A PROOF OF THE KUDLA–RAPOPORT CONJECTURE FOR KRÄMER MODELS

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Abstract. We prove the Kudla–Rapoport conjecture for Krämer models of unitary Rapoport–Zink spaces at ramified places. It is a precise identity between arithmetic intersection numbers of special cycles on Krämer models and modified derived local densities of hermitian forms. As an application, we relax the local assumptions at ramified places in the arithmetic Siegel–Weil formula for unitary Shimura varieties, which is in particular applicable to unitary Shimura vartieties associated to unimodular hermitian lattices over imaginary quadratic fields.

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1. Introduction

1.1. **Background.** The classical *Siegel–Weil formula* ([Sie35, Sie51, Wei65]) relates certain Siegel Eisenstein series to the arithmetic of quadratic forms, namely it expresses special *values* of these series as theta functions — generating series of representation numbers of quadratic forms. Kudla ([Kud97, Kud04]) initiated an influential program to establish the *arithmetic Siegel–Weil formula* relating certain Siegel Eisenstein series to objects in arithmetic geometry, which among others, aims to express the *central derivative* of these series as the arithmetic analogue of theta functions — generating series of arithmetic intersection numbers of n special divisors on Shimura varieties associated to $\mathrm{SO}(n-1,2)$ or $\mathrm{U}(n-1,1)$.

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For $\mathrm{U}(n-1,1)$–Shimura varieties with hyperspecial level at an *unramified* place, Kudla–Rapoport [KR11] conjectured a local arithmetic Siegel–Weil formula, now known as the (local) *Kudla–Rapoport conjecture*. It is a precise identity between the *central derivative* of local representation densities of hermitian forms (the analytic side) and the arithmetic intersection number of special cycles on unitary Rapoport–Zink spaces (the geometric side). This conjecture was recently proved by Zhang and one of us [LZ22a], and we refer to the introduction of [LZ22a] for more background and related results.

It is a natural question, which is also important for global applications, to formulate and prove an analogue of the Kudla–Rapoport conjecture at a *ramified* place. At a ramified place, there are two well-studied level structures for unitary Rapoport–Zink spaces, one gives rise to the *exotic smooth model* which has good reduction, and the other one gives rise to the *Krämer model* which has bad (semistable) reduction. For the even dimensional exotic smooth model, the analogue of Kudla–Rapoport conjecture was formulated and proved by Liu and one of us [LL22] using a strategy similar to [LZ22a].

For the Krämer model, however, the situation is more complicated — it is expected that the analytic side of the conjecture requires nontrivial modification, by a certain linear combination of *central values* of local representation densities. The necessity of such modification in the presence of bad reduction was first discovered by Kudla–Rapoport [KR00] via explicit computation in the context of the Drinfeld p-adic half plane. In [HSY21], three of us formulated *the Kudla–Rapoport conjecture for Krämer models* (recalled in §1.2) by providing a conceptual recipe for the precise modification needed for the analytic side. Moreover, this conjecture was proved for $n = 2$ (based on the previous works [Shi20, HSY20]) and $n = 3$ in [HSY21].

The main theorem of the present paper settles this conjecture for any n (and the proof is new even for $n = 2, 3$). As a first application, we relax the local assumptions in the arithmetic Siegel–Weil formula for $\mathrm{U}(n-1,1)$–Shimura varieties by allowing Krämer models at ramified places. The main theorem should also be useful to relax the local assumptions at ramified places in the arithmetic inner product formula [LL21, LL22] and its p-adic avatar by Disegni–Liu [DL22].

1.2. **Kudla–Rapoport conjecture for Krämer models.** Let p be an odd prime. Let F_0 be a finite extension of $\mathbb{Q}_p$ with residue field $\kappa = \mathbb{F}_q$. Let F be a ramified quadratic extension of F_0. Let π be a uniformizer of F such that $\mathrm{Tr}_{F/F_0}(\pi) = 0$. Then $\pi_0 := \pi^2$ is a uniformizer of F_0. Let $\breve{F}$ be the completion of the maximal unramified extension of F. Let $O_F, O_{\breve{F}}$ be the ring of integers of $F, \breve{F}$ respectively.

Let $n \geq 2$ be an integer. To define the Krämer model of the unitary Rapoport–Zink space, we fix a (principally polarized) supersingular hermitian O_F-modules $\mathbb{X}$ of signature $(1, n-1)$ over $\bar{\kappa}$ (Definition 2.1). The *Krämer model* $\mathcal{N} = \mathcal{N}_n$ is the formal scheme over $\mathrm{Spf}\, O_{\breve{F}}$ parameterizing hermitian formal O_F-modules X of signature $(1, n-1)$ within the quasi-isogeny class of $\mathbb{X}$, together with a rank 1 filtration $\mathcal{F} \subseteq \mathrm{Lie}\, X$ satisfying the Krämer condition (Definition 2.2). The space $\mathcal{N}$ is locally of finite type, and semistable of relative dimension $n-1$ over $\mathrm{Spf}\, O_{\breve{F}}$. There are two choices of the framing object $\mathbb{X}$ (up to quasi-isogeny), giving rise to two non-isomorphic (resp. isomorphic) spaces $\mathcal{N}$ when n is even (resp. odd) (§2.2).

Let $\mathbb{Y}$ be the framing hermitian O_F-modules of signature $(1,0)$ over $\bar{\kappa}$ defined as in Definition 2.1. The *space of quasi-homomorphisms* $\mathbb{V} = \mathbb{V}_n := \mathrm{Hom}_{O_F}(\mathbb{Y}, \mathbb{X}) \otimes_{O_F} F$ carries a natural F/F_0-hermitian form, which makes $\mathbb{V}$ a non-degenerate F/F_0-hermitian space of dimension n (§2.2). The two choices of the framing object $\mathbb{X}$ exactly correspond to the two isomorphism classes of $\mathbb{V}$, classified by $\chi(\mathbb{V}) := \chi((-1)^{\frac{n(n-1)}{2}} \det(\mathbb{V})) \in \{\pm 1\}$, where $\chi : F_0^\times \to \{\pm 1\}$ is the quadratic character associated to F/F_0. For any subset $L \subseteq \mathbb{V}$, the *special cycle* $\mathcal{Z}(L)$ (§2.3) is a closed formal subscheme of $\mathcal{N}$, over which each quasi-homomorphism $x \in L$ deforms to homomorphisms.

Let $L \subseteq \mathbb{V}$ be an O_F-lattice (of full rank n). We will associate to L two integers: the *arithmetic intersection number* $\mathrm{Int}(L)$ and the *modified derived local density* $\partial\mathrm{Den}(L)$.

Definition 1.1. Let $L \subseteq \mathbb{V}$ be an O_F-lattice. Let $x_1, \ldots, x_n$ be an O_F-basis of L. Define the *arithmetic intersection number*

$$\mathrm{Int}(L) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}) \in \mathbb{Z}, \tag{1.1}$$

where $\mathcal{O}_{\mathcal{Z}(x_i)}$ denotes the structure sheaf of the special divisor $\mathcal{Z}(x_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{N}$, and χ denotes the Euler–Poincaré characteristic (Definition 2.10). By Howard [How19, Corollary D]), we know that $\mathrm{Int}(L)$ is independent of the choice of the basis $x_1, \ldots, x_n$ and hence is a well-defined invariant of L itself.

For M another hermitian O_F-lattice (of arbitrary rank), denote by $\mathrm{Herm}_{L,M}$ the O_{F_0}-scheme of hermitian O_F-module homomorphisms from L to M (Definition 5.1) and define its *local density* to be

$$\mathrm{Den}(M, L) := \lim_{d \to +\infty} \frac{|\mathrm{Herm}_{L,M}(O_{F_0}/\pi_0^d)|}{q^{N \cdot d_{L,M}}},$$

where $d_{L,M}$ is the dimension of $\mathrm{Herm}_{L,M} \otimes_{O_{F_0}} F_0$. Let H be the standard hyperbolic hermitian O_F-lattice of rank 2 (given by the hermitian matrix $\left(\begin{smallmatrix} 0 & \pi^{-1} \\ -\pi^{-1} & 0 \end{smallmatrix}\right)$). It is well-known that there exists a *local density polynomial* $\mathrm{Den}(M, L, X) \in \mathbb{Q}[X]$ such that for any integer $k \geq 0$,

$$\mathrm{Den}(M, L, q^{-2k}) = \mathrm{Den}(H^k \oplus M, L). \tag{1.2}$$

When M has also rank n and $\chi(M) = -\chi(L)$, we have $\mathrm{Den}(M, L) = 0$ (Lemma 5.7) and in this case we write

$$\mathrm{Den}'(M, L) := -2 \cdot \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}(M, L, X).$$

Define the (normalized) *derived local density*

$$\mathrm{Den}'(L) := \frac{\mathrm{Den}'(I_n, L)}{\mathrm{Den}(I_n, I_n)} \in \mathbb{Q}. \tag{1.3}$$

Here I_n is the unimodular lattice of rank n with $\chi(I_n) = -\chi(L)$. Recall that a hermtian O_F-lattice L is *unimodular*[1] if $L = L^\sharp$, where $L^\sharp$ is the dual lattice of L with respect to the hermitian form (see §1.5 for notation).

The naive analogue of the Kudla–Rapoport conjecture for Krämer model states that

$$\mathrm{Int}(L) \stackrel{?}{=} \mathrm{Den}'(L).$$

[1]We refrain from using the terminology *self-dual* in the ramified case to avoid possible confusion with a lattice L such that $L = L^\vee$, where $L^\vee$ is the dual lattice with respect to the underlying quadratic form, see §4.2.

However, as explained in [HSY21] this naive analogue does not hold for trivial reasons. In fact, by definition $\mathrm{Int}(L)$ vanishes unless L is integral (i.e., $L \subseteq L^\sharp$), while $\mathrm{Den}'(L)$ does not vanish for non-integral lattices L which are dual to vertex lattices. More precisely, recall that an integral O_F-lattice $\Lambda \subseteq \mathbb{V}$ is called a *vertex lattice (of type t)* if $\Lambda^\sharp/\Lambda$ is a κ-vector space (of dimension t). For a vertex lattice $\Lambda \subseteq \mathbb{V}$ of type $t > 0$, we have $\Lambda^\sharp$ is non-integral so $\mathrm{Int}(\Lambda^\sharp) = 0$, while $\mathrm{Den}'(\Lambda^\sharp) \neq 0$ in general (see e.g. (5.7)). In general, we define the type $t(L)$ of L to be the number of positive fundamental invariants of L (see §1.5).

To account for these discrepancies, we will define $\partial\mathrm{Den}(L)$ by modifying $\mathrm{Den}'(L)$ with a linear combination of the (normalized) *local densities* (Corollary 5.8)

$$\mathrm{Den}_t(L) := \frac{\mathrm{Den}(\Lambda_t^\sharp, L)}{\mathrm{Den}(\Lambda_t^\sharp, \Lambda_t^\sharp)} \in \mathbb{Z}. \tag{1.4}$$

Here $\Lambda_t \subseteq \mathbb{V}$ is a vertex lattice of type t (in particular $\chi(\Lambda_t^\sharp) = \chi(L)$). Recall that the possible vertex type t is given by any even integer such that $0 \leq t \leq t_{\max}$, where

$$t_{\max} = \begin{cases} n, & \text{if } n \text{ even}, \chi(\mathbb{V}) = +1, \\ n-1, & \text{if } n \text{ odd}, \\ n-2, & \text{if } n \text{ even}, \chi(\mathbb{V}) = -1. \end{cases}$$

Definition 1.2. Let $L \subseteq \mathbb{V}$ be an O_F-lattice. Define the *modified derived local density* (Corollary 7.2)

$$\partial\mathrm{Den}(L) := \mathrm{Den}'(L) + \sum_{j=1}^{t_{\max}/2} c_{2j} \cdot \mathrm{Den}_{2j}(L) \in \mathbb{Z}. \tag{1.5}$$

Here the coefficients $c_{2j} \in \mathbb{Q}$ are chosen to satisfy

$$\partial\mathrm{Den}(\Lambda_{2i}^\sharp) = 0, \quad 1 \leq i \leq t_{\max}/2, \tag{1.6}$$

which turns out to be a linear system in $(c_2, c_4, \ldots, c_{t_{\max}})$ with a unique solution ([HSY21, Theorem 6.1].

The main purpose of this paper is to prove the following local arithmetic Siegel-Weil formula, settling the main conjecture of [HSY21]. We will prove this theorem in §9.

Theorem 1.3 (Kudla–Rapoport conjecture for Krämer models)**.** *Let $L \subseteq \mathbb{V}$ be an O_F-lattice. Then*

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

1.3. **The arithmetic Siegel–Weil formula.** Next let us describe some global applications of our main theorem, following the setting of [LZ22a, §1.3]. We now switch to global notations. Let F be a CM number field with maximal total real subfield F_0. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and fix a CM type $\Phi \subseteq \mathrm{Hom}(F, \overline{\mathbb{Q}}) = \mathrm{Hom}(F, \mathbb{C})$ of F. We aslo identify the CM type Φ with the set of archimedean places of F_0. Let V be an F/F_0-hermitian space of dimension n and $G = \mathrm{Res}_{F_0/\mathbb{Q}} \mathrm{U}(V)$. Assume the signatures of V are $\{(n-1,1)_{\phi_0}, (n,0)_{\phi\in\Phi-\{\phi_0\}}\}$ for some distinguished element $\phi_0 \in \Phi$. Define a torus $Z^{\mathbb{Q}} = \{z \in \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m : \mathrm{Nm}_{F/F_0}(z) \in \mathbb{G}_m\}$. Associated to $\widetilde{G} := Z^{\mathbb{Q}} \times G$ there is a natural

Shimura datum $(\widetilde{G}, \{h_{\widetilde{G}}\})$ of PEL type ([LZ22a, §11.1]). Let $K = K_{Z^{\mathbb{Q}}} \times K_G \subseteq \widetilde{G}(\mathbb{A}_f)$ be a compact open subgroup. Then the associated Shimura variety $\mathrm{Sh}_K = \mathrm{Sh}_K(\widetilde{G}, \{h_{\widetilde{G}}\})$ is of dimension $n-1$ and has a canonical model over its reflex field E.

