Direct numerical simulation tests of eddy viscosity in two dimensions

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Two-parametric eddy viscosity (TPEV) and other spectral characteristics of two-dimensional (2-D) turbulence in the energy transfer subrange are calculated from direct numerical simulation (DNS) with 512^2 resolution. The DNS-based TPEV is compared with those calculated from the test field model (TFM) and from the renormalization group (RG) theory. Very good agreement between all three results is observed.

Two-dimensional (2-D) incompressible turbulent flows are described by the vorticity equation:

$$\frac{\partial \zeta}{\partial t}+\frac{\partial(\nabla^{-2}\zeta,\ \zeta)}{\partial(x,\ y)}=\nu_0\nabla^2\zeta, \tag{1}$$

where ζ is fluid vorticity and ν_0 is molecular viscosity. It is well known that the existence of inviscid invariants $\int d^2x\,\zeta^{2n}$ of (1) results in the flux of energy towards the largest spatial scales. The presence of this inverse cascade complicates the large-scale description of 2-D flows and requires refinement of the classical hydrodynamic notion of "eddy viscosity." The concept of eddy viscosity is well defined for three-dimensional (3-D) turbulent flows, where energy cascades towards the smallest flow scales where it is dissipated. To achieve an adequate coarse-grained description of 3-D flow, one can introduce increased "effective" viscosity which accounts for the unresolved dissipation.

In 2-D flows, the inverse flux of energy at large scales and enstrophy dissipation at small scales make the eddy viscosity concept more subtle. It was argued by Kraichnan[1] that a 2-D eddy viscosity should include two parameters: a cutoff wave number k_c (which essentially determines the size of the coarse grain), and the wave number of a given mode, k. The two-parameter eddy viscosity (TPEV), denoted by $\nu(k|k_c)$, describes the energy exchange between a given resolved vorticity mode with the wave number k and all subgrid, or unresolved, modes with $k>k_c$; it provides correct account for the energy and enstrophy fluxes between resolved and unresolved scales. The TPEV is derived from the evolution equation for the spectral enstrophy density $\Omega(k,t) \equiv \pi k\langle\zeta(\mathbf{k},t)\zeta(-\mathbf{k},t)\rangle$, where $\langle\ldots\rangle$ denotes averaging over thin circular shells:

$$(\partial_t+2\nu k^2)\Omega(k,t)=T_\Omega(k,t). \tag{2}$$

Here, the enstrophy transfer function $T_\Omega(k,t)$ is given by

$$T_\Omega(k,t)=2\pi k\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}\frac{\mathbf{p}\times\mathbf{q}}{p^2}\times\langle\zeta(\mathbf{p},t)\zeta(\mathbf{q},t)\zeta(-\mathbf{k},t)\rangle[d\mathbf{p}\,d\mathbf{q}/(2\pi)^2]\ . \tag{3}$$

Assuming that the system is in statistical steady state and extending integration in (3) only over all such triangles $(\mathbf{k},\mathbf{p},\mathbf{q})$ that p and/or q are greater than k_c, one defines the two-parametric transfer $T_\Omega(k|k_c)$ and TPEV:[1]

$$\nu(k|k_c)=-T_\Omega(k|k_c)/2k^2\Omega(k)\ . \tag{4}$$

In a wide class of quasinormal approximations[2] the transfer $T_\Omega(k|k_c)$ in two dimensions is given by

$$T_\Omega(k|k_c)=\frac{2k}{\pi}\int\int_\Delta\Theta_{-k,p,q}(p^2-q^2)\sin\alpha \times\left[\frac{p^2-q^2}{p^3\,q^3}\,\Omega(p)\Omega(q)-\frac{k^2-q^2}{k^3q^3}\,\Omega(q)\Omega(k)+\frac{k^2-p^2}{k^3p^3}\,\Omega(p)\Omega(k)\right]dp\,dq, \tag{5}$$

where $\Theta_{-k,\,p,\,q}$ is the triad relaxation time. Here, the angle α is formed by the vectors $\mathbf{p}$ and $\mathbf{q}$, and $\int\int_\Delta$ denotes integration over the area defined above (4).

