Direct numerical simulation tests of eddy viscosity in two dimensions

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Two-parametric eddy viscosity (TPEV) and other spectral characteristics of two-dimensional (2-D) turbulence in the energy transfer subrange are calculated from direct numerical simulation (DNS) with 512$^2$ resolution. The DNS-based TPEV is compared with those calculated from the test field model (TFM) and from the renormalization group (RG) theory. Very good agreement between all three results is observed.

Two-dimensional (2-D) incompressible turbulent flows are described by the vorticity equation:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (v \zeta)}{\partial x} = v_0 \nabla^2 \zeta,$$

where $\zeta$ is fluid vorticity and $v_0$ is molecular viscosity. It is well known that the existence of inviscid invariants $\int d^2 x \zeta^2$ of (1) results in the flux of energy towards the largest spatial scales. The presence of this inverse cascade complicates the large-scale description of 2-D flows and requires refinement of the classical hydrodynamic notion of “eddy viscosity.” The concept of eddy viscosity is well defined for three-dimensional (3-D) turbulent flows, where energy cascades towards the smallest flow scales where it is dissipated. To achieve an adequate coarse-grained description of 3-D flow, one can introduce increased “effective” viscosity which accounts for the unresolved dissipation.

In 2-D flows, the inverse flux of energy at large scales and enstrophy dissipation at small scales make the eddy viscosity concept more subtle. It was argued by Kraichnan that a 2-D eddy viscosity should include two parameters: a cutoff wave number $k_c$ (which essentially determines the size of the coarse grain), and the wave number of a given mode, $k$. The two-parameter eddy viscosity (TPEV), denoted by $v(k|k_c)$, describes the energy exchange between a given resolved vorticity mode with the wave number $k$ and all subgrid, or unresolved, modes with $k > k_c$; it provides correct account for the energy and enstrophy fluxes between resolved and unresolved scales. The TPEV is derived from the evolution equation for the spectral enstrophy density $\Omega(k,t) = \pi k^2 \zeta(k,t) \zeta(-k,t)$, where $\langle \ldots \rangle$ denotes averaging over thin circular shells:

$$\left( \partial_t + 2v_k^2 \right) \Omega(k,t) = T_\Omega(k,t).$$

(2)

Here, the enstrophy transfer function $T_\Omega(k,t)$ is given by

$$T_\Omega(k,t) = \pi k^2 \int_{p+q=k} \frac{p^2 q^2}{p^2-q^2} \langle \zeta(p,t) \zeta(q,t) \zeta(-k,t) \rangle dp dq/(2\pi)^2.$$

(3)

Assuming that the system is in statistical steady state and extending integration in (3) only over all such triangles $(k, p, q)$ that $p$ and/or $q$ are greater than $k_c$, one defines the two-parametric transfer $T_\Omega(k|k_c)$ and TPEV:

$$v(k|k_c) = -\frac{T_\Omega(k|k_c)}{2k^2 \Omega(k)}.$$

(4)

In a wide class of quasinormal approximations the transfer $T_\Omega(k|k_c)$ in two dimensions is given by

$$T_\Omega(k|k_c) = \frac{2k}{\pi} \int_{k_c}^{\infty} \Theta_{-k,p,q}(p^2-q^2) \sin \alpha \times \left[ \frac{p^2-q^2}{p^3} \Omega(p) \Omega(q) - \frac{k^2-q^2}{k^3} \Omega(q) \Omega(k) \right] dp dq,$$

(5)

where $\Theta_{-k,p,q}$ is the triad relaxation time. Here, the angle $\alpha$ is formed by the vectors $p$ and $q$, and $\int_{k_c}^{\infty} \Theta_{-k,p,q}$ denotes integration over the area defined above (4).

The main difference between various spectral closure models is in specification of $\Theta_{-k,p,q}$. In Ref. 1, $T_\Omega(k|k_c)$ was evaluated using TFM. It was found that TPEV is a sign-changing function of the form $v(k|k_c) = |v(0|k_c)| N(k/k_c)$, with $v(0|k_c) < 0$, $N(0) = -1$, and $N(1) \approx 2.1$. The derivation of $\Theta_{-k,p,q}$ using the RG theory was given in Ref. 3 and adapted for 2-D isotropic and anisotropic turbulence in Refs. 4 and 5, respectively. In the present work, we compare $v(k|k_c)$ for the inverse energy cascade regime calculated from DNS data with those predicted by TFM and the RG theory. Let us mention that for the enstrophy transfer subrange of 2-D turbulence, eddy viscosity was calculated by Maltrud and Vallis.