Assume that $K_{Z^{\mathbb{Q}}} \subseteq Z^{\mathbb{Q}}(\mathbb{A}_f)$ is the unique maximal open compact subgroup. Assume that $K_G = \prod_v K_{G,v}$, where v runs over the finite places of F_0 such that $K_{G,v} \subseteq \mathrm{U}(V)(F_{0,v})$ is given by

- the stabilizer of a self-dual or almost self-dual lattice $\Lambda_v \subseteq V_v$ if v is inert in F,
- the stabilizer of a unimodular lattice $\Lambda_v \subseteq V_v$ if v is ramified in F,
- a principal congruence subgroup of $\mathrm{U}(V)(F_{0,v}) \simeq \mathrm{GL}_n(F_{0,v})$ if v is split in F.

Let $\mathcal{V}_{\mathrm{ram}}$ (resp. $\mathcal{V}_{\mathrm{asd}}$) be the set of finite places v of F_0 such that v is ramified in F (resp. v is inert in F and Λ_v is almost self-dual). Further assume that all places of E above $\mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$ are unramified over F. Then we obtain a global regular integral model $\mathcal{M}_K$ of Sh_K over O_E as in [LZ22a, §14.1-14.2], which is semistable at all places of E above $\mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$ (for more precise technical conditions required, see (G0)-(G4)). When K_G is the stabilizer of a global unimodular lattice, the regular integral model $\mathcal{M}_K$ recovers that in [HSY21], and that in [BHK+20] if $F_0 = \mathbb{Q}$.

Let $\mathbb{V}$ be the *incoherent* hermitian space over $\mathbb{A}_F$ associated to V, namely $\mathbb{V}$ is totally positive definite and $\mathbb{V}_v \cong V_v$ for all finite places v. Let $\varphi_K \in \mathscr{S}(\mathbb{V}_f^n)$ be a K-invariant (where K acts on $\mathbb{V}_f$ via the second factor K_G) factorizable Schwartz function such that $\varphi_{K,v} = \mathbf{1}_{(\Lambda_v)^n}$ at all v nonsplit in F. Let $T \in \mathrm{Herm}_n(F_0)$ be a nonsingular F/F_0-hermitian matrix of size n. Associated to (T, φ_K) we have *arithmetic special cycles* $\mathcal{Z}(T, \varphi_K)$ over $\mathcal{M}_K$ ([LZ22a, §14.3]) generalizing the $\mathcal{Z}(T)$ in [KR14]. Analogous to the local situation (1.1), we can define its local arithmetic intersection numbers $\mathrm{Int}_{T,v}(\varphi_K)$ at finite places v. Using the star product of Kudla's Green functions, we can also define its local arithmetic intersection number $\mathrm{Int}_{T,v}(\mathsf{y}, \varphi_K)$ at infinite places ([LZ22a, §15.3]), which depends on a parameter $\mathsf{y} \in \mathrm{Herm}_n(F_{0,\infty})_{>0}$ where $F_{0,\infty} = F_0 \otimes_{\mathbb{Q}} \mathbb{R}$. Combining all the local arithmetic numbers together, define the *global arithmetic intersection number*, or the *arithmetic degree* of the special cycle $\mathcal{Z}(T, \varphi_K)$ in the arithmetic Chow group of $\mathcal{M}_K$,

$$\widehat{\deg}_T(\mathsf{y}, \varphi_K) := \sum_{v \nmid \infty} \mathrm{Int}_{T,v}(\varphi_K) + \sum_{v \mid \infty} \mathrm{Int}_{T,v}(\mathsf{y}, \varphi_K).$$

On the other hand, associated to $\varphi := \varphi_K \otimes \varphi_\infty \in \mathscr{S}(\mathbb{V}^n)$, where φ_∞ is the Gaussian function, there is a classical *incoherent Eisenstein series* $E(\mathsf{z}, s, \varphi)$ ([LZ22a, §12.4]) on the hermitian upper half space

$$\mathbb{H}_n = \{\mathsf{z} = \mathsf{x} + i\mathsf{y} : \ \mathsf{x} \in \mathrm{Herm}_n(F_{0,\infty}), \ \mathsf{y} \in \mathrm{Herm}_n(F_{0,\infty})_{>0}\}.$$

This is essentially the Siegel Eisenstein series associated to a standard Siegel–Weil section of the degenerate principal series ([LZ22a, §12.1]). The Eisenstein series here has a meromorphic continuation and a functional equation relating $s \leftrightarrow -s$. The central value $E(\mathsf{z}, 0, \varphi) = 0$ by the incoherence. We thus consider its *central derivative*

$$\mathrm{Eis}'(\mathsf{z}, \varphi_K) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\mathsf{z}, s, \varphi).$$

Analogous to the local situation, we need to modify $\mathrm{Eis}'(\mathsf{z}, \varphi_K)$ by central values of coherent Eisenstein series. For $v \in \mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$, let ${}^v\mathbb{V}$ be the *coherent* hermitian space over $\mathbb{A}_F$ nearby $\mathbb{V}$ at v, namely $({}^v\mathbb{V})_w \simeq \mathbb{V}_w$ exactly for all places $w \neq v$. For any vertex lattice $\Lambda_{t,v} \subseteq ({}^v\mathbb{V})_v$ of type t, the Schwartz function $\varphi^v \otimes \mathbf{1}_{(\Lambda^\sharp_{t,v})^n} \in \mathscr{S}(({}^v\mathbb{V})^n)$ gives a classical *coherent Eisenstein series* $E(\mathsf{z}, s, \varphi^v \otimes \mathbf{1}_{(\Lambda^\sharp_{t,v})^n})$. Analogous to (1.4), define the (normalized) *central values*

$$
{}^v\mathrm{Eis}_t(\mathsf{z}, \varphi_K) := \frac{\mathrm{vol}(K_{G,v})}{\mathrm{vol}(K_{\Lambda^\sharp_{t,v}})} \cdot E(\mathsf{z}, 0, \varphi^v \otimes \mathbf{1}_{(\Lambda^\sharp_{t,v})^n}). \tag{1.7}
$$

Here $K_{\Lambda^\sharp_{t,v}} \subseteq \mathrm{U}({}^v\mathbb{V})(F_{0,v})$ is the stabilizer of $\Lambda^\sharp_{t,v}$, and the volumes are taken with respect to the Haar measures on $\mathrm{U}(V)(F_{0,v})$ and $\mathrm{U}({}^v\mathbb{V})(F_{0,v})$ as defined in [LL21, Definition 3.8]. When $v \in \mathcal{V}_{\mathrm{ram}}$, analogous to (1.5), define the linear combination

$$
{}^v\mathrm{Eis}(\mathsf{z}, \varphi_K) := \sum_{j=1}^{t_{\max,v}/2} c_{2j,v} \cdot {}^v\mathrm{Eis}_{2j}(\mathsf{z}, \varphi_K), \tag{1.8}
$$

where $t_{\max,v}$ and $c_{2j,v}$ are the numbers $t_{\max}$ and c_{2j} respectively in (1.5) for the local hermitian space $({}^v\mathbb{V})_v$ over the ramified extension $F_v/F_{0,v}$. When $v \in \mathcal{V}_{\mathrm{asd}}$, define

$$
{}^v\mathrm{Eis}(\mathsf{z}, \varphi_K) := c_{0,v} \cdot {}^v\mathrm{Eis}_0(\mathsf{z}, \varphi_K), \tag{1.9}
$$

where $c_{0,v} = -\frac{1}{1+q_v}$ and q_v is the size of the residue field of $F_{0,v}$. Define the *modified central derivative*

$$
\partial\mathrm{Eis}(\mathsf{z}, \varphi_K) := \mathrm{Eis}'(\mathsf{z}, \varphi_K) + (-1)^n \sum_{v \in \mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}} {}^v\mathrm{Eis}(\mathsf{z}, \varphi_K). \tag{1.10}
$$

Associated to an additive character $\psi : \mathbb{A}_{F_0}/F_0 \to \mathbb{C}^\times$ (as explained in [LZ22a, §12.2] we assume that ψ is unramified outside the set of finite places of F_0 split in F), it has a decomposition into Fourier coefficients

$$
\partial\mathrm{Eis}(\mathsf{z}, \varphi_K) = \sum_{T \in \mathrm{Herm}_n(F_0)} \partial\mathrm{Eis}_T(\mathsf{z}, \varphi_K). \tag{1.11}
$$

The following result asserts an identity between the arithmetic degrees of special cycles and the nonsingular Fourier coefficients of the modified central derivative of the incoherent Eisenstein series, which generalizes [LZ22a, Theorem 1.3.1] from inert places to all nonsplit places. In particular, when F is an imaginary quadratic field of discriminant $d \equiv 1 \pmod 8$, we have an unconditional arithmetic Siegel-Weil formula for all unimodular lattices of signature $(n-1,1)$ at non-singular coefficients, i.e., [HSY21, Theorem 1.5] holds without conditions.

Theorem 1.4 (Arithmetic Siegel–Weil formula: nonsingular terms)**.** *Let* $\mathrm{Diff}(T, \mathbb{V})$ *be the set of places v such that $\mathbb{V}_v$ does not represent T (*[LZ22a, §12.3]*). Let $T \in \mathrm{Herm}_n(F_0)$ be nonsingular such that* $\mathrm{Diff}(T, \mathbb{V}) = \{v\}$ *where v is nonsplit in F and not above 2. Then*

$$
\widehat{\deg}_T(\mathsf{y}, \varphi_K) q^T = c_K \cdot \partial\mathrm{Eis}_T(\mathsf{z}, \varphi_K),
$$

where $q^T := \psi_\infty(\mathrm{Tr}\, T\mathsf{z})$, c_K is a nonzero constant independent of T and φ_K (to be specified in Theorem 10.1).

We form the *generating series of arithmetic degrees*

$$\widehat{\deg}(\mathsf{z},\varphi_K) := \sum_{\substack{T\in \mathrm{Herm}_n(F_0)\\ \det T\neq 0}} \widehat{\deg}_T(\mathsf{y},\varphi_K)q^T. \tag{1.12}$$

The following result relates this generating series to the modified central derivative of the incoherent Eisenstein series, which removes the assumption that F/F_0 is unramified at all finite places from [LZ22a, Theorem 1.3.2].

Theorem 1.5 (Arithmetic Siegel–Weil formula)**.** *Assume that F/F_0 is split at all places above 2. Further assume that φ_K is nonsingular (*[LZ22a, §12.3]*) at two places split in F. Then*

$$\widehat{\deg}(\mathsf{z},\varphi_K) = c_K\cdot \partial\mathrm{Eis}(\mathsf{z},\varphi_K).$$

In particular, $\widehat{\deg}(\mathsf{z},\varphi_K)$ is a nonholomorphic hermitian modular form of genus n.

1.4. **Strategy and novelty of the proof of the Main Theorem 1.3.** Our general strategy is closest to the unramified orthogonal case proved in [LZ22b]. More precisely, fix an O_F-lattice $L^\flat\subseteq \mathbb{V}$ of rank $n-1$ and denote by $\mathbb{W}=(L^\flat_F)^\perp\subseteq\mathbb{V}$. Consider functions on $\mathbb{V}\setminus L^\flat_F$,

$$\mathrm{Int}_{L^\flat}(x) := \mathrm{Int}(L^\flat+\langle x\rangle),\quad \partial\mathrm{Den}_{L^\flat}(x) := \partial\mathrm{Den}(L^\flat+\langle x\rangle).$$

Then it remains to show the equality of the two functions $\mathrm{Int}_{L^\flat}=\partial\mathrm{Den}_{L^\flat}$. To show this equality, we find a decomposition

$$\mathrm{Int}_{L^\flat}=\mathrm{Int}_{L^\flat,\mathscr{H}}+\mathrm{Int}_{L^\flat,\mathscr{V}},\quad \partial\mathrm{Den}_{L^\flat}=\partial\mathrm{Den}_{L^\flat,\mathscr{H}}+\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$$

into "horizontal" and "vertical" parts such that the horizontal identity $\mathrm{Int}_{L^\flat,\mathscr{H}}=\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ holds and that the vertical parts $\mathrm{Int}_{L^\flat,\mathscr{V}}$ and $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ behaves well under Fourier transform along $L^\flat_F$.

The horizontal identity essentially reduces to the horizontal computation for $n=2$ in [Shi20, HSY20]. For the vertical identity, we perform a *partial Fourier transform* along $L^\flat_F$ and consider new functions on $\mathbb{W}\setminus\{0\}$,

$$\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}(x) := \int_{L^\flat_F}\mathrm{Int}_{L^\flat,\mathscr{V}}(y+x)\mathrm{d}y,\quad \partial\mathrm{Den}^\perp_{L^\flat,\mathscr{V}}(x) := \int_{L^\flat_F}\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(y+x)\mathrm{d}y.$$

The key is to show that $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}$ and $\partial\mathrm{Den}^\perp_{L^\flat,\mathscr{V}}$ are both *constant* on $\mathbb{W}^{\geq 0}\setminus\{0\} := \{x\in\mathbb{W}\setminus\{0\} : \mathrm{val}(x)\geq 0\}$ of $\mathbb{W}$ (see §1.5 for notation) as in Theorem 4.16 and Theorem 8.2. Using an induction on the valuation of $L^\flat$, we show that the difference $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}-\partial\mathrm{Den}^\perp_{L^\flat,\mathscr{V}}$ vanishes on $\mathbb{W}^{\leq 0} := \{x\in\mathbb{W} : \mathrm{val}(x)\leq 0\}$, and hence it vanishes identically and allows us to conclude that $\mathrm{Int}_{L^\flat,\mathscr{V}}=\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$.