The main difference between various spectral closure models is in specification of $\Theta_{-k,p,q}$. In Ref. 1, $T_\Omega(k|k_c)$ was evaluated using TFM. It was found that TPEV is a sign-changing function of the form $\nu(k|k_c)=|\nu(0|k_c)|N(k/k_c)$, with $\nu(0|k_c)<0$, $N(0)=-1$, and $N(1)\approx 2.1$. The derivation of $\Theta_{-k,p,q}$ using the RG theory was given in Ref. 3 and adapted for 2-D isotropic and anisotropic turbulence in Refs. 4 and 5, respectively. In the present work, we compare $\nu(k|k_c)$ for the inverse energy cascade regime calculated from DNS data with those predicted by TFM and the RG theory. Let us mention that for the enstrophy transfer subrange of 2-D turbulence, eddy viscosity was calculated by Maltrud and Vallis.[6]

We solve Eq. (1) numerically in a periodic box $2\pi\times 2\pi$ using 512^2 Fourier modes, utilizing a Fourier–Galerkin pseudospectral spatial approximation with implicit Adams-type second-order stiffly stable time-stepping scheme.[7] To increase the effective inertial range, mode-selective

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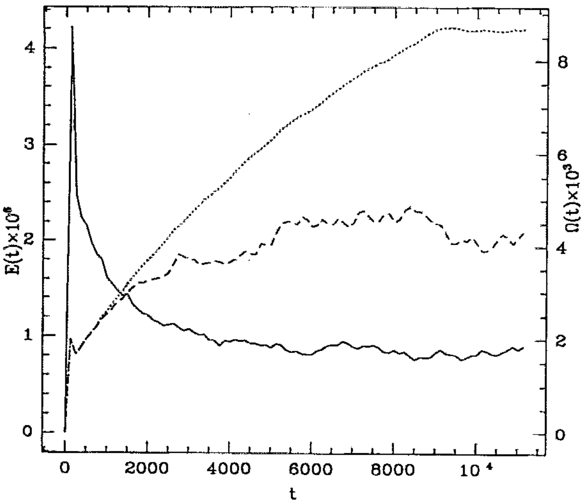

FIG. 1. Evolution of the total energy $E(k)$ (dotted line) and enstrophy $\Omega(k)$ (solid line) towards the steady state. Dashed line denotes the total energy with the first 6 modes excluded.

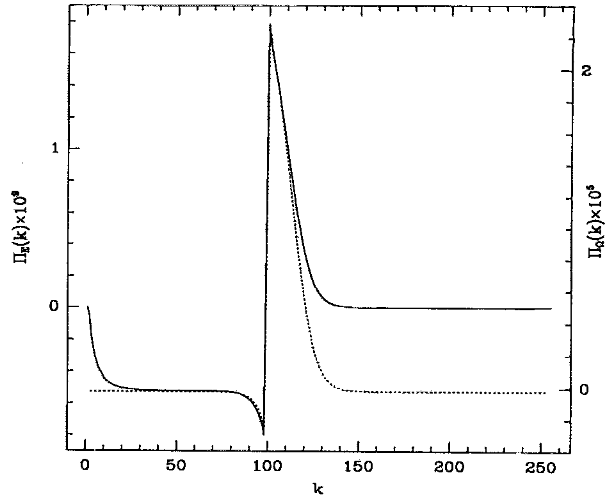

FIG. 3. The energy flux $\Pi_E(k)$ (solid line) and the enstrophy flux $\Pi_\Omega(k)$ (dotted line).

hyperviscosity[8] $\nu(k)=\nu_L(k)+\nu_S(k)=A_L k^{-10}+A_S k^{14}$ has been introduced in the vorticity equation (1) instead of the molecular viscosity.

The white noise, high wave number forcing was introduced in three consecutive wave numbers, k_f-1, k_f, and k_f+1, where $k_f=98$.

