G}^{(\delta)}_{\Phi}(s_1,u;\ s_0,s;\ \phi)\ \frac{du}{2\pi i}\ \frac{ds_1}{2\pi i} \tag{5.10}$$

has exponential decay as $|t_0|\to\infty$, where

$$|t_1|\ >\ \log^2(3+|t_0|)\qquad\text{or}\qquad|v-t_0|\ >\ \log^2(3+|t_0|). \tag{5.11}$$

Proof. In case of (5.11), we have

$$\mathcal{E}(t_0,t_1,v)\ >\ \log^2(3+|t_0|)+|t_1|+|t_0-v| \tag{5.12}$$

from (5.9). The polynomial part $\mathcal{P}^{\Phi}_s(t_0,t_1,v)$ can be crudely bounded by

$$\mathcal{P}^{\Phi}_s(t_0,t_1,v)\ \ll_{\Phi,\sigma_1,T}\ [(1+|t_1|)\,(1+|v-t_0|)\,(1+|t_0|)]^{A(\sigma_1)}, \tag{5.13}$$

where $A(\sigma_1)>0$ is some constant.

Putting (5.12), (5.13), and the bound $e^{-\left(\frac{\pi}{2}-\phi\right)|v|}\leqslant 1$ ($\phi\in(0,\pi/2]$) into (5.6), we obtain

$$\left|\mathcal{G}^{(\delta)}_{\Phi}(s_1,u;\ s_0,s;\ \phi)\right|\ \ll_{\Phi,\sigma_1,T}\ (1+|t_0|)^{A(\sigma_1)}\, e^{-\frac{\pi}{4}\log^2(3+|t_0|)}\cdot[(1+|t_1|)\,(1+|v-t_0|)]^{A(\sigma_1)}e^{-\frac{\pi}{4}[|t_1|+|t_0-v|]} \tag{5.14}$$

[12] Note that the domain (5.2) for (σ,σ_0) is bounded and thus the estimate is uniform in σ,σ_0,ϵ. This will be assumed for all estimates in the rest of this section.

whenever (5.11), (5.4), and $|t| \leqslant T$ hold. The boundedness of $\widetilde{h}$ on vertical strips implies that (5.10) is

$$\ll_{\sigma_1,\Phi,T} \ (1+|t_0|)^{A(\sigma_1)} e^{-\frac{\pi}{4}\log^2(3+|t_0|)}. \tag{5.15}$$

This proves Claim 5.4. □

Now, let $\phi \in (0,\pi/2]$ and consider $\left(\mathcal{F}_\Phi^{(\delta)} H\right)(s_0,\ s;\ \phi)$ as a function on the bounded domain

$$(\sigma_0,\sigma) \ \in \ (5.2), \qquad |t|,\ |t_0| \ \leqslant \ T. \tag{5.16}$$

When $|t_1| > \log^2(3+T)$ or $|v| > T + \log^2(3+T)$, observe that (5.11) is satisfied and from (5.14),

$$\left|\mathcal{G}_\Phi^{(\delta)}\left(s_1,u;\ s_0,s;\ \phi\right)\right| \ \ll_{\Phi,T} \ \left[\left(1+|t_1|\right)\left(1+|v|\right)\right]^{A(15)} \cdot e^{-\frac{\pi}{4}[|t_1|+|v|]}. \tag{5.17}$$

The last function is clearly jointly integrable with respect to t_1, v, and by Remark 5.2, $\left(\mathcal{F}_\Phi^{(\delta)} H\right)(s_0,\ s;\ \phi)$ is a holomorphic function on (5.16). Since the choice of T is arbitrary, we arrive at the first conclusion of Proposition 5.1.

In the remaining part, we prove the second assertion of Proposition 5.1. We estimate the contribution from

$$|t_1| \ \leqslant \ \log^2(3+|t_0|) \qquad \text{and} \qquad |v-t_0| \ \leqslant \ \log^2(3+|t_0|). \tag{5.18}$$

(The complementary part has been treated in Claim 5.4.)

It suffices to restrict ourselves to the effective support (5.8). The polynomial part can be essentially computed by substituting $t_1 := 0$ and $v := t_0$. More precisely, when (5.18) and $|t_0| \gg_T 1$ hold, there are only two possible scenarios for the factors $1+|(\cdots)|$ in (5.5): either $1+|(\cdots)| \ \asymp \ |t_0|$, or $\log^{-C}(3+|t_0|) \ \ll \ 1+|(\cdots)| \ \ll \ \log^{C}(3+|t_0|)$ for some absolute constant $C>0$.

In case of (5.18), apply the bounds $e^{-\frac{\pi}{4}\mathcal{E}(t_0,t_1,v)} \leqslant 1$ and $e^{-\left(\frac{\pi}{2}-\phi\right)|v|} \leqslant e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|}$ for $|t_0| \gg 1$ to (5.6). As a result, if we also have (5.4), $|t| < T$, and $|t_0| > 8T$, then

$$\left|\mathcal{G}_\Phi^{(\delta)}\left(s_1,u;\ s_0,s;\ \phi\right)\right| \ \ll_{\sigma_1,\Phi,T} \ |t_0|^{7-\frac{\sigma_1}{2}} e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|} \log^{B(\sigma_1)}|t_0| \tag{5.19}$$

and

$$\iint\limits_{\substack{(\operatorname{Re} s_1,\operatorname{Re} u)=(\sigma_1,\epsilon),\\ (t_1,v):\ (5.18)\text{ holds}}} \widetilde{h}\left(s-s_1-\frac{1}{2}\right)\cdot \mathcal{G}_\Phi^{(\delta)}\left(s_1,u;\ s_0,s;\ \phi\right) \frac{du}{2\pi i}\frac{ds_1}{2\pi i}$$

$$\ll_{\sigma_1,\Phi,T} \ |t_0|^{7-\frac{\sigma_1}{2}} e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|} \log^{4+B(\sigma_1)}|t_0|, \tag{5.20}$$

where $B(\sigma_1) > 0$ is some constant. If $\phi < \pi/2$, then there is exponential decay in (5.20) as $|t_0| \to \infty$. Therefore, the second conclusion of the proposition follows from (5.20) and (5.15) (putting $\sigma_1 = 15$). □