On the geometric side, we prove a *Bruhat–Tits stratification* for the Krämer model (Theorem 3.19), analogous to the case of the Pappas model treated in Rapoport–Terstiege–Wilson [RTW14]. We make use of the linear invariance of special cycles [How19] to express $\mathrm{Int}_{L^\flat,\mathscr{V}}$ as a linear combination of functions on $\mathbb{V}$ which are *translation invariant* under vertex lattices. A new observation in our ramified case is that the translation invariance already allows us to control the support of its Fourier transform well enough (Lemma 4.13) to conclude the desired key constancy of $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}$ on $\mathbb{W}^{\geq 0}$. Compared to the unramified case, we *completely avoid* the Tate conjecture of generalized Deligne-Lusztig varieties and explicit computation of their intersections with special divisors. It is not clear that the Deligne-Lusztig cycles span the Tate cycles in this case. §3 studies the structure of $\mathcal{N}_{\mathrm{red}}$ and special cycles, and should be of independent interest (in addition to preparation for §4) .

On the analytic side, we make use of the *primitive decomposition* of the local density polynomial into primitive local density polynomials and obtain a decomposition.

$$\partial\mathrm{Den}(L) = \sum_{L\subseteq L'} \partial\mathrm{Pden}(L'), \tag{1.13}$$

where L' runs over O_F-lattices in L_F containing L, and the symbol Pden stands for the primitive version of Den (Corollary 5.4). Unlike the unramified or exotic smooth case, the primitive local density *polynomial* itself seems rather complicated (see e.g. Corollary 6.2). Nevertheless we manage to prove a simple formula for its *modified central derivative* $\partial\mathrm{Pden}(L)$, which we find quite striking.

Theorem 1.6 (Theorem 7.1)**.** *Let $L \subset \mathbb{V}$ be an O_F-lattice (of full rank n).*

(1) If L is not integral, then $\partial\mathrm{Pden}(L) = 0$.

(2) If L is unimodular, then

$$\partial\mathrm{Pden}(L) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

(3) If L is integral and of type $t > 0$, then

$$\partial\mathrm{Pden}(L) = \begin{cases} \displaystyle\prod_{\ell=1}^{\frac{t-1}{2}} (1 - q^{2\ell}), & \text{if } t \text{ is odd,} \\ \displaystyle(1 - \chi(L')q^{\frac{t}{2}}) \prod_{\ell=1}^{\frac{t}{2}-1} (1 - q^{2\ell}), & \text{if } t \text{ is even.} \end{cases}$$

Here we write $L \simeq I_{n-t} \obot L'$ with I_{n-t} unimodular of rank $n - t$.

The proof of this theorem occupies the entire §6 and §7, and is our major technical innovation. One key difference between our case and the unramified or exotic smooth case is that in our case I_n and H (see (1.2) and (1.3)) have different fundamental invariants, hence it is not clear how to reduce the calculation of $\partial\mathrm{Pden}$ into the embedding-counting problems over finite fields in the style of [CY20, §3]. To deal with this difficulty, we first decompose $\partial\mathrm{Pden}(L)$ according to orbits of Hermitian embeddings (Theorem 6.1). Now a new observation is that the primitive local density polynomial becomes simpler when L is "very integral" (i.e., when its fundamental invariants are all ≥ 1, see Proposition 7.6) in which case the decomposition in Theorem 6.1 is simple. The primitive local density polynomial vanishes when L is "very non-integral" (e.g., when one of its fundamental invariants is ≤ -2, see the proof of Lemma 5.2). When L is the dual of a vertex lattice of positive type, this is just our assumption (1.6). The remaining cases (in particular the unimodular lattice case) are much trickier to handle, whose proof occupies most of §7 and is summarized in §7.2. The proof relies on a series of non-trivial polynomial identities (e.g., Lemma 7.15 and Lemma 7.16) involving algebraic combinatorics of quadratic spaces over finite fields, which should be of independent interest.

With the simple formula for $\partial\mathrm{Pden}(L)$ at hand, we finally prove the desired key constancy of $\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}$ on $\mathbb{W}^{\geq 0}\setminus\{0\}$ via involved lattice-theoretic computation in §8, in a fashion similar to [LZ22b]. The techniques developed here on the analytic side seem to have wide applicability and we hope that they may shed new light on the Kudla–Rapoport conjecture in the context of more general level structures, e.g., for minuscule parahoric levels at unramified places formulated by Cho [Cho22].

1.5. **Notation and terminology.** In this paper, a lattice means a hermitian O_F-lattice without explicit mentioning. Unless otherwise stated, L always means a non-degenerate lattice of rank n with a hermitian form $(\ ,\ )$.

- We say a sublattice $L^\flat$ of a hermitian space is non-degenerate if the restriction of the hermitian form to it is non-degenerate.
- We define $L^\sharp$ to be the dual lattice of L with respect to the hermitian form $(\ ,\ )$. If $L\subset L^\sharp$, we say L is integral.
- Following [LL22, Definition 2.11], for a lattice L with hermitian form $(\ ,\ )$, we say that a basis $\{\ell_1,\dots,\ell_n\}$ of L is a normal basis (which always exists by [LL22, Lemma 2.12]) if its moment matrix $T=((\ell_i,\ell_j))_{1\leq i,j\leq n}$ is conjugate to
$$\left(\beta_1\pi^{2b_1}\right)\oplus\cdots\oplus\left(\beta_s\pi^{2b_s}\right)\oplus\begin{pmatrix}0 & \pi^{2c_1+1}\\ -\pi^{2c_1+1} & 0\end{pmatrix}\oplus\cdots\oplus\begin{pmatrix}0 & \pi^{2c_t+1}\\ -\pi^{2c_t+1} & 0\end{pmatrix}$$
by a permutation matrix, for some $\beta_1,\dots,\beta_s\in O_{F_0}^\times$ and $b_1,\dots,b_s,c_1,\dots,c_t\in\mathbb{Z}$. Moreover, we define its (unitary) fundamental invariants $(a_1,\cdots,a_n)$ to be the unique nondecreasing rearrangement of $(2b_1,\cdots,2b_s,2c_1+1,\cdots,2c_t+1)$.
- We define the type $t(L)$ of L to be the number of positive fundamental invariants of L. We use $r(L)$ to denote the rank of L and call L a *full type* lattice if $r(L)=t(L)$.
- We define the valuation of L to be $\mathrm{val}(L)\coloneqq\sum_{i=1}^n a_i$, where $(a_1,\cdots,a_n)$ are the fundamental invariants of L. For $x\in L$, we define $\mathrm{val}(x)=\mathrm{val}((x,x))$, where $\mathrm{val}(\pi_0)=1$.
- For a hermitian space $\mathbb{V}$, we let $\mathbb{V}^{?i}\coloneqq\{x\in\mathbb{V}\mid \mathrm{val}(x)?i\}$ where ? can be $\geq$, $\leq$ or $=$.
- For a ring R, we use $\langle\ell_1,\cdots,\ell_n\rangle_R$ to denote $\mathrm{Span}_R\{\ell_1,\cdots,\ell_n\}$. When $R=O_F$, we simply write $\langle\ell_1,\cdots,\ell_n\rangle$. We use L_F to denote $L\otimes_{O_F}F$.
- For a hermitian lattice of rank n, we define its sign as
$$\chi(L)\coloneqq\chi((-1)^{\frac{n(n-1)}{2}}\det(L))=\pm 1$$
where χ is the quadratic character of $F_0^\times$ associated to F/F_0. Without explicit mentioning, we always use ϵ to denote $\chi(L)$.
- Let I_m^ϵ denote a unimodular lattice of rank m with $\chi(I_m^\epsilon)=\epsilon$. We also simply denote a unimodular lattice of rank m by I_m if we do not need to remember its sign or its sign is clear in the context. In particular, when we consider $\mathrm{Den}'(I_n,L)$, we mean $I_n=I_n^{-\epsilon}$.
- We call a sublattice $N\subset M$ primitive in M if $\dim_{\mathbb{F}_q}\overline{N}=r(N)$, where $\overline{N}=(N+\pi M)/\pi M$. We also use $\overline{L}$ to denote $L\otimes_{O_F}O_F/(\pi)$.
- For two lattices L,L' of same rank, let $n(L',L)=\#\{L''\subset L_F\mid L\subset L'',L''\cong L'\}$.
- We let $\delta_{\mathrm{odd}}(n)=1$ if n is an odd integer and $\delta_{\mathrm{odd}}(n)=0$ if n is an even integer.

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2. Krämer models of Rapoport-Zink spaces and special cycles

We denote $\bar{a}$ the Galois conjugate of $a \in F$ over F_0. Denote by $\mathrm{Nilp}\, O_{\breve{F}}$ be the category of $O_{\breve{F}}$-schemes S such that π is locally nilpotent on S. For such an S, denote its special fiber $S \times_{\mathrm{Spf}\, \mathcal{O}_{\breve{F}}} \mathrm{Spec}\, \bar{\kappa}$ by $\bar{S}$. Let $\sigma \in \mathrm{Gal}(\breve{F}_0/F_0)$ be the Frobenius element. We fix an injection of rings $i_0 : O_{F_0} \to \mathcal{O}_{\breve{F}_0}$ and an injection $i : O_F \to O_{\breve{F}}$ extending i_0. Denote by $\bar{i} : O_F \to O_{\breve{F}}$ the map $a \mapsto i(\bar{a})$.

2.1. **RZ spaces.** Let $S \in \mathrm{Nilp}\, O_{\breve{F}}$. A p-divisible strict O_{F_0}-module over S is a p-divisible group over S with an O_{F_0}-action whose induced action on its Lie algebra is via $O_{F_0} \xrightarrow{i_0} O_{\breve{F}_0} \to \mathcal{O}_S$.

Definition 2.1. A formal hermitian O_F-module of dimension n over S is a triple (X, ι, λ) where X is a supersingular p-divisible strict O_{F_0}-module over S of dimension n and F_0-height $2n$ (supersingular means the O_{F_0}-relative Dieudonné module of X at each geometric point of S has slope $\frac{1}{2}$), $\iota : O_F \to \mathrm{End}(X)$ is an O_F-action and $\lambda : X \to X^\vee$ is a principal polarization in the category of strict O_{F_0}-modules such that the Rosati involution induced by λ is the Galois conjugation of F/F_0 when restricted on O_F.

Definition 2.2. Fix a formal hermitian O_F-module $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of dimension n over $\bar{\kappa}$. The moduli space $\mathcal{N}_n$ is the functor such that $\mathcal{N}_n(S)$ for any $S \in \mathrm{Nilp}\, O_{\breve{F}}$ is the set of isomorphism classes of quintuples $(X, \iota, \lambda, \rho, \mathcal{F})$ such that

(1) (X, ι, λ) is a formal hermitian O_F-module over S.
(2) $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{\kappa}} \bar{S}$ is a morphism of formal O_F-modules of height 0.
(3) $\mathcal{F}$ satisfies Krämer's ([Krä03]) signature condition: it is a local direct summand of $\mathrm{Lie}\, X$ of rank $n-1$ as an $\mathcal{O}_S$-module such that O_F acts on $\mathcal{F}$ by $O_F \xrightarrow{\bar{i}} O_{\breve{F}} \to \mathcal{O}_S$ and acts on $\mathrm{Lie}\, X/\mathcal{F}$ by $O_F \xrightarrow{i} O_{\breve{F}} \to \mathcal{O}_S$.

An isomorphism between two such quintuples $(X, \iota, \lambda, \rho, \mathcal{F})$ and $(X', \iota', \lambda', \rho', \mathcal{F}')$ is an isomorphism $\alpha : X \to X'$ such that $\rho' \circ (\alpha \times_S \bar{S}) = \rho$, $\alpha^*(\lambda')$ is a $O_{F_0}^\times$-multiple of λ and $\alpha_*(\mathcal{F}) = \mathcal{F}'$.

Notice that $\mathcal{N}_n$ is a relative Rapoport-Zink space in the sense of [Mih22]. By [How19, Proposition 2.2], $\mathcal{N}_n$ is representable by a flat formal scheme of relative dimension $n-1$ over $\mathrm{Spf}\, O_{\breve{F}}$. We drop the subscript n in $\mathcal{N}_n$ when there is no ambiguity.

2.2. **Associated hermitian spaces.** For a strict O_{F_0}-module X over $\bar{\kappa}$, let $M(X)$ be the O_{F_0}-relative Dieudonné module of X. Let $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be the framing object as in Definition 2.2, and $N = M(\mathbb{X}) \otimes_{O_{F_0}} F_0$ be its rational relative Dieudonne module. Then N is a $2n$-dimensional $\breve{F}_0$-vector space equipped with a σ-linear operator $\mathbf{F}$ and a σ^{-1}-linear operator $\mathbf{V}$. The O_F-action $\iota_{\mathbb{X}} : O_F \to \mathrm{End}(\mathbb{X})$ induces on N an O_F-action commuting with $\mathbf{F}$ and $\mathbf{V}$. We still denote this induced action by $\iota_{\mathbb{X}}$ and denote $\iota_{\mathbb{X}}(\pi)$ by Π. The principal polarization of $\mathbb{X}$ induces a skew-symmetric $\breve{F}_0$-bilinear form $\langle\, ,\, \rangle$ on N satisfying

$$\langle \mathbf{F}x, y\rangle = \langle x, \mathbf{V}y\rangle^{\sigma}, \quad \langle \iota(a)x, y\rangle = \langle x, \iota(\bar{a})y\rangle,$$

for any $x, y \in N$, $a \in O_F$. Then N is an n-dimensional $\breve{F}$-vector space equipped with a $\breve{F}/\breve{F}_0$-hermitian form $(\, ,\,)$ defined by (see [Shi18, (2.6)])

$$(x, y) := \delta(\langle \Pi x, y\rangle + \pi\langle x, y\rangle), \tag{2.1}$$

where δ is a fixed element in $\mathcal{O}_{\breve{F}_0}^{\times}$ such that $\sigma(\delta) = -\delta$. We can use the relation

$$\langle x, y\rangle = \frac{1}{2\delta}\operatorname{Tr}_{\breve{F}/\breve{F}_0}(\pi^{-1}(x, y)). \tag{2.2}$$

to recover $\langle\, ,\, \rangle$. Let $\tau := \Pi\mathbf{V}^{-1}$ and $C := N^{\tau}$. Then C is an F-vector space of dimension n and $N = C \otimes_{F_0} \breve{F}_0$. The restriction of $(\, ,\,)$ to C is a F/F_0-hermitian form which we still denote by $(\, ,\,)$. There are two choices of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ up to quasi-isogenies preserving the polarization on the nose, according to the sign $\epsilon = \chi(C)$ of C. Here $\chi : F_0^{\times} \to \{\pm 1\}$ is the character associated to the quadratic extension F/F_0 and we define the sign of C as

$$\chi(C) := \chi((-1)^{n(n-1)/2}\det(C)).$$

When n is odd, two different choices of ϵ give us isomorphic moduli spaces. When n is even, two different choices of ϵ give us two sets of non-isomorphic moduli spaces. See [Shi18, Remark 2.16] and [RTW14, Remark 4.2].

