Towards automorphy lifting for semi-stable representations

§1 Automorphy lifting conjectures and classicality conjectures

let $F^+/\mathbb{Q}$ tot real, F/F^+ CM, $p>2$ prime

- G/F^+ unitary group
  - compact at
  - split at places dividing p

  (+ technical assumptions)

- fix $K^p \subseteq G(\mathbb{A}_{F^+}^{\infty,p})$ compact open (satisfying some tech. assumptions)

  $S=$ set of primes where K^p not hyperspecial

- $L/\mathbb{Q}_p$ fin $\supseteq \mathcal{O}_L$, $k=$ res. field, unif ϖ

  $\bar\rho : \mathcal{G}_{F,S} = \mathrm{Gal}(F_S/F) \longrightarrow GL_n(k)$ polarizable (i.e. $\check\rho^c \cong \rho\otimes\varepsilon^{1-n}$)

  abs. irred $\qquad\qquad \varepsilon=$ cyclotomic char

assume - $\bar\rho$ is the mod ϖ reduction of the Galois rep'n

assoc. to an automorphic rep'n of $G(\mathbb{A}_{F^+})$

(of tame level K^p)

(+ technical assumptions on $\bar\rho$)

Conj. 1 (Automorphy lifting)

Let $\rho : \mathcal{G}_{F,S} \longrightarrow GL_n(L)$ be a polarizable lift of $\bar\rho$

s.th $\rho_v = \rho|_{\mathrm{Gal}(\bar F_v/F_v)}$ is $\begin{cases}\text{crystalline}\\ \text{semi-stable}\end{cases}$ with regular HT weights for all $v\mid p$

then ρ is assoc to an automorphic form $f\in S(K^pK_p, V_\lambda)$ (space of autom forms level K^pK_p, weight λ)

($V_\lambda=$ alg rep'n of highest wt $\lambda \longleftrightarrow$ HT wts) with $K_p = \begin{cases} \text{max compact} & , \text{ all } \rho_v \text{ crystalline} \\ \prod \text{Iwahoris} & \rho_v \text{ semi-stable} \end{cases}$

<u>Conj. 2</u> (Classicality)

Assume $\rho = \rho_f : \mathcal{G}_{F,S} \longrightarrow GL_n(L)$ is a lift of $\bar{\rho}$ that is assoc. to a p-adic automorphic form f, overconvergent, of fin slope

If ρ is semi-stable w. regular Hodge-Tate weights at places dividing p ($\Rightarrow$ f has algebraic wt.) + f has dominant alg wt.

$\Rightarrow$ f is classical.

<u>Aim of this talk:</u>

<u>Thm</u> A (H.-Weirich)

Assume $\rho = \rho_f : \mathcal{G}_{F,S} \longrightarrow GL_n(L)$ is a lift of $\bar{\rho}$ that is assoc. to a p-adic automorphic form f, overconvergent, of fin slope

ρ semi-stable w. pw. distinct Frob. eigenvalues + reg HT weights

$\Rightarrow$ $\exists$ classical automorphic form f' s.th. $\rho = \rho_{f'}$.

use this to show:

<u>Thm</u> B (Conj 1 for crystalline repn's) $\Rightarrow$ (Conj 2 for semi-stable repn's)

<u>Rem</u>: - past years: lot of progress on Conj 1 for crystalline repn's; in the semi-stab case: not much known beyond 2-dim case and ordinary case.

- Thm A proved in jt work with Breuil + Schraen for ρ s.th. Frob evals φ_i on $WD(\rho_v)$ satisfy $\frac{\varphi_i}{\varphi_j} \notin \{1, q_v\}$ ($\Rightarrow \rho_v$ crystalline)

- idea to prove Thm B:

show that $\left(\begin{array}{c}\text{Conj. for crystalline} \\ \text{rep'n's}\end{array}\right) \Rightarrow$ every semi-stable ρ is assoc. to a p-adic autom form of fin. slope.

+ use Thm A.

§2 Taylor-Wiles' construction

assume <u>for simplicity</u>: $F^+ = \mathbb{Q}$ + ignore bad primes away from p

let $\bar{r} = \bar{\rho}|_{G_{\mathbb{Q}_p}}$, $R_{\bar{r}}$ universal lifting ring of $\bar{r}$

$\mathfrak{X}_{\bar{\rho}}$ = rigid analytic generic fiber

$\cup$

$\mathfrak{X}_{\bar{\rho}}^{\underline{k}-st}$ closed subspace of semi-stable rep'n's of HT wt $\underline{k}$

$\cup$

$\mathfrak{X}_{\bar{\rho}}^{\underline{k}-cr}$ —— " —— crystalline —"—

TW, Kisin:

(*) If $\rho_p \in \mathfrak{X}_{\bar{r}}^{\underline{k}-st}$ lies on the same irred. compo as $(\rho')_p$ for ρ' assoc. to an automorphic rep'n $\Rightarrow \rho$ assoc. to an autom. rep'n

("ρ_p lies on an automorphic component")

$\to$ variant for overconv. p-adic autom forms of fin slope (BHS)

$\mathcal{T}$ = space of cont. chars of $\mathbb{Q}_p^\times$

$\mathfrak{X}_{\bar{r}} \times \mathcal{T}^n \supseteq X_{tri}$ = Zariski closure of $\{ (r, \delta_1, \dots, \delta_n) \mid r$ crystalline wt $k_1', \dots, k_n'$, Frobenius $\varphi_1, \dots, \varphi_n$, $\delta_i = \mathrm{unr}(\varphi_i) \cdot z^{k_i'} \}$

"space of trianguline rep'n's"

Then: $\quad \rho$ is assoc. to an o.c. autom form of fin slope
with system of Hecke evals at p given by $\delta \in \mathcal{T}^n$
$\Leftrightarrow \quad (\rho_p, \delta)$ lies on an automorphic component of X_{tri}
(i.e. on same irred compo as (ρ'_p, δ') for $\rho' = \rho_{f'}$,
f' p-adic autom form, ...)