6. Analytic Continuation of the Off-diagonal

6.1. **Step 1:** We first obtain a holomorphic continuation of $OD_\Phi(s;\phi)$ up to $\operatorname{Re} s > \frac{1}{2}+\epsilon$ by shifting the s_0-integral to the left.

Fix any $\phi \in (0,\pi/2)$ and $T \geqslant 1000$. We first restrict ourselves to

$$1+\frac{\theta}{2}+2\epsilon \ < \ \sigma \ < \ 4, \qquad |t| \ < \ T. \tag{6.1}$$

Clearly, the pole $s_0 = 2s-1$ of $\zeta(2s-s_0)$ is on the right of the contour $\operatorname{Re} s_0 = 1+\theta+2\epsilon$ of the integral (4.4).

Let $T_0 \gg 1$. The rectangle with vertices $2\epsilon \pm iT_0$ and $(1+\theta+2\epsilon)\pm iT_0$ in the s_0-plane lies inside the region of holomorphy (5.2) of $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$. The contribution from the horizontal segments $[2\epsilon \pm iT_0, (1+\theta+2\epsilon)\pm iT_0]$ tends to 0 as $T_0 \to \infty$ by the exponential decay of $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,\ s;\ \phi)$ (see Proposition 5.1.[13]) As a result, we

[13] which surely counteracts the polynomial growth from $L(s_0,\Phi)$ and $\zeta(2s-s_0)$.

may shift the line of integration to $\operatorname{Re} s_0 = 2\epsilon$ and no pole is crossed. Hence,

$$OD_\Phi(s;\phi) \;=\; \frac{1}{4}\int_{(2\epsilon)} \zeta\left(2s-s_0\right) L\left(s_0,\Phi\right)\cdot \sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0,\; s;\; \phi\right)\,\frac{ds_0}{2\pi i} \tag{6.2}$$

on (6.1). The right side of (6.2) is holomorphic on

$$\frac{1}{2}+\epsilon \;<\; \sigma \;<\; 4, \qquad |t| \;<\; T \tag{6.3}$$

and serves as an analytic continuation of $OD_\Phi(s;\phi)$ to (6.3) by using (5.2). Note that $\sigma > \frac{1}{2}+\epsilon$ implies the holomorphy of $\zeta(2s-s_0)$.

6.2. **Step 2: Crossing the Polar Line (Shifting the s_0-integral again).** Consider a subdomain of (6.3):

$$\frac{1}{2}+\epsilon \;<\; \sigma \;<\; \frac{3}{4}, \qquad |t| \;<\; T. \tag{6.4}$$

Different from Step 1, the pole $s_0 = 2s-1$ is now inside the rectangle with vertices $2\epsilon \pm iT_0$ and $\frac{1}{2}\pm iT_0$ provided $T_0 > 4T$. Such a rectangle lies in the region of holomorphy (5.2) of $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$. When $\phi<\pi/2$, the exponential decay of $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$ once again allows us to shift the line of integration from $\operatorname{Re} s_0 = 2\epsilon$ to $\operatorname{Re} s_0 = 1/2$, crossing the pole of $\zeta(2s-s_0)$ which has residue -1. In other words,

$$\begin{aligned} OD_\Phi(s;\phi) \;&=\; \frac{1}{4}\, L(2s-1,\Phi) \sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(2s-1,\; s;\; \phi\right) \\ &\quad + \frac{1}{4}\int_{(1/2)} \zeta\left(2s-s_0\right) L\left(s_0,\Phi\right)\cdot \sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0,\; s;\; \phi\right)\,\frac{ds_0}{2\pi i}. \end{aligned} \tag{6.5}$$

On the line $\operatorname{Re} s_0 = 1/2$, observe that $s \mapsto \left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0,\; s;\; \phi\right)$ is holomorphic on $\sigma > \frac{1}{4}+\frac{\epsilon}{2}$ by (5.2) ; whereas $s\mapsto \zeta(2s-s_0)$ is holomorphic on $\sigma < 3/4$ as $2\sigma-\sigma_0 < 1$. As a result, the function $s \mapsto \int_{(1/2)}(\cdots)\,\frac{ds_0}{2\pi i}$ in (6.5) is holomorphic on the vertical strip

$$\frac{1}{4}+\frac{\epsilon}{2} \;<\; \sigma \;<\; \frac{3}{4}, \tag{6.6}$$

which is sufficient for our purpose.

However, Proposition 5.1 only asserts that the function $s \mapsto \left(\mathcal{F}_\Phi^{(\delta)}H\right)(2s-1,s;\phi)$ is holomorphic on $\frac{1}{2}+\epsilon<\sigma<4$. This issue will be addressed in Section 6.4.