Fix a formal hermitian O_F-module $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ of dimension 1 over $\operatorname{Spec}\bar{\kappa}$. Define

$$\mathbb{V}_n = \operatorname{Hom}_{O_F}(\mathbb{Y}, \mathbb{X}) \otimes \mathbb{Q}. \tag{2.3}$$

We drop the subscript n of $\mathbb{V}_n$ unless we need to specify the dimension. The vector space $\mathbb{V}$ is equipped with a hermitian form $(\, ,\,)_{\mathbb{V}}$ such that for any $x, y \in \mathbb{V}$

$$(x, y)_{\mathbb{V}} = \lambda_{\mathbb{Y}}^{-1} \circ y^{\vee} \circ \lambda_{\mathbb{X}} \circ x \in \operatorname{End}(\mathbb{Y}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} F \tag{2.4}$$

where $y^{\vee}$ is the dual quasi-homomorphism of y. The hermitian spaces $(\mathbb{V}, (\, ,\,)_{\mathbb{V}})$ and $(C, (,))$ are related by the F-linear isomorphism

$$b : \mathbb{V} \to C, \quad x \mapsto x(e) \tag{2.5}$$

where e is a generator of τ-fixed points of the O_{F_0}-relative Dieudonné module $M(\mathbb{Y})$. The relative Diedonné module $M(\mathbb{Y})$ is equipped with a hermitian form $(\, ,\,)_{\mathbb{Y}}$ such that $(e, e)_{\mathbb{Y}} \in O_{F_0}^{\times}$. By [Shi18, Lemma 3.6], we have

$$(x, x)_{\mathbb{V}} \cdot (e, e)_{\mathbb{Y}} = (b(x), b(x)). \tag{2.6}$$

By scaling the polarization $\lambda_{\mathbb{Y}}$ by a factor in $O_{F_0}^{\times}$ we can assume that

$$(e, e)_{\mathbb{Y}} = 1,$$

so $\mathbb{V}$ and C are isomorphic as hermitian spaces. When the context is clear we often drop the subscript $\mathbb{V}$ in $(\, ,\,)_{\mathbb{V}}$.

2.3. **Special cycles.** We fix a canonical lift $(\mathcal{G}, \iota_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ to $O_{\breve{F}}$ in the sense of [Gro86] such that the action of O_F on $\mathrm{Lie}\,\mathcal{G}$ is via the inclusion $\bar{i}$. Such lift is unique up to isomorphism by [How19, Proposition 2.1].

Definition 2.3. For an O_F-lattice L of $\mathbb{V}$, define $\mathcal{Z}(L)$ to be the subfunctor of $\mathcal{N}$ such that $\mathcal{N}(S)$ is the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}(S)$ such that for any $x \in L$ the quasi-homomorphism

$$\rho^{-1} \circ x \circ \rho_{\mathcal{G}} : \mathbb{Y} \times_{\mathrm{Spec}\,\bar{\kappa}} \bar{S} \to X \times_S \bar{S}$$

entends to a homomorphism $\mathcal{G} \times_{\mathrm{Spf}\,\mathcal{O}_{\breve{F}}} S \to X$. By Grothendieck-Messing theory $\mathcal{Z}(L)$ is a closed formal subscheme of $\mathcal{N}$. For $x \in \mathbb{V}$, we let $\mathcal{Z}(x) \coloneqq \mathcal{Z}(L)$ when $L = \langle x \rangle$.

2.4. **Bruhat-Tits stratification of $\mathcal{N}_{\mathrm{red}}$.** We say a lattice $\Lambda \subset C$ (resp. $\Lambda \subset \mathbb{V}$) is a vertex lattice if $\pi\Lambda^\sharp \subseteq \Lambda \subseteq \Lambda^\sharp$ where $\Lambda^\sharp$ is dual lattice of Λ with respect to $(\ ,\ )$ (resp. $(\ ,\ )_{\mathbb{V}}$)[2]. Using the isomorphism of hermitian spaces (2.5), we often identify Λ with $b^{-1}(\Lambda)$ and use the same notation to denote both. We call $t = \dim_{\mathbb{F}_q}(\Lambda^\sharp/\Lambda)$ the type of Λ. Recall from [RTW14, Lemma 3.2] that t has to be an even integer. To each vertex lattice Λ of type $2m$, we can associate to it a subscheme $\mathcal{N}_\Lambda$ which is a subscheme of the minuscule special cycle $\mathcal{Z}(\Lambda)$, see Definition 3.5 below. Let $V = \Lambda^\sharp/\Lambda$, we can define a (modified) Deligne-Lusztig variety Y_V over $\bar{\kappa}$, see (3.2) below. We prove that Y_V is projective and smooth, see Proposition 3.2. When $m \neq 0$ the scheme $\mathcal{N}_\Lambda$ is isomorphic to Y_V, see Theorem 3.16.

For vertex lattices of type 0, we define Exc_Λ following the idea of [How19, Appendix].

Definition 2.4. The exceptional divisor Exc of $\mathcal{N}$ is the set of all points $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}(\bar{\kappa})$ such that the action

$$\iota : O_F \to \mathrm{End}(\mathrm{Lie}\,X)$$

factor through $O_F \xrightarrow{i} \mathcal{O}_{\breve{F}} \to \bar{\kappa}$ where $\mathcal{O}_{\breve{F}} \to \bar{\kappa}$ is the quotient map. For a vertex lattice Λ in C of type 0, define Exc_Λ to be is the set of all points $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathrm{Exc}$ such that $\rho(M(X)) = \Lambda \otimes_{O_F} O_{\breve{F}}$. Both Exc and Exc_Λ are closed subset of $\mathcal{N}$ and we endow them the structure of reduced schemes over $\bar{\kappa}$.

The following is a refinement of [How19, Proposition A.2].

Lemma 2.5. *Each* Exc_Λ *is a Cartier divisor of* $\mathcal{N}$ *isomorphic to* $\mathbb{P}^{n-1}_{\bar{\kappa}}$. *The scheme* Exc *is a disjoint union of* Exc_Λ *over all type* 0 *lattices* Λ *in* C.

Proof. Let $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathrm{Exc}(\bar{\kappa})$ and $M = \rho(M(X)) \subset N$. Then the action of $\iota(\pi)$ on $\mathrm{Lie}\,X$ is trivial. Hence $\Pi M \subset \mathbf{V}M$ as $\mathrm{Lie}\,X = M/\mathbf{V}M$. Since $\dim_{\bar{\kappa}}(M/\mathbf{V}M) = \dim_{\bar{\kappa}}(M/\Pi M) = n$, we know $\mathbf{V}M = \Pi M$ which is equivalent to $\tau(M) = M$. By [RTW14, Proposition 4.1], $M = \Lambda \otimes_{O_F} O_{\breve{F}}$ for some vertex lattice Λ. Since M is self-dual, Λ is of type 0. Hence $z \in \mathrm{Exc}_\Lambda(\bar{\kappa})$. Moreover for any $\bar{\kappa}$-algebra R, every rank $n-1$ locally direct summand of $\mathrm{Lie}\,X_R$ satisfies the Krämer's signature condition as in Definition 2.2 and determines a point of $\mathrm{Exc}_\Lambda(R)$ uniquely. So we get an isomorphism $\mathbb{P}^{n-1}_{\bar{\kappa}} \to \mathrm{Exc}_\Lambda$. Since $\mathcal{N}$ is regular and Exc_Λ has codimension 1, Exc_Λ is a Cartier divisor in $\mathcal{N}$. By

[2] Notice that the vertex lattice Λ in the sense of [RTW14], [HSY20] or[HSY21] corresponds to $\Lambda^\sharp$ in our convention.

looking at $\rho(M(X))$, it is clear that $\mathrm{Exc}_\Lambda \cap \mathrm{Exc}_{\Lambda'}(k) = \emptyset$ if $\Lambda \neq \Lambda'$. Hence Exc is a disjoint union of over all type 0 lattices Λ. $\square$

Remark 2.6. *The proof of Lemma 2.5 shows that the definition of* Exc_Λ *above agrees with that of* [HSY21, §2].

By Proposition 3.17 below, $\mathcal{N}_\Lambda = \mathrm{Exc}_\Lambda$ for type 0 lattices Λ. The reduced locus $\mathcal{N}_{\mathrm{red}}$ has a decomposition (see Theorem 3.19)

$$\mathcal{N}_{\mathrm{red}} = \bigcup_{\Lambda} \mathcal{N}_\Lambda,$$

where the union is over all vertex lattices. The reduced subscheme $\mathcal{Z}(L)_{\mathrm{red}}$ is a union of Bruhat-Tits strata (see Proposition 3.20)

$$\mathcal{Z}(L)_{\mathrm{red}} = \bigcup_{\Lambda \supset L} \mathcal{N}_\Lambda. \tag{2.7}$$

2.5. **Horizontal and vertical part.** A formal scheme X over $\mathrm{Spf}\, O_{\breve{F}}$ is called horizontal (resp. vertical) if it is flat over $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ (resp. π is locally nilpotent on $\mathcal{O}_X$). For a formal scheme X over $\mathrm{Spf}\, O_{\breve{F}}$, its horizontal part $X_{\mathscr{H}}$ is canonically defined by the ideal sheaf $\mathcal{O}_{X,\mathrm{tor}}$ of torsion sections on $\mathcal{O}_X$. If X is noetherian, there exists a $m \in \mathbb{Z}_{>0}$ such that $\pi^m \mathcal{O}_{X,\mathrm{tor}} = 0$. We define the vertical part $X_{\mathscr{V}} \subset X$ to be the closed formal subscheme defined by the ideal sheaf $\pi^m \mathcal{O}_X$. Since $\mathcal{O}_{X,\mathrm{tor}} \cap \pi^m \mathcal{O}_X = \{0\}$, we have the following decomposition by primary decomposition

$$X = X_{\mathscr{H}} \cup X_{\mathscr{V}} \tag{2.8}$$

as a union of horizontal and vertical formal subschemes. Notice that the horizontal part $X_{\mathscr{H}}$ is canonically defined while the vertical part $X_{\mathscr{V}}$ depends on the choice of m.

Lemma 2.7. *For a lattice $L^\flat \subset \mathbb{V}$ of rank greater or equal to $n-1$ with non-degenerate hermitian form, $\mathcal{Z}(L^\flat)$ is noetherian.*

Proof. First we know that $\mathcal{Z}(L)$ is locally noetherian since it is a closed formal subscheme of $\mathcal{N}$ which is locally noetherian. Since the rank of L is greater or equal to $n-1$, the number of vertex lattices Λ containing L is finite. By (2.7), we know that $\mathcal{Z}(L)_{\mathrm{red}}$ is a closed subset in finitely many irreducible components of $\mathcal{N}_{\mathrm{red}}$. Since each irreducible component of $\mathcal{N}_{\mathrm{red}}$ is quasi-compact, we know that $\mathcal{Z}(L)$ is quasi-compact, hence noetherian. $\square$

Lemma 2.8. *For a rank $n-1$ lattice $L^\flat \subset \mathbb{V}$ with non-degenerate hermitian form, $\mathcal{Z}(L)_{\mathscr{V}}$ is supported on the reduced locus $\mathcal{N}_{\mathrm{red}}$ of $\mathcal{N}$, i.e., $\mathcal{O}_{\mathcal{Z}(L)_{\mathscr{V}}}$ is annihilated by a power of the ideal sheaf of $\mathcal{N}_{\mathrm{red}}$.*

Proof. We remark here that $\mathcal{N}_{\mathrm{red}}$ is exactly the supersingular locus of $\mathcal{N}$. Hence the proof of the lemma is the same as that of [LZ22a, Lemma 5.1.1]. $\square$

2.6. **Derived special cycles.** For a locally noetherian formal scheme X together with a formal subscheme Y, denote by $K_0^Y(X)$ the Grothendieck group of finite complexes of coherent locally free $\mathcal{O}_X$-modules acyclic outside Y. For such a complex $A^\bullet$, denote by $[A^\bullet]$ the element in $K_0^Y(X)$ represented by it. We use $K_0(X)$ to denote $K_0^X(X)$. Let $K_0'(Y)$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_Y$-modules on Y. We have a group homomorphism $K_0^Y(X) \to K_0'(Y)$ which is an isomorphism when X is regular.