In Figs. 1(a) and 1(b) we plot the total energy $E_{\text{tot}}(t)=\int_0^{+\infty}k^{-2}\Omega(k,t)dk$ and enstrophy $\Omega_{\text{tot}}(t)=\int_0^{+\infty}\Omega(k,t)dk$. One can see that the energy grows with time and eventually tends to reach a steady state. However, the drift towards the energy steady state is significantly slower than towards that of the enstrophy. Defining the RMS velocity as $V^2_{\text{RMS}}=\Sigma_{\mathbf{k}}|\mathbf{u}(\mathbf{k})|^2$ and the characteristic turnover time of the largest eddies as $\tau_{tu}=2\pi/V_{\text{RMS}}$, we infer that a steady state for the total enstrophy was achieved after about $1.2\tau_{tu}$, while about $5\tau_{tu}$ were required to attain a steady state for the total energy. Note however that all the modes with $k>5$ have reached the steady state after $t\approx 2\tau_{tu}$, and only the largest modes were still developing. The results presented below pertain to the integration time $t\leqslant 10^4$, before the energy saturates at low wave numbers.

In Fig. 2 we plot the time-averaged energy spectrum after about $5\tau_{tu}$. The inertial range $E\propto k^{-x}$ extends over more than a decade in wave number space. Mean square line-fitting over the interval $k\in(12, 50)$ gives the scaling exponent close to the Kolmogorov value of $\frac{5}{3}$. Note that good agreement with the Kolmogorov scaling in the energy subrange has been reported recently in Ref. 6 for 256^2 simulations and in Ref. 9 for very high resolution simulations with 2048^2 Fourier modes. In Fig. 2 we also plot a compensated energy spectrum, $k^{5/3}\epsilon^{-2/3}E(k)$, where ϵ is the energy transfer rate. The value of the Kolmogorov constant calcu-

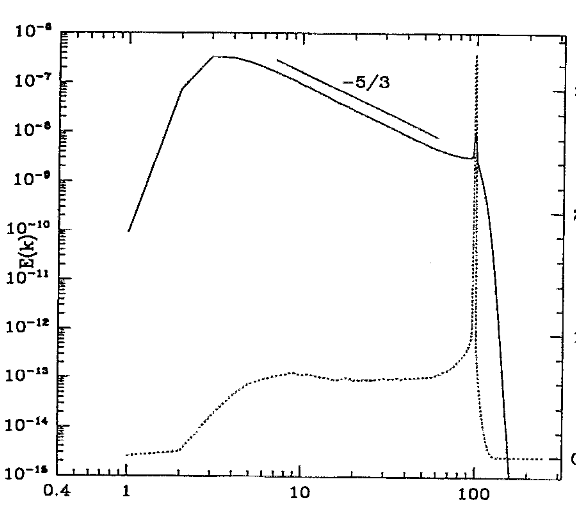

FIG. 2. Energy spectrum $E(k)$ (solid line) and compensated energy spectrum $E(k)k^{5/3}\epsilon^{-2/3}$ (dotted line).

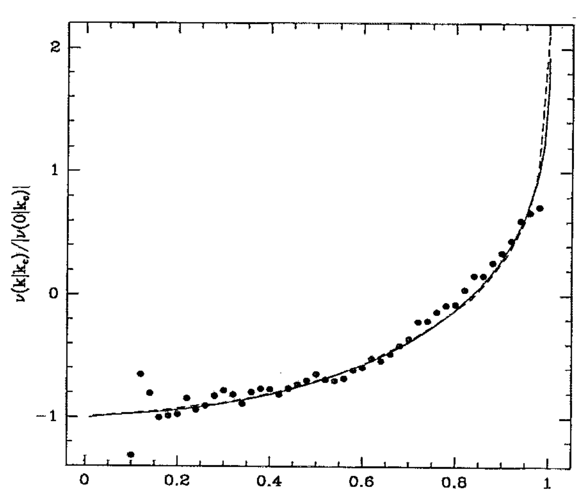

FIG. 4. Normalized two-parametric eddy viscosity from DNS (dots), from TFM (dashed line), and from RG (solid line).