6.3. **Step 3: Putting Back $\phi\to\pi/2$ — Shifting the s_1-integral and Refining Step 1-2.** By using estimate (5.14) and Dominated Convergence,

$$\lim_{\phi\to\pi/2}\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(2s-1,\; s;\; \phi\right) \;=\; \left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(2s-1,\; s;\; \pi/2\right) \tag{6.7}$$

for $\frac{1}{2}+\epsilon<\sigma<4$ and $|t|<T$. However, for the continuous part of (6.5), we need a follow-up of Proposition 5.1 in order to pass to the limit $\phi\to\pi/2$. Essentially, thanks to the structure of the Γ's in Proposition 2.1 and the analytic properties of $\widetilde{h}$, it is possible to shift the line of integration of the s_1-integral to gain sufficient polynomial decay.

Proposition 6.1. *Let $H\in\mathcal{C}_\eta$. There exists a constant $B=B_\eta$ such that whenever $(\sigma_0,\sigma)\in$ (5.2), $|t|<T$, and $|t_0|\gg_T 1$, we have the estimate*

$$\left|\left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0,\; s;\; \pi/2\right)\right| \;\ll\; |t_0|^{8-\frac{\eta}{2}}\log^B|t_0|, \tag{6.8}$$

where the implicit constant depends only on η, T, Φ.

Proof. On domain (5.2), observe that the vertical strip $\operatorname{Re} s_1 \in [15,\eta-\frac{1}{2}]$ contains no pole of the function $s_1\mapsto \mathcal{G}_\Phi^{(\delta)}\left(s_1,u;\; s_0,s;\; \phi\right)$, and it lies within the region of holomorphy of $\widetilde{h}$ (see Remark 5.2). The estimate (5.14) allows us to shift the line of integration from $\operatorname{Re} s_1 = 15$ to $\operatorname{Re} s_1 = \eta-\frac{1}{2}$ in (4.5). Notice that the estimates done in Proposition 5.1 works for $\phi=\pi/2$ too. In particular, from (5.20) and (5.15), the bound (6.8) follows by taking $\sigma_1 := \eta-\frac{1}{2}$ therein (upon the contour shift). This completes the proof. ∎

Suppose $(3+\theta)/2 < \sigma < 4$. By Proposition 4.4, equation (4.4) and equation (6.2), we have

$$\begin{aligned} OD_\Phi(s) &= \lim_{\phi\to\pi/2} OD_\Phi(s;\phi) \\ &= \lim_{\phi\to\pi/2} \frac{1}{4}\int_{(1+\theta+2\epsilon)} \zeta(2s-s_0) L(s_0,\Phi)\cdot \sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,\ s;\ \phi)\ \frac{ds_0}{2\pi i} \\ &= \lim_{\phi\to\pi/2} \frac{1}{4}\int_{(2\epsilon)} \zeta(2s-s_0) L(s_0,\Phi)\cdot \sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,\ s;\ \phi)\ \frac{ds_0}{2\pi i}. \end{aligned} \tag{6.9}$$

Proposition 6.1 ensures enough polynomial decay and hence the absolute convergence of (6.10) at $\phi = \pi/2$:

$$OD_\Phi(s) = \frac{1}{4}\int_{(2\epsilon)} \zeta(2s-s_0) L(s_0,\Phi)\cdot \sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,\ s;\ \pi/2)\ \frac{ds_0}{2\pi i}. \tag{6.10}$$

Now, (6.10) serves as an analytic continuation of $OD_\Phi(s)$ to the domain $1/2+\epsilon < \sigma < 4$.

On the smaller domain $1/2+\epsilon<\sigma<3/4$, the expressions (6.9) and (6.5) are equal. Then

$$\begin{aligned} OD_\Phi(s) = (6.9) &= \frac{1}{4} L(2s-1,\Phi) \sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)(2s-1,\ s;\ \pi/2) \\ &\quad + \frac{1}{4}\int_{(1/2)} \zeta(2s-s_0) L(s_0,\Phi)\cdot \sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,\ s;\ \pi/2)\ \frac{ds_0}{2\pi i} \end{aligned} \tag{6.11}$$

by Dominated Convergence and Proposition 5.1. The last integral is holomorphic on $\frac{1}{4}+\frac{\epsilon}{2} < \sigma < \frac{3}{4}$.

To ease the notations, let's write $(\mathcal{F}_\Phi H)(s_0,s) := \left(\mathcal{F}_\Phi^{+}H\right)(s_0,s;\ \pi/2) + \left(\mathcal{F}_\Phi^{-}H\right)(s_0,s;\ \pi/2)$. Then by the duplication and the reflection formula of Γ-functions, we have

$$\begin{aligned} (\mathcal{F}_\Phi H)(s_0,s) &= \sqrt{\pi}\int_{(\eta-1/2)} \widetilde{h}\left(s-s_1-\frac{1}{2}\right)\pi^{-s_1}\frac{\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)} \\ &\quad \cdot\int_{(\epsilon)} \frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{s_1-(s_0-u)}{2}+\frac{1}{2}-s\right)\cdot\prod_{i=1}^{3}\Gamma\left(\frac{(s_0-u)+\alpha_i}{2}\right)\Gamma\left(s-\frac{s_0+u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)\Gamma\left(\frac{(s_0-u)+s_1}{2}\right)}\ \frac{du}{2\pi i}\ \frac{ds_1}{2\pi i} \end{aligned} \tag{6.12}$$

6.4. **Step 4: Continuation of the Residual Term — Shifting the u-integral.**

Proposition 6.2. *Let $H\in\mathcal{C}_\eta$. The function $s\mapsto(\mathcal{F}_\Phi H)(2s-1,s)$ can be holomorphically continued to the vertical strip $\epsilon<\sigma<4$ except at the three simple poles: $s=(1-\alpha_i)/2$ $(i=1,2,3)$, where $(\alpha_1,\alpha_2,\alpha_3)$ are the Langlands parameters of the Maass cusp form Φ.*