Denote by $\mathrm{F}^i K_0^Y(X)$ the codimension i filtration on $K_0^Y(X)$ and $\mathrm{Gr}^i K_0^Y(X)$ its i-th graded piece. When X is regular, we have a cup product $\cdot$ on $K_0^Y(X)_\mathbb{Q}$ defined by tensor product of complexes. Under the identification $K_0^Y(X) \xrightarrow{\sim} K_0'(Y)$, the cup product is nothing but derived tensor product:

$$[A] \cdot [B] = [A \otimes^{\mathbb{L}}_{\mathcal{O}_X} B].$$

When X is a scheme, the cup product satisfies ([SABK94, Section I.3, Theorem 1.3])

$$\mathrm{F}^i K_0^Y(X)_\mathbb{Q} \cdot \mathrm{F}^j K_0^Y(X)_\mathbb{Q} \subset \mathrm{F}^{i+j} K_0^Y(X)_\mathbb{Q}. \tag{2.9}$$

It is expected that (2.9) is also true when X is a formal scheme, see [Zha21, (B.3)], however we do not need this fact in this paper. Throughout the paper, we assume $X = \mathcal{N}$ unless stated otherwise.

Recall that for $x \in \mathbb{V}$, $\mathcal{Z}(x)$ is a Cartier divisor ([How19, Proposition 4.3]).

Definition 2.9. Let $L \subset \mathbb{V}$ be a rank r lattice with a basis $\{x_1, \ldots, x_r\}$. Define ${}^{\mathbb{L}}\mathcal{Z}(L)$ to be

$$[\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_N} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_N} \mathcal{O}_{\mathcal{Z}(x_r)}] \in K_0^{\mathcal{Z}(L)}(\mathcal{N}) \tag{2.10}$$

where $\otimes^{\mathbb{L}}$ is the derived tensor product of complexes of coherent locally free sheaves on $\mathcal{N}$. By [How19, Corollary C], ${}^{\mathbb{L}}\mathcal{Z}(L)$ is independent of the choice of the basis $\{x_1, \ldots, x_r\}$.

Definition 2.10. When L has rank n, we define the intersection number

$$\mathrm{Int}(L) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L)), \tag{2.11}$$

where χ is the Euler characteristic.

Lemma 2.11. *When L is a rank n lattice in $\mathbb{V}$, $\mathcal{Z}(L)$ is a proper scheme over $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$. In particular,* $\mathrm{Int}(L)$ *is finite.*

Proof. By Lemma 2.8 $\mathcal{Z}(L)_{\mathscr{V}}$ is a scheme. We show that $\mathcal{Z}(L)_{\mathscr{H}}$ is empty. If not, there exists $z \in \mathcal{Z}(L)(\mathcal{O}_K)$ for some finite extension K of $\breve{F}$. Let X be the corresponding formal hermitian O_F-module of signature $(1, n-1)$ over $\mathcal{O}_K$. Since L has rank n and $\mathcal{G}$ has signature $(0,1)$, this would imply that X has signature $(0, n)$, which is a contradiction. Hence $\mathcal{Z}(L)$ is a scheme. Since $\mathcal{Z}(L)_{\mathrm{red}}$ is contained in finitely many irreducible components of $\mathcal{N}_{\mathrm{red}}$ and each irreducible component is proper over $\mathrm{Spec}\, \bar{\kappa}$, it follows that $\mathcal{Z}(L)$ is proper over $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$. The finiteness of $\mathrm{Int}(L)$ then follows from the same argument before [Zha21, (B.4)]. $\square$

2.7. **A geometric cancellation law.** Recall that for two lattices $L, L' \subset \mathbb{V}$ of rank n, we define

$$n(L', L) = \#\{L'' \subset L_F \mid L \subset L'', L'' \cong L'\}.$$

Also recall that $\delta_{\mathrm{odd}}(n) = 1$ or 0 depending on whether n is odd or not.

Proposition 2.12. *Let $L = I_\ell \oplus L_2 \subset \mathbb{V}$ where L_2 is of rank r, I_ℓ is unimodular of rank ℓ and $n = \ell + r$. Let I_r be a unimodular lattice that contains L_2. Then*

$$\mathrm{Int}(I_\ell \oplus L_2) - \mathrm{Int}(L_2) = n(I_r, L_2) \cdot (\delta_{\mathrm{odd}}(n) - \delta_{\mathrm{odd}}(r)). \tag{2.12}$$

Moreover,

$$\mathrm{Int}(I_n) = \delta_{\mathrm{odd}}(n). \tag{2.13}$$

Proof. If L_2 is unimodular and $r = 2$, then $\mathrm{Int}(L_2) = 0$ by [Shi20, Theorem 1.3] and [HSY20, Theorem 1.3]. Combining this with (2.12), we obtain (2.13). In order to prove (2.12), we prove the following equation,

$$\mathrm{Int}(I_1 \oplus L_2) - \mathrm{Int}(L_2) = (-1)^r n(I_r, L_2). \tag{2.14}$$

which is the special case of (2.12) when $\ell = 1$. The general case then follows from an easy induction on n using (2.14) and the fact

$$n(I_n, I_\ell \oplus L_2) = n(I_r, L_2). \tag{2.15}$$

By Proposition 3.2 of [HSY21], we have the following decomposition of Cartier divisors on $\mathcal{N}_n$

$$\mathcal{Z}(I_1) = \tilde{\mathcal{Z}}(I_1) + \sum_{\Lambda_0 \supset I_1} \mathrm{Exc}_{\Lambda_0},$$

where the summation is over vertex lattices of type 0 in $\mathbb{V}_n$ and $\tilde{\mathcal{Z}}(I_1) \cong \mathcal{N}_{n-1}$ by [HSY21, Corollary 2.7]. By the same corollary, we know that

$$\chi(\mathcal{N}_n, [\mathcal{O}_{\tilde{\mathcal{Z}}(I_1)}] \cdot {}^{\mathbb{L}}\mathcal{Z}(L_2)) = \chi(\mathcal{N}_r, {}^{\mathbb{L}}\mathcal{Z}(L_2)) = \mathrm{Int}(L_2).$$

Hence we have

$$\mathrm{Int}(L) - \mathrm{Int}(L_2) = \sum_{\Lambda_0 \supset I_1} \chi(\mathcal{N}_n, [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}] \cdot {}^{\mathbb{L}}\mathcal{Z}(L_2)).$$

If $L_2 \not\subset \Lambda_0$, then $\mathrm{Exc}_{\Lambda_0} \cap \mathcal{Z}(L_2)$ is empty by Proposition 3.20 below. If $L_2 \subset \Lambda_0$, then by [HSY21, Corollary 3.6], we have

$$\chi(\mathcal{N}_n, [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}] \cdot {}^{\mathbb{L}}\mathcal{Z}(L_2)) = (-1)^r.$$

Hence

$$\mathrm{Int}(L) - \mathrm{Int}(L_2) = \sum_{\Lambda_0 \supset I_1 \oplus L_2} (-1)^r.$$

Combining this with (2.15) finishes the proof of (2.14) and the proposition. □

3. Bruhat–Tits Stratification of Krämer models

We prove a Bruhat–Tits stratification for the Krämer model (Theorem 3.19), analogous to the case of the Pappas model treated in [RTW14]. More precisely, we define closed subschemes $\mathcal{N}_\Lambda$ (Definition 3.5) and show that the reduced locus of $\mathcal{N}$ is stratified by $\mathcal{N}_\Lambda$ (Theorem 3.19). From this stratification we obtain a stratification of the reduced locus of $\mathcal{Z}(L)$ (Proposition 3.20). We also show that $\mathcal{N}_\Lambda$ is isomorphic to the (modified) Deligne-Lusztig variety $Y_{V,\bar{\kappa}}$ defined in §3.1 (Theorem 3.16), and is in particular a smooth projective variety over $\bar{\kappa}$. We remark here that for the purpose of our main result (Theorem 9.7), only a weeker version of Proposition 3.20 is needed (namely we do not need the reducedness of $\mathcal{N}_\Lambda$). However we believe the rest of this section contributes to the theory of Rapoport-Zink space and is of independent interest.

3.1. **Deligne-Lusztig varieties.** Through out this subsection we assume $m \geq 1$. Let V be a $2m$-dimensional symplectic space over $\kappa = \mathbb{F}_q$ equipped with the symplectic form $\langle\, , \,\rangle$. Let $V_{\bar\kappa} = V \otimes_\kappa \bar\kappa$ and denote the bilinear extension of $\langle\, , \,\rangle$ to $V_{\bar\kappa}$ still by $\langle\, , \,\rangle$. Let $\mathrm{Gr}(i, V)$ be the Grassmannian variety parametrizing rank i locally direct summands of V_R for any κ-algebra R. Let L_V be the subvariety of $\mathrm{Gr}(m, V)$ whose R-points are rank m locally direct summands of V_R that are Lagrangian. Let S_V be the subvariety of L_V as in [RTW14, Equation (5.3)] whose $\bar\kappa$-points are specified by

$$S_V(\bar\kappa) = \{U \in L_V(\bar\kappa) \mid \dim(U \cap \Phi(U)) \geq m-1\}, \tag{3.1}$$

where Φ is the Frobenius endormophism. By [RTW14, Proposition 5.3] and its remark, S_V has isolated singularities which are exactly the κ-points of L_V. By Proposition 5.5 of loc.cit., S_V is irreducible of dimension m. To resolve the singularities of S_V, define Y_V to be the subvariety of $L_V \times \mathrm{Gr}(m-1, V)$ whose $\bar\kappa$-points are specified by

$$Y_V(\bar\kappa) = \{(U, U') \in (L_V \times \mathrm{Gr}(m-1, V))(\bar\kappa) \mid U' \subseteq U \cap \Phi(U)\}. \tag{3.2}$$

Then the variety Y_V is a projective subscheme of $\mathrm{Gr}(m, V) \times \mathrm{Gr}(m-1, V)$. The forgetful map $(U, U') \to U$ defines a morphism $\pi_m : Y_V \to S_V$.

Lemma 3.1. *The morphism π_m is a projective morphism. It is an isomorphism outside the singular locus of S_V. For a singular point z of S_V, $\pi_m^{-1}(z) \cong \mathbb{P}^{m-1}_{\bar\kappa}$.*

Proof. First we know π_m is projective as it is a morphism between projective schemes. Consider a $\bar\kappa$-point $z = U$ of S_V outside its singular locus. Then $U \cap \Phi(U)$ has dimension $m-1$, this entails $U' = U \cap \Phi(U)$. In fact for any affine open subset $\operatorname{Spec} R$ of S_V outside the singular locus, define $U'_R := U_R \cap \mathrm{Span}_R\{\Phi(U_R)\} \in \mathrm{Gr}(m-1, V_R)$ where $U_R \in L_V(R)$ is the universal object over $\operatorname{Spec} R$. The map $U_R \mapsto (U_R, U'_R)$ defines the inverse morphim of π_m outside the singular locus of S_V.

If $z = U$ is a singular $\bar\kappa$-point, then $U = \Phi(U)$ and U' can be any element in $\mathrm{Gr}(m-1, U) \cong \mathbb{P}^{m-1}_{\bar\kappa}$. This finishes the proof of the lemma. □

Proposition 3.2. *The projective variety Y_V is smooth of dimension m.*

Proof. By Lemma 3.1 and the fact that S_V has dimension m, it suffices to show that the tangent space $T_z(Y_V)$ of Y_V at z has dimension less or equal to m at any point $z = (U, U')$ of $Y_V(\bar\kappa)$ such that $U = \Phi(U)$ is rational (a singlular point of S_V). We choose a κ-basis $\{e_1, \ldots, e_{2m}\}$ of V such that

$$U = \mathrm{Span}_{\bar\kappa}\{e_1, \ldots, e_m\}, \ \langle e_i, e_{m+j}\rangle = \delta_{ij}, \quad 1 \leq i, j \leq m,$$

and all other pairings between basis vectors are zero. We assume that U' is the hyperplane of U defined by

$$\sum_{i=1}^m b_i e_i^* = 0$$

where $\{e_i^*\}_{i=1}^m$ is the dual basis of $\{e_i\}_{i=1}^m$ and b_i are not all zero. Without loss of generality we can assume $b_m = 1$.

Let $\tilde{V} := V \otimes_\kappa (\bar{\kappa}[\epsilon]/(\epsilon^2))$ and $\tilde{e}_i := e_i \otimes 1 \in \tilde{V}$. The tangent space $T_z(Y_V)$ can be identified with $\mathrm{Hom}(\mathcal{O}_{Y_V,z}, \bar{\kappa}[\epsilon]/(\epsilon^2))$ which parameterizes lifts of (U, U'). A lift $(\tilde{U}, \tilde{U}')$ of (U, U') to $T_z(Y_V)$ can be determined as follows. First $\tilde{U}$ can be spanned by vectors $x_1, \dots, x_m$ defined by

$$(x_1, \dots, x_m) = (\tilde{e}_1, \dots, \tilde{e}_{2m}) \begin{pmatrix} I_m \\ A\epsilon \end{pmatrix}$$

where $A \in M_{m \times m}(\bar{\kappa})$ is symmetric as U is Lagrangian. The filtration $\tilde{U}'$ is a hyperplane of $\tilde{U}$ defined by

$$\sum_{i=1}^m \tilde{b}_i x_i^* = 0$$

where $\tilde{b}_i = b_i + \epsilon c_i$ for $1 \leq i \leq m-1$ and $\tilde{b}_m = 1$ where $c_i \in \bar{\kappa}$. Then $\tilde{U}'$ is spanned by vectors

$$x_j - \tilde{b}_j x_m = \tilde{e}_j - \tilde{b}_j \tilde{e}_m + \epsilon \sum_{i=1}^m (a_{ij} - \tilde{b}_j a_{im}) \tilde{e}_{i+m},\ 1 \leq j \leq m-1.$$

Since $\Phi(\epsilon) = \epsilon^q = 0$, $\Phi(\tilde{U})$ is spanned by

$$(\tilde{e}_1, \dots, \tilde{e}_{2m}) \begin{pmatrix} I_m \\ 0 \end{pmatrix} = (\tilde{e}_1, \dots, \tilde{e}_m).$$

Hence the condition $\tilde{U}' \subset \Phi(\tilde{U})$ is equivalent to

$$a_{ij} = \tilde{b}_j a_{im},\ 1 \leq i \leq m, 1 \leq j \leq m-1.$$

In particular $a_{mj} = \tilde{b}_j a_{mm}$. Together with the fact $A = {}^tA$, we can deduce $a_{jm} = a_{mj} = \tilde{b}_j a_{mm}$ for $1 \leq j \leq m$. Hence

$$a_{ij} = \tilde{b}_j a_{im} = \tilde{b}_j \tilde{b}_i a_{mm}$$

for all $1 \leq i, j \leq m$. In conclusion, a point in $\mathrm{Hom}(\mathcal{O}_{Y_V,z}, \bar{\kappa}[\epsilon]/(\epsilon^2))$ is determined by

$$c_1, \dots, c_{m-1}, a_{mm}.$$

This shows that $T_z(Y_V)$ of Y_V at z has dimension less or equal to m. □

Remark 3.3. *One can show that Y_V is in fact the blow-up of S_V along its singular locus.*

For later use, we record the following lemma.