§3 On the geometry of X_{tri}

Thm A' Let $x = (r, \delta) \in X_{tri}$,
$r : \mathcal{G}_{\mathbb{Q}_p} \longrightarrow GL_n(L)$ semi stable with reg HT wts
+ pw. distinct Frob evals on $D_{st}(r)$
then X_{tri} is normal + Cohen-Macaulay at x

Thm B' Let r be semi stable w. reg HT wts $k_1 > \dots > k_n$
$\varphi_1, \dots, \varphi_n$ ordering of φ-evals on $D_{st}(r)$ corresp. to a (φ, N)-stable flag
$\delta_i := \mathrm{unr}(\varphi_i) z^{k_i} : \mathbb{Q}_p^\times \longrightarrow L^\times$
$\Rightarrow (r, \delta_1, \dots, \delta_n) \in X_{tri}$.

- Thm A' + TW construction (+ some locally analytic rep'n theory) $\Rightarrow$ Thm A
- ad Thm B:
  $\left(\text{Conj 1 for crystalline rep'ns}\right) \Rightarrow$ all components of X_{tri} are automorphic
  then Thm B' + Thm A $\Rightarrow$ Thm B
- on a technical level note that:
  Thm A' implies: in $(*)$ can replace "irred. compo" by "connected compo"

§4 Sketch of proof of Thm A', B'

(R. Liu; Kedlaya-Pottharst-Xiao) $\Rightarrow$

$(r, \delta_1, \ldots, \delta_n) \in X_{tri}$ $\Rightarrow$ $D^{\dagger}_{rig}(r)$ assoc. (φ, Γ)-module over Robba ring $\mathcal{R}$
is a successive extension of rank 1 objects
gr_i s.th. $gr_i[\frac{1}{t}] = \mathcal{R}(\delta_i)[\frac{1}{t}]$

in order to control X_{tri}: need to control families of extensions

$n=2$: given $\tilde{\delta}_1, \tilde{\delta}_2$ univ char's $/ \mathcal{T}^2$

look at $\mathcal{M} = \mathrm{Ext}^1_{\varphi,\Gamma}(\mathcal{R}(\tilde{\delta}_2), \mathcal{R}(\tilde{\delta}_1))$ coh sheaf $/ \mathcal{T}^2$

main problem: if $\exists$ $D \in \mathrm{Ext}^1(\mathcal{R}(\delta_2), \mathcal{R}(\delta_1))$ semi-stable non-crystalline

$\Rightarrow$ $\mathcal{M}$ not locally free in nbhd of $(\delta_1, \delta_2) \in \mathcal{T}^2$
(reason: $\mathrm{Ext}^2 \neq 0$)

<u>Idea</u>: given $(r, \delta_1, \ldots, \delta_n)$ with semi-stable r HT wt $k_1 > \ldots > k_n$
$\delta_i = \mathrm{unr}(\varphi_i)\, z^{k_i}$,

let $\delta_i' = \mathrm{unr}(\varphi_i)$, $\mathcal{U} \subseteq \mathcal{T}^n$ small nbhd of $(\delta_1', \ldots, \delta_n')$
(s.th. all $\mathrm{Ext}^2(\mathcal{R}(\tilde{\delta}_i), \mathcal{R}(\tilde{\delta}_j))$ vanish on $\mathcal{U}$)

construct - $X' \longrightarrow \mathcal{U}$ vb param successive extensions D_x of $\mathcal{R}(\tilde{\delta}_i)$

- $X \longrightarrow X'$ space parametrizing Γ-stable lattices in $D_x[\frac{1}{t}]$ w. elementary divisors $k_1, \ldots, k_n$

in order to prove Thm A' + Thm B' prove that

- $x = (D^{\dagger}_{rig}(r), \delta_1, \dots, \delta_n) \in X$

- X is normal and Cohen-Macaulay at x

  ($\rightsquigarrow$ study a "linear algebra model" of $\mathrm{Spf}\, \hat{\mathcal{O}}_{X,x}$ $\leftarrow$ irred compo of an explicit moduli space)

- miracle flatness $\Rightarrow$ $X \longrightarrow \mathcal{T}^n \longrightarrow \mathcal{W}^n$ ($\mathcal{W}$ = space of chars of $\mathbb{Z}_p^\times$)

  is flat at x $\rightsquigarrow$ hence open

  $\Rightarrow$ every nbhd of x in X contains many crystalline points

  $\Rightarrow$ $(r, \delta_1, \dots, \delta_n) \in X_{tri}$

- identify $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X_{tri}, (r, \delta_1, \dots, \delta_n)}$

  (using that LHS is an irred compo of an explicit deformation space)