Proof. We will prove a stronger result in Proposition 7.3. However, a simpler argument suffices for the time being. Suppose $\frac{1}{2}+\epsilon<\sigma<4$ and $s_0=2s-1$. In (6.12), we shift the line of integration from $\operatorname{Re}u=\epsilon$ to $\operatorname{Re}u=-1.9$:

$$\begin{aligned} (\mathcal{F}_\Phi H)(2s-1,s) &= 2\sqrt{\pi}\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)\int_{(\eta-\frac{1}{2})}\widetilde{h}\left(s-s_1-\frac{1}{2}\right)\frac{\pi^{-s_1}\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)\Gamma\left(\frac{s_1}{2}+1-2s\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)\Gamma\left(s-\frac{1}{2}+\frac{s_1}{2}\right)}\ \frac{ds_1}{2\pi i} \\ &\quad + \sqrt{\pi}\int_{(\eta-\frac{1}{2})}\int_{(-1.9)}\widetilde{h}\left(s-s_1-\frac{1}{2}\right)\pi^{-s_1}\frac{\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)} \\ &\quad\quad \cdot\frac{\Gamma\left(\frac{u+s_1}{2}+1-2s\right)\cdot\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}-\frac{u}{2}\right)\Gamma\left(\frac{u}{2}\right)}{\Gamma\left(s-\frac{1}{2}+\frac{s_1-u}{2}\right)}\ \frac{du}{2\pi i}\ \frac{ds_1}{2\pi i}. \end{aligned}$$

By Stirling's formula and the same argument following (5.17), the integrals above represent holomorphic functions on $\epsilon<\sigma<4$. ∎

Apply Proposition 6.2 to (6.11) and observe that the poles of $s \mapsto (\mathcal{F}_\Phi H)(2s-1,s)$ are exactly the trivial zeros of the the arithmetic factor $L(2s-1,\Phi)$ in (6.5). We conclude that the product of functions $s \mapsto L(2s-1,\Phi)\cdot(\mathcal{F}_\Phi H)(2s-1,s)$ is holomorphic on $\epsilon<\sigma<4$ and thus (6.11) provides a holomorphic continuation of $OD_\Phi(s)$ to the vertical strip $\frac{1}{4}+\frac{\epsilon}{2}<\sigma<\frac{3}{4}$. By the rapid decay of Φ at ∞, the inner product $s\mapsto \left\langle P,\ \mathbb{P}_2^3\Phi\cdot|\det *|^{\bar{s}-\frac{1}{2}}\right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))}$ represents an entire function. Putting (2.19), (3.8) and (6.11) together, we arrive at Theorem 1.1.

7. Explicit Evaluations of the Transform

The power of spectral summation formulae (including Theorem 1.1) is encoded in the archimedean transformations involved. Therefore, it is important to obtain very explicit expressions for the transformations, usually in terms of *special functions*. It is well-known that the special functions for $GL(2)$ possess lots of symmetries and identities under various transforms. However, this ceases to be true when it comes to higher-rank groups and there remain plenty of prospects for in-depth investigations.

Nevertheless, there have been some successes in higher-rank groups. For example, Stade [St01, St02] was able to compute the Mellin transforms and certain Rankin-Selberg integrals of Whittaker functions for $GL_n(\mathbb{R})$; Goldfeld et. al. [GK13, GSW21] obtained (harmonic-weighted) spherical Weyl laws of $GL_3(\mathbb{R})$ and $GL_4(\mathbb{R})$ with strong power-saving error terms; and there is the work of Buttcane [Bu13, Bu16] on Kuznetsov formulae for $GL(3)$. What lies at the core of the aforementioned results are various *Mellin-Barnes* integrals which represent the special functions of higher-rank. Judging from their experiences, this way of handling the archimedean aspects of problems is more likely to generalize.

In this final section, we continue such investigation and record several formulae for the archimedean transform $(\mathcal{F}_\Phi H)(s_0,s)$.

Lemma 7.1. *Suppose $H\in\mathcal{C}_\eta$ and $h:=H^\flat$. On the vertical strip $-\frac{1}{2}<\operatorname{Re} w<\eta$, we have*

$$\widetilde{h}(w) \ := \ \int_0^\infty h(y)y^w\, d^\times y \ = \ \frac{\pi^{-w-\frac{1}{2}}}{4}\int_{(0)} H(\beta)\cdot\frac{\Gamma\left(\frac{w+\frac{1}{2}+\beta}{2}\right)\Gamma\left(\frac{w+\frac{1}{2}-\beta}{2}\right)}{|\Gamma(\beta)|^2}\,\frac{d\beta}{2\pi i}, \tag{7.1}$$

Proof. Since $H\in\mathcal{C}_\eta$, both sides of (7.1) converge absolutely on the strip $-1/2<\operatorname{Re} w<\eta$ by Stirling's formula and Proposition 2.7. Substituting the definition of h as in (2.6) into $\widetilde{h}(w)$, the result follows from

$$\int_0^\infty W_\beta(y)y^w\, d^\times y \ = \ \frac{\pi^{-w-\frac{1}{2}}}{2}\,\Gamma\left(\frac{w+\frac{1}{2}+\beta}{2}\right)\Gamma\left(\frac{w+\frac{1}{2}-\beta}{2}\right), \tag{7.2}$$

where $\operatorname{Re}\beta \ = \ 0$ and $\operatorname{Re} w \ > \ -1/2$.