Lemma 3.4. *Let $U_0 \in L_V(\bar{\kappa})$. Define a subvariety $L_V^{U_0}$ of $L_{V,\bar{\kappa}}$ by specifying its $\bar{\kappa}$-points*

$$L_V^{U_0}(\bar{\kappa}) = \{U \in L_V(\bar{\kappa}) \mid \dim_{\bar{\kappa}}(U \cap U_0) \geq m-1\}.$$

Then $\dim_{\bar{\kappa}} T_z(L_V^{U_0}) \leq m$ for any $z \in L_V^{U_0}(\bar{\kappa}) - \{U_0\}$.

Proof. Fix a point $z = U \in L_V^{U_0}(\bar{\kappa}) - \{U_0\}$. Take a basis $\{e_1, \dots, e_{m-1}\}$ of the isotropic subspace $U \cap U_0$. Enlarge it to a basis $\{e_1, \dots, e_m\}$ of U, and further to a basis $\{e_1, \dots, e_{2m}\}$ of $V_{\bar{\kappa}}$. We assume that $\langle e_i, e_{m+j} \rangle = \delta_{ij}$ for $1 \leq i, j \leq m$ and all other pairings between these basis vectors are zero. This implies that $U_0 = \mathrm{Span}_{\bar{\kappa}}\{e_1, \dots, e_{m-1}, ae_m + be_{2m}\}$ for some $a, b \in \bar{\kappa}$. By further adjust basis we can assume $a = 0$, $b = 1$.

Let $\tilde{V} := V \otimes_\kappa (\bar{\kappa}[\epsilon]/(\epsilon^2))$ and $\tilde{U}_0 := U_0 \otimes_{\bar{\kappa}} (\bar{\kappa}[\epsilon]/(\epsilon^2))$. Let $\tilde{e}_i := e_i \otimes 1 \in \tilde{V}$. The tangent space $T_z(L_V^{U_0})$ can be identified with $\mathrm{Hom}(\mathcal{O}_{L_V^{U_0},z}, \bar{\kappa}[\epsilon]/(\epsilon^2))$ which parameterizes lifts of U such that $\mathrm{rank}_{\bar{\kappa}[\epsilon]/(\epsilon^2)}(U \cap U_0) \geq m-1$. A lift $\tilde{U}$ of U to $T_z(L_V^{U_0})$ can be parameterized as follows. First $\tilde{U}$ can be spanned by vectors $x_1, \ldots, x_m$ defined by

$$(x_1, \ldots, x_m) = (\tilde{e}_1, \ldots, \tilde{e}_{2m}) \begin{pmatrix} I_m \\ A\epsilon \end{pmatrix}$$

where $A \in M_m(\bar{\kappa})$ and $A = {}^tA$ as $\tilde{U}$ is Lagrangian. A hyperplane in $\tilde{U}_0$ lifting $U \cap U_0$ is defined by one single equation

$$\tilde{e}_{2m}^* - \sum_{j=1}^{m-1} \epsilon b_j \tilde{e}_j^* = 0,$$

where $b_j \in \bar{\kappa}$. Hence $\tilde{U} \cap \tilde{U}_0$ is spanned by vectors

$$\tilde{e}_j + \epsilon b_j \tilde{e}_{2m},\ 1 \leq j \leq m-1.$$

If in addition the above vectors are in $\tilde{U}$, there must exists $\lambda_j^i \in \bar{\kappa}[\epsilon]/(\epsilon^2)$ ($1 \leq i \leq m$, $1 \leq j \leq m-1$) such that

$$\tilde{e}_j + \epsilon b_j \tilde{e}_{2m} = \sum_{i=1}^m \lambda_j^i x_i = \sum_{i=1}^m \lambda_j^i (\tilde{e}_i + \sum_{l=1}^m \epsilon a_{li} \tilde{e}_{m+l}).$$

Compare coefficients we see that $\lambda_j^i = \delta_{ij}$ and

$$a_{mj} = b_j,\ a_{li} = 0,$$

for $1 \leq j \leq m-1$, $1 \leq l, i \leq m-1$. Hence a point in $\mathrm{Hom}(\mathcal{O}_{L_V^{U_0},z}, \bar{\kappa}[\epsilon]/(\epsilon^2))$ is determined by

$$b_1, \ldots, b_{m-1}, a_{mm},$$

Hence the tangent space $T_z(L_V^{U_0})$ has dimension $\leq m$. □

3.2. **Minuscule cycle $\mathcal{N}_\Lambda$ and its tangent space.** In this subsection, we identify Λ with $b^{-1}(\Lambda)$ using the isomorphism of hermitian spaces (2.5).

Definition 3.5. For a vertex lattice $\Lambda \subset \mathbb{V}$ of type $t(\Lambda) = 2m$, define the subfunctor $\mathcal{N}_\Lambda$ to be the subfunctor of $\mathcal{N}$ such that for a $O_{\breve{F}}$-scheme S, $\mathcal{N}_\Lambda(S)$ is the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho, \mathcal{F})$ satisfying the following conditions.

(1) $(X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{Z}(\Lambda)(S)$.
(2) If $m \geq 1$, we require in addition that $x_*(\mathrm{Lie}(\mathcal{G} \times_{\mathrm{Spf}\, \mathcal{O}_{\breve{F}}} S)) \subset \mathcal{F}$ for any $x \in \Lambda$.

We first describe the $\bar{\kappa}$-points of $\mathcal{N}$ and $\mathcal{N}_\Lambda$.

Proposition 3.6. *There is a bijection between $\mathcal{N}_{\mathrm{red}}(\bar{\kappa})$ and the set of pairs of $\mathcal{O}_{\breve{F}}$-lattices (M, M') in N satisfying*

$$M^\sharp = M, \quad \Pi\tau(M) \subset M \subset \Pi^{-1}\tau(M), \quad M' \subset \tau^{-1}(M) \cap M, \quad \text{and} \quad \mathrm{length}(M/M') = 1.$$

Proof. Let $(X, \iota, \lambda, \rho, \mathcal{F})$ be a $\bar{\kappa}$-point of $\mathcal{N}$ and $M(X)$ be the Dieudonné module of X. Define $M = \rho(M(X)) \subset N$ and $M' = \rho(\mathrm{Pr}^{-1}(\mathcal{F})) \subset N$ where $\mathrm{Pr} : M(X) \to \mathrm{Lie}\, X = M(X)/\mathbf{V}M(X)$ is the natural quotient map. The condition $M^\sharp = M$ is equivalent to the fact that λ is a principal polarization. The condition $\Pi\tau(M) \subset M \subset \Pi^{-1}\tau(M)$ is equivalent to $\pi_0 M \subset \mathbf{V}M \subset M$. The condition $M' \subset \tau^{-1}(M) \cap M$ and $\mathrm{length}_{\bar{\kappa}}(M/M') = 1$ is equivalent to the condition

$$\mathbf{V}M \subset M' \subset M,\ \Pi M' \subset \mathbf{V}M,\ \dim_{\bar{\kappa}}(M/M') = 1.$$

Hence we have translated all conditions in the definition of $\mathcal{N}$ in term of Dieudonné modules. The proposition now follows from Dieudonné theory. $\square$

For a vertex lattice Λ in C, define

$$\breve{\Lambda} := \Lambda \otimes_{O_F} O_{\breve{F}},\ \breve{\Lambda}^\sharp := \Lambda^\sharp \otimes_{O_F} O_{\breve{F}}. \tag{3.3}$$

Corollary 3.7. *Let Λ be a vertex lattice in $\mathbb{V}$. There is a bijection between $\mathcal{N}_\Lambda(\bar{\kappa})$ and the set of pairs of $\mathcal{O}_{\breve{F}}$-lattices (M, M') in N satisfying the conditions in Proposition 3.6 and the following condition.*

(1) If $t(\Lambda) = 0$, then $M = \breve{\Lambda}$.
(2) If $t(\Lambda) \geq 2$, then $\breve{\Lambda} \subset M' \subset M$.

Proof. The condition that $\breve{\Lambda} \subset M$ is a direct translation of Condition (1) in Definition 3.5. When $t(\Lambda) = 0$, both M and $\breve{\Lambda}$ are self-dual, and thus $M = \breve{\Lambda}$. (2) is a direct consequence of Condition (2) in Definition 3.5. $\square$

To study the tangent space of $\mathcal{N}_\Lambda$, we recall the Grothendieck-Messing deformation theory of $\mathcal{N}$ from [How19, §3]. We remark here that although [How19] deals with the case $F_0 = \mathbb{Q}_p$, the argument in fact applies to general F_0 using the relative display theory of [ACZ16]. Let $R \in \mathrm{Nilp}\, O_{\breve{F}}$. For a strict O_{F_0}-module X over $\mathrm{Spec}\, R$, we denote by $D'(X)$ the Lie algebra of the universal vector extension of X (a vector extension of X is an extension of X by $\mathbb{G}_a^n$). Denote by $D(X)$ the subspace of $D'(X)$ on which O_{F_0} acts by the structural morphism $O_{F_0} \to \mathcal{O}_S$. A point $z \in \mathcal{N}(R)$ corresponds to a strict O_{F_0}-module (X, ι, λ) over R together with filtration $\mathcal{F} \subset \mathrm{Lie}\, X$ satisfying Definition 2.2. We have the following exact sequence of locally free R-modules

$$0 \to \mathrm{Fil}(X) \to D(X) \to \mathrm{Lie}\, X \to 0, \tag{3.4}$$

where $\mathrm{Fil}(X)$ and $\mathrm{Lie}\, X$ are of rank n and $D(X)$ is of rank $2n$. The principal polarization λ induces a symplectic form $\langle\, ,\, \rangle$ on $D(X)$ such that

$$\langle \iota(a)x, y\rangle = \langle x, \iota(\bar{a})y\rangle$$

for all $a \in O_F$ and $x, y \in D(X)$. With respect to $\langle\, ,\, \rangle$ the Hodge filtration $\mathrm{Fil}(X)$ is maximal isotropic. Hence $\langle\, ,\, \rangle$ induces a perfect pairing (still denoted by $\langle\, ,\, \rangle$):

$$\langle\, ,\, \rangle : \mathrm{Fil}(X) \times \mathrm{Lie}\, X \to R. \tag{3.5}$$

The submodule $\mathcal{F} \subset \mathrm{Lie}\, X$ and its perpendicular complement $\mathcal{F}^\perp$ (which is locally a direct summand of $\mathrm{Fil}(X)$ of rank one) with respect to (3.5) determine each other. The condition on $\mathcal{F}$ in

Definition 2.2 is

$$O_F \text{ acts on } \mathcal{F} \text{ by } O_F \xrightarrow{\bar{i}} O_{\breve{F}} \to \mathcal{O}_S \text{ and on } \operatorname{Lie} X/\mathcal{F} \text{ by } O_F \xrightarrow{i} O_{\breve{F}} \to \mathcal{O}_S. \tag{3.6}$$

This is equivalent to the condition that O_F acts on $\mathcal{F}^\perp$ by $O_F \xrightarrow{\bar{i}} O_{\breve{F}} \to \mathcal{O}_S$ and on $\mathrm{Fil}(X)/\mathcal{F}^\perp$ by $O_F \xrightarrow{i} O_{\breve{F}} \to \mathcal{O}_S$. Since O_{F_0} acts on $D(X)$ by i_0 and $O_F = O_{F_0}[\pi]$, (3.6) is further equivalent to

$$(\Pi+\pi)\cdot\mathcal{F}^\perp = 0,\ (\Pi-\pi)\cdot \mathrm{Fil}(X) \subset \mathcal{F}^\perp, \tag{3.7}$$

where we use Π to denote the action $\iota(\pi)$ on $D(X)$.

Definition 3.8. Let $\mathscr{C}$ be the following category. Objects in $\mathscr{C}$ are triples $(\mathcal{O}, \mathcal{O}\to\bar{\kappa}, d)$ where $\mathcal{O}$ is an Artinian $O_{\breve{F}}$-algebra, $\mathcal{O}\to\bar{\kappa}$ is an $O_{\breve{F}}$-algebra homomorphism, and d is a nilpotent divided power structure on $\mathrm{Ker}(\mathcal{O}\to\bar{\kappa})$. Morphisms in $\mathscr{C}$ are $O_{\breve{F}}$-algebra homomorphisms compatible with structure maps to $\bar{\kappa}$ and divided power structures.