We solve Eq. (1) numerically in a periodic box $2\pi \times 2\pi$ using 512$^2$ Fourier modes, utilizing a Fourier–Galerkin pseudospectral spatial approximation with implicit Adams-type second-order stiffly stable time-stepping scheme. To increase the effective inertial range, mode-selective
hyperviscosity\(^*\) \( \nu(k) = \nu_L(k) + \nu_s(k) - \Lambda_1 k^{-10} + \Lambda_2 k^{14} \) has been introduced in the vorticity equation (1) instead of the molecular viscosity.

The white noise, high wave number forcing was introduced in three consecutive wave numbers, \( k_f - 1 \), \( k_f \), and \( k_f + 1 \), where \( k_f = 98 \).

In Figs. 1(a) and 1(b) we plot the total energy \( E_{\text{tot}}(t) = \int_0^T \langle k \rangle \Omega(k,t) dk \) and enstrophy \( \Omega_{\text{tot}}(t) = \int_0^T \Omega(k,t) dk \). One can see that the energy grows with time and eventually tends to reach a steady state. However, the drift towards the energy steady state is significantly slower than towards that of the enstrophy. Defining the RMS velocity as \( V_{\text{RMS}}^2 = \sum_k \langle \mathbf{u}(k) \rangle^2 \) and the characteristic turnover time of the largest eddies as \( \tau_{uu} = 2 \pi / V_{\text{RMS}} \), we infer that a steady state for the total enstrophy was achieved after about \( 5 \tau_{uu} \), while about \( 1.2 \tau_{uu} \) were required to attain a steady state for the total energy. Note however that all the modes with \( k > 5 \) have reached the steady state after \( t = 2 \tau_{uu} \), and only the largest modes were still developing. The results presented below pertain to the integration time \( t \leq 10^4 \) before the energy saturates at low wave numbers.

In Fig. 2 we plot the time-averaged energy spectrum after about \( 5 \tau_{uu} \). The inertial range \( E \propto k^{-5/3} \) extends over more than a decade in wave number space. Mean square line-fitting over the interval \( k \in (12, 50) \) gives the scaling exponent close to the Kolmogorov value of \( 5/3 \). Note that good agreement with the Kolmogorov scaling in the energy subrange has been reported recently in Ref. 6 for 256\(^2 \) simulations and in Ref. 9 for very high resolution simulations with 2048\(^2 \) Fourier modes. In Fig. 2 we also plot a compensated energy spectrum, \( k^{5/3} e^{-12s} E(k) \), where \( s \) is the energy transfer rate. The value of the Kolmogorov constant calcu-
The physics leading to this cusp is as follows. As closer $k$ approaches $k_c$, as more elongated triads with either $p$ or $q$ become involved in the energy exchange between the mode $k$ and the subgrid scale modes. The contribution from these triads results in the cusp behavior of the theoretical TPEV. However, in finite box DNS with large-scale energy removal, the energy of small-wavenumber modes is reduced (see Fig. 2), which implies that instead of the sharp growth, the TPEV should saturate at $k = k_c$. To illustrate and quantify this explanation, we recalculated the RG-based TPEV with the enstrophy spectrum in (5) corrected at $k = 5$ according to Fig. 2. In Fig. 5, we compare the DNS- and RG-based TPEV in their actual values, whereas the RG calculations were based upon the value of $\epsilon$ found from DNS. The agreement between the two is very good. Also, we have calculated TPEV for $k_c = 35, 45$, and $55$ and found that the DNS-inferred TPEV scales with $k_c^{-4/3}$, in full agreement with the Kolmogorov and Richardson laws. At all values of $k_c$ tested an equally good agreement between DNS data and RG predictions was observed.

The good agreement demonstrated in Figs. 4 and 5 provides an indirect validation of TFM and RG results for isotropic 2-D turbulence in the energy transfer subrange.

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