□

Proposition 7.2. *Suppose $H\in\mathcal{C}_\eta$. On the domain* [14]

$$\sigma_0 \ > \ \epsilon, \quad \sigma \ < \ 4, \quad 2\sigma-\sigma_0-\epsilon \ > \ 0, \quad \sigma_0+2\sigma-1-\epsilon \ > \ 0, \quad 1+\epsilon-\sigma_0-\sigma \ > \ 0, \tag{7.3}$$

we have

$$(\mathcal{F}_\Phi H)(s_0,s) \ = \ \frac{\pi^{\frac{1}{2}-s}}{4}\int_{(0)}\frac{H(\beta)}{|\Gamma(\beta)|^2}\cdot\mathcal{K}_\Phi(s_0,s;\beta)\,\frac{d\beta}{2\pi i}, \tag{7.4}$$

where the kernel function $\mathcal{K}_\Phi(\cdots)$ is given explicitly by the double Barnes integrals

$$\mathcal{K}_\Phi(s_0,s;\beta) \ := \ \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\Gamma\left(\frac{s-s_1+\beta}{2}\right)\Gamma\left(\frac{s-s_1-\beta}{2}\right)\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)} \cdot\frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{s_1-(s_0-u)}{2}+\frac{1}{2}-s\right)\cdot\prod_{i=1}^{3}\Gamma\left(\frac{(s_0-u)+\alpha_i}{2}\right)\Gamma\left(s-\frac{s_0+u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)\Gamma\left(\frac{(s_0-u)+s_1}{2}\right)}\,\frac{du}{2\pi i}\,\frac{ds_1}{2\pi i}, \tag{7.5}$$

[14] Certainly non-empty — it includes our point of interest $(\sigma_0,\sigma)=(1/2,1/2)$.

and the contours follow Barnes' convention. Explicitly, they can be taken as $\operatorname{Re} u = \epsilon$ *and* $\operatorname{Re} s_1 = \sigma_1$ *with*

$$\sigma_0 + 2\sigma - 1 - \epsilon \ < \ \sigma_1 \ < \ \sigma. \tag{7.6}$$

Proof. Suppose

$$\sigma_0 \ > \ \epsilon, \qquad \sigma \ < \ 4, \qquad \text{and} \qquad 2\sigma - \sigma_0 - \epsilon \ > \ 0 \tag{7.7}$$

as in Proposition 5.1. Recall the expression (6.12) for $(\mathcal{F}_\Phi H)(s_0, s)$. This time, we shift the line of integration of the s_1-integral to $\operatorname{Re} s_1 = \sigma_1$ satisfying

$$\sigma_1 \ < \ \sigma \tag{7.8}$$

and no pole is crossed during this shift as long as

$$\sigma_1 \ > \ 0 \qquad \text{and} \qquad \sigma_1 \ > \ \sigma_0 + 2\sigma - 1 - \epsilon. \tag{7.9}$$

Now, assume (7.3). The restrictions (7.7), (7.8), (7.9) hold and such a line of integration for the s_1-integral exists. Upon shifting the line of integration to such a position, substituting (7.1) into (6.12) and the result follows. □

In Section 4, the u-integral was introduced for various technical reasons. However, the u-integral turns out containing nice symmetries upon bringing in new Γ-factors and is an integral part of the archimedean computation. Indeed, recall the Second Barnes Lemma: for $a, b, c, d, e, f \in \mathbb{C}$ with $f = a + b + c + d + e$,

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(w+a)\Gamma(w+b)\Gamma(w+c)\Gamma(d-w)\Gamma(e-w)}{\Gamma(w+f)} \, \frac{dw}{2\pi i} = \frac{\Gamma(d+a)\Gamma(d+b)\Gamma(d+c)\Gamma(e+a)\Gamma(e+b)\Gamma(e+c)}{\Gamma(f-a)\Gamma(f-b)\Gamma(f-c)} \tag{7.10}$$

(see Theorem 2.4.3 of [AAR99]). We then have

Proposition 7.3. *Suppose* $\frac{1}{2} + \epsilon < \sigma < 1$. *Then*

$$(\mathcal{F}_\Phi H)(2s-1, s) \ = \ \pi^{\frac{1}{2}-s} \cdot \prod_{i=1}^{3} \frac{\Gamma\left(s - \frac{1}{2} + \frac{\alpha_i}{2}\right)}{\Gamma\left(1 - s - \frac{\alpha_i}{2}\right)} \cdot \int_{(0)} \frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot \prod_{i=1}^{3} \prod_{\pm} \Gamma\left(\frac{1 - s + \alpha_i \pm \beta}{2}\right) \frac{d\beta}{2\pi i}. \tag{7.11}$$

Proof. Suppose $\frac{1}{2} + \epsilon < \sigma < \frac{2+\epsilon}{3}$. Then (7.3) is satisfied with $\sigma_0 = 2\sigma - 1$ and by Proposition 7.2,

$$(\mathcal{F}_\Phi H)(2s-1, s) \ = \ \frac{\pi^{\frac{1}{2}-s}}{4} \int_{(0)} \frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot \int_{(\sigma_1)} \frac{\Gamma\left(\frac{s-s_1+\beta}{2}\right)\Gamma\left(\frac{s-s_1-\beta}{2}\right)\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} + 1 - 2s\right)} \cdot \int_{(\epsilon)} \frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{u+s_1}{2} + 1 - 2s\right) \cdot \prod_{i=1}^{3} \Gamma\left(s - \frac{1}{2} + \frac{\alpha_i - u}{2}\right)}{\Gamma\left(s - \frac{1}{2} + \frac{s_1 - u}{2}\right)} \, \frac{du}{2\pi i} \, \frac{ds_1}{2\pi i} \, \frac{d\beta}{2\pi i}. \tag{7.12}$$