Let $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{Z}(\Lambda)(\bar{\kappa})$ and $M = \rho(M(X)) \subset N$. Then $\Lambda \subset M$ by Corollary 3.7. We can identify (3.4) with

$$0 \to \mathbf{V}M/\pi_0 M \to M/\pi_0 M \to M/\mathbf{V}M \to 0.$$

Let $\mathcal{F}^\perp \subset \mathbf{V}M/\pi_0 M$ be the perpendicular complement of $\mathcal{F}$ as described above. Denote by $\mathcal{Z}(\Lambda)_z$ (resp. $\mathcal{N}_{\Lambda,z}$) the completion of $\mathcal{Z}(\Lambda)$ (resp. $\mathcal{N}_\Lambda$) at z. For any $\mathcal{O}\in\mathscr{C}$ and $\tilde{z} = (\tilde{X}, \cdots) \in \mathcal{Z}(\Lambda)_z(\mathcal{O})$, we can identify $D(\tilde{X})$ with $M_{\mathcal{O}} := M \otimes_{O_{\breve{F}_0}} \mathcal{O}$ and by Grothendieck-Messing theory $\tilde{z}$ corresponds to a filtration of free $\mathcal{O}$-module direct summands

$$\tilde{\mathcal{F}}^\perp \subset \widetilde{\mathrm{Fil}} \subset M_{\mathcal{O}},$$

which lifts the filtration $\mathcal{F}^\perp \subset \mathrm{Fil} \subset M_{\bar{\kappa}} = M/\pi_0 M$. Let $f_{\mathcal{O}}$ be the map

$$f_{\mathcal{O}} : \tilde{z} \mapsto (\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}}). \tag{3.8}$$

Lemma 3.9. *Let the notations be as above. Denote by $\Lambda_{M,\mathcal{O}}$ the image of the composition of maps $\breve{\Lambda} \to M \to M_{\mathcal{O}}$, and let $\Lambda^\perp_{M,\mathcal{O}}$ be its perpendicular complement in $M_{\mathcal{O}}$ under the alternating form $\langle\, ,\rangle$.*

(1) The map $f_{\mathcal{O}}$ defines a bijection from $\mathcal{Z}(\Lambda)_z(\mathcal{O})$ to the set consisting of pairs $(\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}})$ lifting $(\mathcal{F}^\perp, \mathrm{Fil})$ satisfying the following conditions:

- *(a) $\tilde{\mathcal{F}}^\perp$ and $\widetilde{\mathrm{Fil}}$ are free $\mathcal{O}$-module direct summands of $M_{\mathcal{O}}$ of rank 1 and n respectively and $\tilde{\mathcal{F}}^\perp \subset \widetilde{\mathrm{Fil}}$;*
- *(b) $\widetilde{\mathrm{Fil}}$ is isotropic with respect to $\langle\, ,\rangle$;*
- *(c) $(\Pi+\pi)\cdot\tilde{\mathcal{F}}^\perp = 0$ and $(\Pi-\pi)\cdot\widetilde{\mathrm{Fil}} \subset \tilde{\mathcal{F}}^\perp$;*
- *(d) $\widetilde{\mathrm{Fil}}$ contains $\widetilde{\mathrm{Fil}}^- := (\Pi+\pi)\cdot\Lambda_{M,\mathcal{O}}$.*

(2) The restriction of $f_{\mathcal{O}}$ to $\mathcal{N}_{\Lambda,z}(\mathcal{O})$ defines a bijection from $\mathcal{N}_{\Lambda,z}(\mathcal{O})$ to the set consisting of pairs $(\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}})$ satisfying the above conditions together with the extra condition:

- *(e) $\tilde{\mathcal{F}}^\perp \subset \Lambda^\perp_{M,\mathcal{O}}$.*

Proof. Proof of (1): By the previous discussion, $(\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}})$ satisfies conditions (a), (b) and (c) for any $\tilde{z} \subset \mathcal{N}_z(\mathcal{O})$ ($\mathcal{N}_z$ is the completion of $\mathcal{N}$ at z). Conversely by Grothendieck-Messing theory any pair $(\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}})$ lifting $(\mathcal{F}^\perp, \mathrm{Fil})$ satisfying (a), (b) and (c) gives rise to a unique point $\tilde{z} \in \mathcal{N}_z(\mathcal{O})$. Since the action of O_F on $\operatorname{Lie}\mathcal{G}$ is via the inclusion $\bar{i}$, the Hodge filtration of $\mathcal{G}_{\mathcal{O}}$ is $\mathrm{Span}_{\mathcal{O}}\{(\Pi+\pi)\cdot e\otimes 1\}$ where e is a generator of $M(\mathbb{Y})$ as in §2.2. The image of the $\mathrm{Span}_{\mathcal{O}}\{(\Pi+\pi)\cdot e\otimes 1\}$ under Λ in $M_{\mathcal{O}}$ is exactly $\widetilde{\mathrm{Fil}}^{-}$. By Grothendick-Messing theory again, $\tilde{z}\in\mathcal{Z}(\Lambda)_z(\mathcal{O})$ if and only if condition (d) holds.

(2) is a corollary of (1). For any $\tilde{z}=(\tilde{X},\dots,\tilde{\mathcal{F}})\in\mathcal{Z}(\Lambda)_z(\mathcal{O})$, let $\tilde{\mathcal{F}}'$ be the preimage of $\tilde{\mathcal{F}}$ under the quotient map $M_{\mathcal{O}}\to M_{\mathcal{O}}/\widetilde{\mathrm{Fil}}$. Condition (2) in Definition 3.5 is equivalent to $\Lambda_{M,\mathcal{O}}\subset\tilde{\mathcal{F}}'$. The perpendicular complement of $\tilde{\mathcal{F}}'$ with respect to $\langle\,,\,\rangle$ is $\tilde{\mathcal{F}}^{\perp}$. Hence condition (2) in Definition 3.5 is equivalent to condition (e). Hence $\tilde{z}\in\mathcal{N}_{\Lambda,z}(\mathcal{O})$ if and only if (e) is satisfied. This finishes the proof of the lemma. ☐

Lemma 3.10. *Let Λ be a vertex lattice of type $2m$ in C and $M\subset N$ be an $O_{\breve{F}}$ lattice such that $\Lambda\subset M$ and $M=M^{\sharp}$. Then there is an $O_{\breve{F}}$-basis $\{e_1,\dots,e_n\}$ of M such that*

$$(e_\alpha,e_{\alpha+m})=1,\ (e_\mu,e_\mu)\in O_{\breve{F}}^{\times}$$

for $1\le\alpha\le m$, $2m+1\le\mu\le n$, the inner product $(\,,\,)$ between any other basis vectors is zero, and

$$\breve{\Lambda}=\mathrm{Span}_{O_{\breve{F}}}\{\Pi e_1,\dots,\Pi e_m,e_{m+1},\dots,e_n\}.$$

Proof. By assumption we have $\Pi M\subset\Pi\breve{\Lambda}^{\sharp}\subset\breve{\Lambda}\subset M$ and $\dim_{\bar{\kappa}}(M/\breve{\Lambda})=m$. With respect to the $\bar{\kappa}$-valued quadratic form $(\,,\,)\pmod{\pi}$ on $M/\Pi M$, $\breve{\Lambda}/\Pi M$ has a decomposition

$$\breve{\Lambda}/\Pi M=R\ominus W,$$

where R is totally isotropic and W is non-degenerate. Then by the nondegeneracy of $(\,,\,)\pmod{\pi}$ on $M/\Pi M$ we know that there is a totally isotropic subspace R' such that

$$M/\Pi M=(R'\oplus R)\ominus W,$$

and $(\,,\,)\pmod{\pi}$ induces a perfect pairing between R and R'. Hence we can find a basis $\{\bar{e}_1,\bar{e}_n\}$ of $M/\Pi M$ such that $R'=\langle\bar{e}_1,\dots,\bar{e}_m\rangle$, $R=\langle\bar{e}_{m+1},\dots,\bar{e}_{2m}\rangle$, $W=\langle\bar{e}_{2m+1},\dots,\bar{e}_n\rangle$, and

$$(\bar{e}_\alpha,\bar{e}_{\alpha+m})=1\pmod{\pi},\ (\bar{e}_\mu,\bar{e}_\mu)\pmod{\pi}\in\bar{\kappa}^{\times}$$

for $1\le\alpha\le m$ and $2m+1\le\mu\le n$ and the pairing between all other basis vectors are zero. We can lift the above basis to a basis $\{e_1,\dots,e_n\}$ of M which will satisfy the assumptions of the lemma. ☐

Proposition 3.11. *The scheme $\mathcal{Z}(\Lambda)$ has no $O_{\breve{F}}/(\pi^2)$-point. In other words, $\mathcal{Z}(\Lambda)$ is a subscheme of $\mathcal{N}\times_{\mathrm{Spf}\,O_{\breve{F}}}\mathrm{Spec}\,\bar{\kappa}$.*

Proof. Let $\mathcal{O}=O_{\breve{F}}/(\pi^2)$ with the reduction map $\mathcal{O}\to\bar{\kappa}$ and the natural divided power structure on $\pi\mathcal{O}$. Then $\mathcal{O}\in\mathscr{C}$. Let $z=(X,\iota,\lambda,\rho,\mathcal{F})\in\mathcal{Z}(\Lambda)(\bar{\kappa})$ and $M=\rho(M(X))\subset N$ as in Proposition 3.6. Then by Corollary 3.7 $\breve{\Lambda}\subset M$, and we can assume there is an $O_{\breve{F}}$-basis $\{e_1,\dots,e_n\}$ of M as in Lemma 3.10. Denote the image of e_i in $M_{\mathcal{O}}$ still by e_i. Then $\{e_1,\dots,e_n,\Pi e_1,\dots,\Pi e_n\}$ is an $\mathcal{O}$-basis of $M_{\mathcal{O}}$. With respect to the alternating form $\langle\,,\,\rangle$, we have by (2.2)

$$\langle e_\alpha,\Pi e_{m+\alpha}\rangle=-1/\delta,\ \langle e_{m+\alpha},\Pi e_\alpha\rangle=-1/\delta,\ \langle e_\mu,\Pi e_\mu\rangle\in O_{\breve{F}_0}^{\times},\tag{3.9}$$

for $1\le\alpha\le m$, $2m+1\le\mu\le n$, and all other pairings between basis vectors are zero.

Assume that z can be lifted to a point $\tilde{z} \in \mathcal{Z}(\Lambda)_z(\mathcal{O})$, which corresponds to a pair $(\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}})$ as in Lemma 3.9. First notice that

$$\widetilde{\mathrm{Fil}}^- = (\Pi+\pi)\cdot\Lambda_{M,\mathcal{O}} = \mathrm{Span}_{O_{\breve{F}}\otimes O_{\breve{F}_0}\mathcal{O}}\{\pi\Pi e_1,\dots,\pi\Pi e_m,(\Pi+\pi)e_{m+1},\dots,(\Pi+\pi)e_n\}. \tag{3.10}$$

With respect to the alternating form $\langle\,,\,\rangle$, its perpendicular complement $(\widetilde{\mathrm{Fil}}^-)^\perp$ in $\Lambda_{M,\mathcal{O}}$ is generated by

$$\{(\Pi+\pi)e_1,\dots(\Pi+\pi)e_m,\Pi e_{m+1},\pi e_{m+1},\dots,\Pi e_{2m},\pi e_{2m},(\Pi+\pi)e_{2m+1},\dots,(\Pi+\pi)e_n\}. \tag{3.11}$$

By Lemma 3.9 (c), $\tilde{\mathcal{F}}^\perp$ is annihilated by $\Pi+\pi$, hence it is spanned by a vector

$$v=\sum_{i=1}^n a_i(\Pi-\pi)e_i,$$

where $a_i\in\mathcal{O}^\times$ for some i as $\tilde{\mathcal{F}}^\perp$ is a direct summand of $M_{\mathcal{O}}$. By Lemma 3.9, we must have $\widetilde{\mathrm{Fil}}^-\subset\widetilde{\mathrm{Fil}}$, $\tilde{\mathcal{F}}^\perp\subset\widetilde{\mathrm{Fil}}$ and $\widetilde{\mathrm{Fil}}$ is isotropic. Hence $\widetilde{\mathrm{Fil}}\subset(\widetilde{\mathrm{Fil}}^-)^\perp$. Moreover $\langle\widetilde{\mathrm{Fil}}^-,\tilde{\mathcal{F}}^\perp\rangle=0$, which implies $a_i\in\pi\mathcal{O}$ for $1\le i\le m$ and $2m+1\le i\le n$. Hence without loss of generality we can assume that $a_{m+1}=1$.

Since $\widetilde{\mathrm{Fil}}$ is a direct summand of $M_{\mathcal{O}}$ we have $M_{\mathcal{O}}=\widetilde{\mathrm{Fil}}\oplus S$ where S is an $\mathcal{O}$-module. We can write $\Pi e_1=w+w'$ where $w\in\widetilde{\mathrm{Fil}}$ and $w'\in S$. Since $\pi\Pi e_1\in\widetilde{\mathrm{Fil}}^-\subset\widetilde{\mathrm{Fil}}$, we must have $\pi w'=0$. This implies that $w'\in\pi M_{\mathcal{O}}$ and w is of the form

$$w=(\Pi+b\pi)e_1+x$$

where $b\in\mathcal{O}$ and $x\in\pi\cdot\mathrm{Span}_{\mathcal{O}}\{e_2,\Pi e_2,\dots,e_n,\Pi e_n\}$. Since $w\in\widetilde{\mathrm{Fil}}\subset(\widetilde{\mathrm{Fil}}^-)^\perp$, by (3.11), we must have $b=1$ and x is of the form

$$x=\sum_{i=2}^m d_i(\Pi+\pi)e_i+\sum_{i=m+1}^{2m}(c_i+d_i\Pi)e_i+\sum_{i=2m+1}^n d_i(\Pi+\pi)e_i,$$

where $c_i\in\pi\mathcal{O}$ for $m+1\le i\le 2m$ and $d_i\in\pi\mathcal{O}$ for $2\le i\le n$. Since $(\Pi+\pi)e_i\in\widetilde{\mathrm{Fil}}$ for $2m+1\le i\le n$, by changing w and x at the same time if necessary we can assume that $d_i=0$ for $2m+1\le i\le n$. By (3.9), we have

$$\langle(\Pi+\pi)e_1,(\Pi-\pi)e_{m+1}\rangle=2\pi\langle e_1,\Pi e_{m+1}\rangle\neq0.$$

Moreover

$$\langle x,v\rangle=\sum_{i=2}^m\langle d_i(\Pi+\pi)e_i,a_{m+i}(\Pi-\pi)e_{m+i}\rangle+\sum_{i=m+1}^{2m}\langle(c_i+d_i\Pi)e_i,a_{i-m}(\Pi-\pi)e_{i-m}\rangle=0$$

Here we have used the fact that $a_i\in\pi\mathcal{O}$ for $1\le i\le m$, $c_i\in\pi\mathcal{O}$ for $m+1\le i\le 2m$ and $d_i\in\pi\mathcal{O}$ for $2\le i\le 2m$. Then $\langle w,v\rangle\neq0$ which contradicts the fact that $\widetilde{\mathrm{Fil}}$ is isotropic. Hence there is no lift of z into $\mathcal{Z}(\Lambda)(\mathcal{O})$. This proves the lemma. □

As $\mathcal{N}_\Lambda$ is a formal subscheme of $\mathcal{Z}(\Lambda)$, the following corollary is immediate.

