Etale fundamental groups

Now we are in a position to define the (no longer mysterious) notion of etale fundamental groups using all finite etale coverings. In the topological setting, the universal covering $\tilde{X}$ "covers" all other coverings $h : Y \to X$. Moreover, for a fixed $x \in X$, giving a cover $\tilde{X} \to Y$ boils down to a choice of a point in $h^{-1}(x)$. Precisely speaking, we have

Proposition 1. Let X be a topological space and $\tilde{h} : \tilde{X} \to X$ be a universal covering (unique up to isomorphism over X). Fix a point $x \in X$, then for any covering $h : Y \to X$, there is a bijection

$$\mathrm{Hom}_X(\tilde{X}, Y) \cong h^{-1}(x) = \mathrm{Hom}_X(x, Y).$$

Namely, giving a map from $\tilde{X}$ to Y is the same as choosing a preimage of x on Y.

In other words, the functor $F_x : \mathrm{Hom}_X(x, -)$ is represented by the universal covering $\tilde{X}$. This universal property can be used similarly to define the etale universal covering in the algebraic setting.

Theorem 1. Let $X = \mathrm{Spec}\, A$ and $x = \mathrm{Spec}\, \Omega \hookrightarrow X$ be a geometric point (i.e., Ω is a separably algebraically closed field). Define the functor

$$F_x : \{\text{finite etale coverings of } X\} \to \mathbf{Sets}, \quad (h : Y \to X) \mapsto h^{-1}(x) := \mathrm{Hom}_X(x, Y).$$

Then the functor F_x is pro-represented, i.e., there exists an inverse system $(X_i)_{i \in I}$ of finite etale coverings such that

$$\varinjlim_{i \in I} \mathrm{Hom}_X(X_i, Y) \cong \mathrm{Hom}_X(x, Y).$$

Remark. As discussed before, the functor F_x is not genuinely representable for the reason that the universal covering usually does not exist in the algebraic category.

Definition 1. We define $\tilde{X} := \varprojlim_{i \in I} X_i$ and the **etale fundamental group**

$$\pi_1(X, x) = \mathrm{Gal}(\tilde{X}/X) := \varprojlim_{i \in I} \mathrm{Gal}(X_i/X).$$

These are all very nice except that we have not computed a single example of etale fundamental groups. Now we will introduce more algebraic number theory, get into the computation and make more concrete sense of the inverse limit constructions if you are not entirely comfortable with them.

Example 1. Let us compute the etale fundamental group of $X = \mathrm{Spec}\, \mathbb{F}_q$ (it is nontrivial even though geometrically $\mathrm{Spec}\, \mathbb{F}_q$ is a single point!). Giving a geometric point x of X is the same as fixing an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$. By definition, a finite etale covering of X is a finite disjoint union of $Y = \coprod_{j=1}^{m} \mathrm{Spec}\, k_j$, where k_j's are finite extensions of $\mathbb{F}_p$. So

$$\mathrm{Hom}_X(x, Y) = \coprod_j \mathrm{Hom}_X(\mathrm{Spec}\, \overline{\mathbb{F}_q}, \mathrm{Spec}\, k_j).$$

Notice that for each j,

$$\mathrm{Hom}_X(\mathrm{Spec}\, \overline{\mathbb{F}_q}, \mathrm{Spec}\, k_j) = \varinjlim_{n} \mathrm{Hom}_X(\mathrm{Spec}\, \mathbb{F}_{q^n}, \mathrm{Spec}\, k_j).$$

So the universal covering is

$$\tilde{X} = \varprojlim_n \operatorname{Spec} \mathbb{F}_{q^n} = \operatorname{Spec} \overline{\mathbb{F}_q}.$$

Recall that $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ generated by the Frobenius automorphism $x \mapsto x^q$, we know that the etale fundamental group is

$$\pi_1(\operatorname{Spec} \mathbb{F}_q, x) = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \varprojlim_n \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}.$$

Concretely,

$$\varprojlim_n \mathbb{Z}/n\mathbb{Z} = \{(a_n)_{n\geq 1} : \phi_{n,m}(a_n) = a_m, \text{ for } m \mid n\},$$

where $\phi_{n,m} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the natural quotient map for $m \mid n$. Therefore $\hat{\mathbb{Z}}$ is a gigantic group compared to $\mathbb{Z}$ (e.g., it is uncountable and contains infinitely many copies of $\mathbb{Z}$). The natural map $\mathbb{Z} \mapsto \hat{\mathbb{Z}}$ carries $1 \in \mathbb{Z}$ to the Frobenius automorphism $(\sigma : x \mapsto x^q) \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

Remark. $\hat{\mathbb{Z}}$ is an example of a **profinite group**, i.e., an inverse limit of finite groups. By definition, every etale fundamental group is a profinite group. We endow a profinite group the topology induced from the product topology. So every profinite group is compact by Tychonoff's theorem. The following fact matches our intuition about "completion".

Exercise. Show that the natural map $\mathbb{Z} \to \hat{\mathbb{Z}}$ is injective and has dense image.

Remark. In general, for any group G, its normal subgroups of finite index naturally form an inverse system (N_i) and the profinite group

$$\hat{G} := \varprojlim_i G/N_i$$

is called the **profinite completion** of G. For example, the profinite completion of $\mathbb{Z}$ is $\hat{\mathbb{Z}}$. Since the group $\mathbb{Z}$ is not profinite (it is not compact), it can never be an etale fundamental group. Therefore as the arithmetic counterpart to $\pi_1(S^1) = \mathbb{Z}$, the profinite completion $\hat{\mathbb{Z}}$ may be the best possible candidate we can hope for. Luckily, we already know that $\pi_1(\operatorname{Spec} \mathbb{F}_p) = \hat{\mathbb{Z}}$.

Remark. A fundamental comparison theorem asserts that for any varieties over $\mathbb{C}$, the profinite completion of its fundamental group (under the complex analytic topology) is the same as its etale fundamental group.

Example 2. In general, for a field F, the etale fundamental group of $\operatorname{Spec} F$ is the **absolute Galois group** $\mathrm{Gal}(F^s/F)$, where F^s is the separable closure of F (by the same argument). The absolute Galois group of $\mathbb{Q}$ contains monstrous information about all number fields and thus is of great interest to study in number theory. People are far from understanding the whole absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as of today.

Next time we will use beautiful facts from algebraic number theory to compute the etale fundamental group of $\operatorname{Spec} \mathbb{Z}$ and use our computation to keep exploring the analogy between knots and primes.