For the u-integral, apply the change of variable $u \to -2u$ and (7.10) with

$$(a, b, c, d, e) \ \to \ \left(s - \frac{1}{2} + \frac{\alpha_1}{2}, \ s - \frac{1}{2} + \frac{\alpha_2}{2}, \ s - \frac{1}{2} + \frac{\alpha_3}{2}, \ 0, \ \frac{s_1}{2} + 1 - 2s\right),$$

we obtain

$$\begin{aligned}
(\mathcal{F}_\Phi H)(2s-1,s) &= \frac{\pi^{\frac12 - s}}{2} \int_{(0)} \frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot \int_{(\sigma_1)} \frac{\Gamma\left(\frac{s-s_1+\beta}{2}\right)\Gamma\left(\frac{s-s_1-\beta}{2}\right) \prod_{i=1}^{3} \Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)} \\
&\qquad \cdot \prod_{i=1}^{3} \frac{\Gamma\left(s-\frac12+\frac{\alpha_i}{2}\right)\Gamma\left(\frac12 - s + \frac{s_1+\alpha_i}{2}\right)}{\Gamma\left(\frac{s_1-\alpha_i}{2}\right)} \frac{ds_1}{2\pi i} \frac{d\beta}{2\pi i} \\
&= \frac{\pi^{\frac12 - s}}{2} \cdot \prod_{i=1}^{3} \Gamma\left(s - \frac12 + \frac{\alpha_i}{2}\right) \\
&\qquad \cdot \int_{(0)} \frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot \int_{(\sigma_1)} \frac{\prod_{i=1}^{3} \Gamma\left(\frac12 - s + \frac{s_1+\alpha_i}{2}\right) \cdot \Gamma\left(\frac{s-s_1+\beta}{2}\right)\Gamma\left(\frac{s-s_1-\beta}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)} \frac{ds_1}{2\pi i} \frac{d\beta}{2\pi i}.
\end{aligned} \tag{7.13}$$

For the s_1-integral, apply the change of variable $s_1 \to 2s_1$ and (7.10) the second time but with

$$(a,b,c,d,e) \;\to\; \left(\frac12 - s + \frac{\alpha_1}{2},\ \frac12 - s + \frac{\alpha_2}{2},\ \frac12 - s + \frac{\alpha_3}{2},\ \frac{s+\beta}{2},\ \frac{s-\beta}{2}\right), \tag{7.14}$$

we obtain

$$(\mathcal{F}_\Phi H)(2s-1,s) = \pi^{\frac12 - s} \cdot \prod_{i=1}^{3} \Gamma\left(s - \frac12 + \frac{\alpha_i}{2}\right) \cdot \int_{(0)} \frac{H(\beta)}{|\Gamma(\beta)|^2} \cdot \prod_{i=1}^{3} \frac{\prod_{\pm} \Gamma\left(\frac{1-s+\alpha_i \pm \beta}{2}\right)}{\Gamma\left(1 - s - \frac{\alpha_i}{2}\right)} \frac{d\beta}{2\pi i}. \tag{7.15}$$

By analytic continuation, (7.11) holds for $\frac12 + \epsilon < \sigma < 1$ and this completes the proof. □

More generally, it is possible to express the transform in terms of *hypergeometric functions* of a special type. We define

$$\begin{aligned}
{}_4\widehat{F}_3\left(\begin{matrix} A_1 & & A_2 & & A_3 & & A_4 \\ & B_1 & & B_2 & & B_3 & \end{matrix} \middle| z\right) &:= \frac{\Gamma(A_1)\Gamma(A_2)\Gamma(A_3)\Gamma(A_4)}{\Gamma(B_1)\Gamma(B_2)\Gamma(B_3)} \cdot {}_4F_3\left(\begin{matrix} A_1 & & A_2 & & A_3 & & A_4 \\ & B_1 & & B_2 & & B_3 & \end{matrix} \middle| z\right) \\
&:= \sum_{n=0}^{\infty} \frac{\Gamma(A_1+n)\Gamma(A_2+n)\Gamma(A_3+n)\Gamma(A_4+n)}{\Gamma(B_1+n)\Gamma(B_2+n)\Gamma(B_3+n)} \frac{z^n}{n!}.
\end{aligned} \tag{7.16}$$

By Theorem 2.1.2 of [AAR99], the series converges absolutely when $|z| < 1$ and $A_1, A_2, A_3, A_4 \notin \mathbb{Z}_{\leqslant 0}$; and on $|z| = 1$ if $\operatorname{Re}(B_1 + B_2 + B_3 - A_1 - A_2 - A_3 - A_4) > 0$. In fact, our hypergeometric functions are of *Saalschütz* type, i.e., the sum of the argument in the second row minus that in the first row is *identically equal to* 1. They possess many functional relations and integral representations.