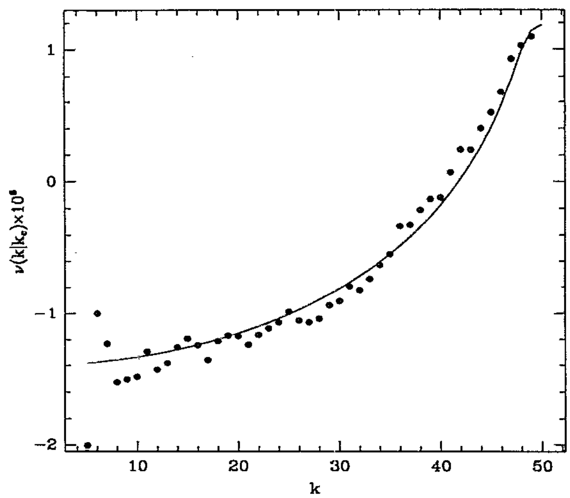

FIG. 5. Actual two-parametric eddy viscosity from DNS (dots) and from RG (solid line). In RG calculations, the energy spectrum for $k<5$ was corrected in accordance with the DNS results (Fig. 2).

lated from this data is about $C_k=6.2$, in reasonable agreement with 5.8, calculated from DNS in Ref. 10 using the 256^2 resolution and 6.69, obtained analytically in Ref. 11 on the basis of TFM.

In Figs. 3(a) and 3(b) we plot the k-dependent energy and enstrophy flux functions, defined as $\Pi_E(k)=\int_0^k T_\Omega(n)n^{-2}dn$ and $\Pi_\Omega(k)=\int_0^k T_\Omega(n)dn$, respectively. As expected, an inverse energy cascade with constant transfer rate ϵ develops for $k<k_f=98$. The resolution employed in this study was insufficient to detect a well defined enstrophy transfer range. The flux of enstrophy in the energy subrange, $k<k_f$, is zero. Figures 3(a) and 3(b) indicate that the numerical scheme used conserves both total energy and enstrophy.

We have calculated k and k_c-dependent enstrophy transfer function $T_\Omega(k|k_c)$ in (4) by computing the third-order vorticity cumulants in (3) extending the integration only over those $\mathbf{p}$ and $\mathbf{q}$ that either $p\geqslant k_c$ or $q\geqslant k_c$. We have set $k_c=50$, well inside the energy inertial subrange.

The DNS-inferred normalized TPEV [viz., the function $N(k/k_c)=\nu(k|k_c)/|\nu(0|k_c)|$] is plotted in Fig. 4, along with the TFM-based[1] and RG-based[4] analytical predictions. The agreement between the DNS-based results and the TFM and RG theories is very good over the entire energy transfer range, up to the wave numbers close to k_c, where the DNS data saturates, while TFM and RG curves exhibit sharp cusp. The physics leading to this cusp is as follows. As closer k approaches k_c, as more elongated triads with either p or $q\ll k_c$ become involved in the energy exchange between the mode k and the subgrid scale modes. The contribution from these triads results in the cusp behavior of the theoretical TPEV. However, in finite box DNS with large-scale energy removal, the energy of small wave number modes is reduced (see Fig. 2), which implies that instead of the sharp growth, the TPEV should saturate at $k\rightarrow k_c$. To illustrate and quantify this explanation, we recalculated the RG-based TPEV with the enstrophy spectrum in (5) corrected at $k\leqslant5$ according to Fig. 2. In Fig. 5, we compare the DNS- and RG-based TPEV in their actual values, whereas the RG calculations were based upon the value of ϵ found from DNS. The agreement between the two is very good. Also, we have calculated TPEV for k_c=35, 45, and 55 and found that the DNS-inferred TPEV scales with $k_c^{-4/3}$, in full agreement with the Kolmogorov and Richardson laws. At all values of k_c tested an equally good agreement between DNS data and RG predictions was observed.

The good agreement demonstrated in Figs. 4 and 5 provides an indirect validation of TFM and RG results for isotropic 2-D turbulence in the energy transfer subrange.

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