Proposition 7.4. *Suppose $H \in \mathcal{C}_\eta$ and $h := H^\flat$. On the region $\sigma_0 > \epsilon$, $\sigma < 4$, and $2\sigma - \sigma_0 - \epsilon > 0$, we have*

$$\begin{aligned}
\frac{1}{2\pi^{3/2}} (\mathcal{F}_\Phi H)(s_0,s) &= \int_{(\eta-1/2)} \widetilde{h}\left(s - s_1 - \frac12\right) \cdot \frac{\prod_{i=1}^{3} \Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)} \cdot \pi^{-s_1} \sec\frac{\pi}{2}(2s+s_0-s_1) \\
&\qquad \cdot {}_4\widehat{F}_3\left(\begin{matrix} s - \frac{s_0}{2} & & \frac{s_0+\alpha_1}{2} & & \frac{s_0+\alpha_2}{2} & & \frac{s_0+\alpha_3}{2} \\ & 1/2 & & \frac{s_0+s_1}{2} & & s + \frac12 + \frac{s_0-s_1}{2} & \end{matrix} \middle| 1\right) \frac{ds_1}{2\pi i} \\
&\quad - \int_{(\eta-1/2)} \widetilde{h}\left(s - s_1 - \frac12\right) \cdot \frac{\prod_{i=1}^{3} \Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)} \cdot \pi^{-s_1} \sec\frac{\pi}{2}(2s+s_0-s_1) \\
&\qquad \cdot {}_4\widehat{F}_3\left(\begin{matrix} \frac12 - s_0 + \frac{s_1}{2} & & \frac12 - s + \frac{s_1+\alpha_1}{2} & & \frac12 - s + \frac{s_1+\alpha_2}{2} & & \frac12 - s + \frac{s_1+\alpha_3}{2} \\ & \frac12 - s + s_1 & & 1 - s - \frac{s_0-s_1}{2} & & \frac32 - s - \frac{s_0-s_1}{2} & \end{matrix} \middle| 1\right) \frac{ds_1}{2\pi i}.
\end{aligned} \tag{7.17}$$

Proof. By Stirling's formula, we can shift the line of integration of the u-integral in (6.12) to $-\infty$. The residual series obtained can then be identified in terms of hypergeometric series as asserted in the present proposition.

This can also be verified by *InverseMellinTransform[]* command in mathematica. More systematically, one rewrites the u-integral in the form of a Meijer's G-function. The conversion between Meijer's G-functions and generalized hypergeometric functions is known as *Slater's theorem*, see Chapter 8 of [PBM90]. □

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Appendix A. Some Further Extensions

The focus of this article has been the archimedean aspect of the spectral moment (1.3). However, it is desirable to study the more general *'twisted moments'*, which are obtained by incorporating extra non-archimedean features. In this Appendix, the twists under consideration are the Hecke eigenvalues of $GL(2)$, see (2.25) with $a \geqslant 1$ being any integer. Not only does this displays new arithmetic structures, but this is also essential for many further applications, such as mollification, amplification, or building moments of L-functions of higher degrees, see [BBFR20, Fr20, BF21].

There will be some book-keeping needed for the dual, non-archimedean calculation of the inner product $\left\langle P^a, \ \mathbb{P}_2^3\Phi \cdot |\det *|^{\overline{s}-\frac{1}{2}}\right\rangle_{L^2(\Gamma_2\backslash GL_2(\mathbb{R}))}$. We will indicate the necessary modifications for interested readers. We begin by revisiting Section 3 & 4. Following the proof of Proposition 3.1, we have

$$\int_0^1 \Phi\left[\begin{pmatrix} 1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] \cdot e(-a\cdot u_{1,2})\, du_{1,2}$$
$$= \sum_{d|a_2} \sum_{\substack{a_0\in\mathbb{Z}\\ \gcd(a_0,d)=1}} \sum_{a_1=-\infty}^{\infty} (\widehat{\Phi})_{(a_1,a/d)}\left[\begin{pmatrix} 1 & & \\ & * & * \\ & -a_0 & d\end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] \tag{A.1}$$

with $\left(\begin{smallmatrix} * & * \\ -a_0 & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

The diagonal contribution comes from $a_0 = 0$. It is given by

$$a^{\frac{1}{2}-s}\left(\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(a,a_1)}{|a_1|^{2s}}\right)\cdot\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac{1}{2}}\cdot W_{-\alpha(\Phi)}\left(y_0,y_1\right)\frac{dy_0dy_1}{y_0y_1^2}. \tag{A.2}$$

The double integral has been computed in Proposition 3.5. On the other hand, we have

$$\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(a,a_1)}{|a_1|^{2s}} \ = \ 2\cdot L(2s,\Phi)\sum_{r|a}\frac{\mu(r)\mathcal{B}_\Phi(a/r,1)}{r^{2s}}. \tag{A.3}$$

by the Hecke relation of Φ (see Theorem 6.4.11 of [Gold]):

$$\mathcal{B}_\Phi(a,a_1) \ = \ \sum_{r|(a,a_1)}\mu(r)\mathcal{B}_\Phi(a/r,1)\mathcal{B}_\Phi(1,a_1/r).$$

The off-diagonal contribution comes from $a_0\neq 0$. In this case,

$$\begin{pmatrix} 1 & & \\ & \alpha & \beta \\ & -a_0 & d\end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix} \equiv \begin{pmatrix} 1 & & \\ & 1 & \frac{\beta}{d}-\frac{1}{da_0}\frac{(a_0y_0/d)^2}{1+(a_0y_0/d)^2} \\ & & 1\end{pmatrix}\begin{pmatrix} \frac{y_0y_1}{\sqrt{(a_0y_0)^2+d^2}} & & \\ & \frac{y_0}{(a_0y_0)^2+d^2} & \\ & & 1\end{pmatrix} \tag{A.4}$$

under the right quotient $O_3(\mathbb{R})\cdot\mathbb{R}^\times$. As a result, Definition 3.4 now generalizes to

$$\begin{aligned} OD_\Phi^{(a)}(s) \;:=\; & a^{\frac12-s}\sum_{d|a} d^{2s}\sum_{\substack{a_0\neq 0\\ \gcd(a_0,d)=1}}\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi\left(a/d,a_1\right)}{|a_0|^{2s-1}|a_1|}\cdot e\left(-\frac{a_1\overline{a_0}}{d}\right)\\ &\cdot\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac12}\cdot W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{da_0}\right|\cdot\frac{y_0}{1+y_0^2},\; y_1\sqrt{1+y_0^2}\right)\\ &\cdot e\left(\frac{a_1}{da_0}\cdot\frac{y_0^2}{1+y_0^2}\right)\frac{dy_0dy_1}{y_0y_1^2}, \end{aligned} \tag{A.5}$$

where $a_0\overline{a_0}\equiv 1 \pmod d$.

Remark A.1. *The essential difference between $OD_\Phi^{(a)}(s)$ and $OD_\Phi(s)$ lies in the arithmetic factor $e\left(-\frac{a_1\overline{a_0}}{d}\right)$.*

The version of Proposition 4.2 here is

$$OD_\Phi^{(a)}(s;\phi) \;=\; \frac14\int_{(1+\theta+2\epsilon)}\mathcal{L}_\Phi^{(a)}(s_0,s)\cdot\sum_{\delta=\pm}\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,\; s;\; \phi)\;\frac{ds_0}{2\pi i}, \tag{A.6}$$

where for $\mathrm{Re}(2s-s_0)>1$ and $\mathrm{Re}\, s_0>1+\theta$, we define

$$\mathcal{L}_\Phi^{(a)}(s_0,s) \;:=\; a^{\frac12-s}\sum_{d|a} d^{2s+s_0-1}\sideset{}{^*}\sum_{\ell\ (\bmod\ d)}\;\sum_{\substack{a_0\neq 0\\ a_0\,\equiv\,\ell\ (\bmod\ d)}}\;\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi\left(a/d,a_1\right)}{|a_0|^{2s-s_0}|a_1|^{s_0}}\cdot e\left(-\frac{a_1\overline{\ell}}{d}\right). \tag{A.7}$$

Remark A.2. *The transform $\left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,\; s;\; \phi)$ is the same as the one defined in (4.5). As a result, the work carried out in Section 5 & 7 can be applied directly to the present context.*

In other words, it remains to consider the analytic properties of $\mathcal{L}_\Phi^{(a)}(s_0,s)$. In view of this, we introduce two special Dirichlet series:

- For $\mathrm{Re}\, s>1$ and $a\neq 0,-1,-2,\ldots$, the Hurwitz ζ-function is defined as

$$\zeta(s,a) \;:=\; \sum_{n=0}^\infty (n+a)^{-s}. \tag{A.8}$$

  It admits a holomorphic continuation to $\mathbb{C}$ except at $s=1$. It has a simple pole at $s=1$ and the residue is 1. Moreover, it satisfies a functional equation

$$\zeta(1-s,a) \;=\; \frac{\Gamma(s)}{(2\pi)^s}\sum_{\pm} e^{\mp\frac{i\pi s}{2}}\sum_{n=1}^\infty\frac{e(\pm na)}{n^s} \tag{A.9}$$

  when $\mathrm{Re}\, s>1$ and $0<a\leqslant 1$. See Chapter 12 of [Ap76].
- For $\mathrm{Re}\, s>1+\theta$ and $a/c\in\mathbb{Q}$, the additively-twisted L-series of Φ by a/c is defined as

$$L\left(s,\frac{a}{c};\Phi\right) \;:=\; \sum_{n=1}^\infty\frac{\mathcal{B}_\Phi(1,n)}{n^s}\cdot e\left(\frac{na}{c}\right). \tag{A.10}$$

  It admits an entire continuation and its functional equation is precisely the *Voronoi formula of $GL(3)$*, see [GoLi06].

The L-series $\mathcal{L}_\Phi^{(a)}(s_0,s)$ can be re-written as

$$2a^{\frac12-s}\sum_{\pm}\sum_{d|a}\sum_{r|(a/d)}\frac{\mu(r)}{r^{s_0}}d^{2s_0-1}\mathcal{B}_\Phi\left(\frac{a}{dr},1\right)\sideset{}{^*}\sum_{\ell\ (\bmod\ d)}\zeta\left(2s-s_0,\frac{\ell}{d}\right)L\left(s_0;\mp\frac{r\overline{\ell}}{d};\Phi\right). \tag{A.11}$$

As a result, it admits a holomorphic continuation to $\mathbb{C}^2$ except on $2s-s_0=1$. When $\mathrm{Re}\, s>1+\theta$, the residue of $\mathcal{L}_\Phi^{(a)}(s_0,s)$ at $s_0=2s-1$ is equal to

$$-2a^{\frac12-s}\sum_{d|a} d^{4s-3}\sum_{\pm}\sum_{a_1=1}^\infty\frac{\mathcal{B}_\Phi\left(a/d,a_1\right)}{a_1^{2s-1}}\cdot S\left(0,\mp a_1;d\right), \tag{A.12}$$

where $S(0, \mp a_1; d) := \sum_{\ell \ (\bmod\ d)}^{*} e\left(\mp a_1 \overline{\ell}/d\right)$ is the Ramanujan sum. Now, the argument of Section 6 carries over to the present case.

Remark A.3. *In conjunction with the Dirichlet series method, the moment identity developed in this Appendix can serve as a different starting point for studying the simultaneous moment of $GL(3) \times GL(2)$ and $GL(2)$ L-functions than that of Li* [Li09] *(a companion paper of* [Li11]*). This approach should provide a better understanding of the sequence of summation formulae and transforms done in* [Li09]*. This is of separate interest and we will leave it to a future paper.*

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