Corollary 3.12. *$\mathcal{N}_\Lambda$ is a formal subscheme of $\mathcal{N}\times_{\mathrm{Spf}\,\mathcal{O}_{\breve{F}}}\mathrm{Spec}\,\bar{\kappa}$.*

Proposition 3.13. *Let Λ be a vertex lattice of type $2m$ ($m \geq 1$) in $\mathbb{V}$ and $z \in \mathcal{N}_\Lambda(\bar{\kappa})$. Then the tangent space $T_z(\mathcal{N}_\Lambda)$ has dimension less or equal to m.*

Proof. Let $z = (X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}_\Lambda(\bar{\kappa})$ and $M = \rho(M(X)) \subset N$ as in Proposition 3.6. Let $\mathcal{O} = \bar{\kappa}[\epsilon]/(\epsilon^2)$, then $\mathcal{O}$ is an $O_{\breve{F}}$-algebra through the map $O_{\breve{F}} \to \bar{\kappa} \to \mathcal{O}$ and the ideal $(\epsilon) \subset \mathcal{O}$ is equipped with a natural divided power structure. Then $\mathcal{O} \in \mathscr{C}$. Any point $\tilde{z} \in T_z(\mathcal{N}_\Lambda) = \mathcal{N}_{\Lambda,z}(\mathcal{O})$ corresponds to a unique pair $(\tilde{\mathcal{F}}^\perp, \widetilde{\mathrm{Fil}})$ lifting $(\mathcal{F}^\perp, \mathrm{Fil})$ as in Lemma 3.9. We prove the lemma in two cases.

Case (a): $\mathrm{Fil} \neq \Pi \cdot M_{\bar{\kappa}}$. Since $M_\mathcal{O}$ is a free $O_F \otimes_{O_{F_0}} \mathcal{O}$-modules of rank n, we have the following exact sequence

$$0 \to \Pi \cdot M_\mathcal{O} \to M_\mathcal{O} \xrightarrow{\Pi} \Pi \cdot M_\mathcal{O} \to 0,$$

where $\Pi \cdot M_\mathcal{O}$ is a free $\mathcal{O}$-module of rank n and the first arrow is the natural injection. This implies the following sequence is exact.

$$0 \to (\Pi \cdot M_\mathcal{O}) \cap \widetilde{\mathrm{Fil}} \to \widetilde{\mathrm{Fil}} \xrightarrow{\Pi} \Pi \cdot \widetilde{\mathrm{Fil}} \to 0. \tag{3.12}$$

Since $\widetilde{\mathrm{Fil}} \neq \Pi \cdot M_\mathcal{O}$, by (3.12) we know that $\Pi \cdot \widetilde{\mathrm{Fil}} \neq \{0\}$. Since by Lemma 3.9, $\Pi \cdot \widetilde{\mathrm{Fil}} \subset \tilde{\mathcal{F}}^\perp$ and $\tilde{\mathcal{F}}^\perp$ has rank 1, we know that $\Pi \cdot \widetilde{\mathrm{Fil}} = \tilde{\mathcal{F}}^\perp$ by Nakayama's lemma. Hence $\tilde{\mathcal{F}}^\perp$ is determined by $\widetilde{\mathrm{Fil}}$. Moreover $\widetilde{\mathrm{Fil}}$ is determined by its image in the $\mathcal{O}$-module $(\widetilde{\mathrm{Fil}}^-)^\perp/\widetilde{\mathrm{Fil}}^-$ where $\widetilde{\mathrm{Fil}}^- = \Pi \cdot \Lambda_{M,\mathcal{O}}$ as in Lemma 3.9. Equation (3.10) is still true and implies that $\widetilde{\mathrm{Fil}}^-$ is an isotropic free $\mathcal{O}$-module direct summand of $M_\mathcal{O}$ of rank $n - m$ (notice that $\pi = 0$ in $\mathcal{O}$). Moreover

$$(\widetilde{\mathrm{Fil}}^-)^\perp = \mathrm{Span}_\mathcal{O}\{\Pi e_1, \dots, \Pi e_m, \Pi e_{m+1}, e_{m+1}, \dots, \Pi e_{2m}, e_{2m}, \Pi e_{2m+1}, \dots, \Pi e_n\}, \tag{3.13}$$

and $(\widetilde{\mathrm{Fil}}^-)^\perp/\widetilde{\mathrm{Fil}}^- = V_\mathcal{O}$ where $V = \Pi \cdot \breve{\Lambda}^\sharp/\Pi \cdot \breve{\Lambda}$ is a symplectic vector space over $\bar{\kappa}$ of dimension $2m$. The proposition in this case follows from Lemma 3.4 if we define $U_0 = \Pi \cdot M/\Pi \cdot \breve{\Lambda} \subset V$.

Case (b): $\mathrm{Fil} = \Pi \cdot M_{\bar{\kappa}}$. By Lemma 3.10, we know $\Pi \cdot M_\mathcal{O} \subset \Lambda_{M,\mathcal{O}}$ and $\Lambda_{M,\mathcal{O}}$ is a free $\mathcal{O}$-module direct summand of $M_\mathcal{O}$ of corank m. Hence $(\Lambda_{M,\mathcal{O}})^\perp$ is a free $\mathcal{O}$-module of rank m and is in $(\Pi \cdot M_\mathcal{O})^\perp = \Pi \cdot M_\mathcal{O}$. As in [Krä03], we assume that we have a $\mathcal{O} \otimes_{O_{F_0}} O_F$-basis $\{e_1, \dots, e_n\}$ of $M_\mathcal{O}$ such that $\langle e_i, \Pi \cdot e_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$ and all other pairings between these basis vectors are zero. The lift $\widetilde{\mathrm{Fil}}$ is spanned by $x_1, \dots, x_n$ where

$$(x_1, \dots, x_n) = (e_1, \dots, e_n, \Pi e_1, \dots, \Pi e_n) \begin{pmatrix} A\epsilon \\ I_n \end{pmatrix}$$

where $A \in M_n(\bar{\kappa})$ and $A = {}^tA$ since $\widetilde{\mathrm{Fil}}$ is isotropic. Assume $\mathcal{F}^\perp \subset \mathrm{Fil} = \Pi \cdot M_{\bar{\kappa}}$ is spanned by

$$\sum_{i=1}^n b_{n+i} \Pi \cdot e_i.$$

Then $b_{n+i} \neq 0$ for some i and we can assume without loss of generality $b_{n+1} = 1$. The lift $\tilde{\mathcal{F}}^\perp$ is spanned by

$$\sum_{i=1}^n \tilde{b}_i e_i + \sum_{i=n+1}^{2n} \tilde{b}_i \Pi \cdot e_i,$$

where $\tilde{b}_{n+1} = 1$ and $\tilde{b}_{n+i} = b_{n+i} + \epsilon c_i$ for $2 \leq i \leq n$ and some $c_i \in \bar{\kappa}$. Let

$$\lambda = {}^t(\tilde{b}_{n+1}, \ldots, \tilde{b}_{2n}).$$

Equations (4.7), (4.8) and (4.10) of [Krä03] tell us that

$$A = \gamma_1 \lambda \cdot {}^t\lambda$$

for some $\gamma_1 \in \bar{\kappa}$. Equation (4.5) of loc.cit. tells us

$${}^t(\tilde{b}_1, \ldots, \tilde{b}_n) = A\lambda,$$

which is equal to $\gamma_1 \lambda \cdot {}^t\lambda \cdot \lambda = 0$ as ${}^t\lambda \cdot \lambda = 0$ by (4.9) of loc.cit.. In particular $\tilde{\mathcal{F}}^\perp \subset \Pi M_{\mathcal{O}}$ and a point in $T_z(\mathcal{N}_\Lambda)$ is determined by the $n-1$ parameters c_i for $2 \leq i \leq n$ together with the additional parameter γ_1. Now the condition $\tilde{\mathcal{F}}^\perp \subset (\Lambda_{M,\mathcal{O}})^\perp$ (condition (e) of Lemma 3.9) imposes further $n-m$ independent linear equations on the parameters c_i for $2 \leq i \leq n$. This shows that the tangent space $T_z(\mathcal{N}_\Lambda)$ has dimension less or equal to m. This finishes the proof of the proposition. □

3.3. **Isomorphism between $\mathcal{N}_\Lambda$ and $Y_{V,\bar{\kappa}}$.** By [RTW14, Lemma 6.1], the lattices $\breve{\Lambda}$ and $\breve{\Lambda}^\sharp$ (see (3.3)) are closed under Π, $\mathbf{V}$ and $\mathbf{F}$, hence determine formal hermitian O_F-modules (see Definition 2.1) X_- and X_+ (denoted by X_{Λ^-} and X_{Λ^+} resp. in [RTW14, §6]) of dimension n over $\bar{\kappa}$ together with quasi-isogenies $\rho_- : X_- \to \mathbb{X}$, $\rho_+ : X_+ \to \mathbb{X}$. The inclusion $\breve{\Lambda} \subset \breve{\Lambda}^\sharp$ also defines an isogeny $\rho_\Lambda : X_- \to X_+$ of height $2m$. Both ρ_- and ρ_+ have height m. Since $X_- \cong \mathbb{Y}^n$ as an O_F-module, for any $\bar{\kappa}$-scheme S, condition (1) in Definition 3.5 is equivalent to the condition

$$(3.14) \qquad (1)' : \text{ The quasi-isogeny } \rho_{X,-} := \rho^{-1} \circ (\rho_-)_S : (X_-)_S \to X \text{ is an isogeny.}$$

This is further equivalent by loc.cit. to the condition

$$(3.15) \qquad (1)'' : \text{ The quasi-isogeny } \rho_{X,+} := (\rho_+)_S^{-1} \circ \rho : X \to (X_+)_S \text{ is an isogeny.}$$

Lemma 3.14. *The functor $\mathcal{N}_\Lambda$ is representable by a projective scheme over $\bar{\kappa}$. The functor monomorphism $\mathcal{N}_\Lambda \to \mathcal{N}$ is a closed immersion.*

Proof. By Proposition 3.11, $\mathcal{Z}(\Lambda)$ is defined over $\bar{\kappa}$. It is clearly a closed formal subscheme of $\mathcal{N}$. Since for any $\bar{\kappa}$-scheme S, condition (1) in Definition 3.5 is equivalent to (3.14), the functor $\mathcal{Z}(\Lambda)$ can be represented by a projective scheme over $\bar{\kappa}$ by exact the same argument as that of [VW11, Lemma 3.2]. Condition (2) defines $\mathcal{N}_\Lambda$ as a closed subscheme of $\mathcal{Z}(\Lambda)$, hence is itself projective over $\bar{\kappa}$ and a closed formal subscheme of $\mathcal{N}$. This finishes the proof of the lemma. □

In the following discussion we assume that Λ has type greater or equal to 2. Let $V = \Lambda^\sharp/\Lambda$ and define a symplectic form $\langle\, ,\, \rangle_V$ on V as follows. For $\bar{x}, \bar{y} \in V$ with lifts $x, y \in \Lambda^\sharp$, define $\langle \bar{x}, \bar{y} \rangle_V$ by the image of $\pi_0 \delta \langle x, y \rangle$ in $\mathbb{F}_q$ (see §2.2). Extend this form bilinearly to $V_{\bar{\kappa}}$. Note that τ induces identity on V and the Frobenius Φ on $V_{\bar{\kappa}}$. Let R be a $\bar{\kappa}$-algebra and $(X, \iota, \lambda, \rho, \mathcal{F}) \in \mathcal{N}_\Lambda(R)$. As in the proof of[VW11, Corollary 3.9], $\mathrm{Image}(D(\rho_{X,+}))$ and $\mathrm{Image}(D((\rho_\Lambda)_R))$ are locally free direct summands of $D((X_+)_R)$ of corank m and $2m$ respectively. Define

$$U(X) := \mathrm{Image}(D(\rho_{X,+}))/\mathrm{Image}(D((\rho_\Lambda)_R)).$$

Then $U(X)$ is a locally free direct summand of $D((X_+)_R)/\mathrm{Image}(D((\rho_\Lambda)_R)) \cong \breve{\Lambda}^\sharp/\breve{\Lambda} \otimes_{\bar{\kappa}} R = V_R$ of corank m. Define

$$\mathcal{F}(X) := (D(\rho_{X,+}))(q_X^{-1}(\mathcal{F}))/\mathrm{Image}(D((\rho_\Lambda)_R)),$$

where $q_X : D(X) \to \mathrm{Lie}\, X$ is the natural quotient homomorphism of R-modules. Then $\mathcal{F}(X)$ is a locally free direct summand of $U(X)$ of rank $m-1$. We define a map $\phi : \mathcal{N}_\Lambda \to \mathrm{Gr}(m, V_{\bar{\kappa}}) \times \mathrm{Gr}(m-1, V_{\bar{\kappa}})$ by

$$\phi : (X, \iota, \lambda, \rho, \mathcal{F}) \mapsto (U(X), \mathrm{Span}_R\{\Phi(\mathcal{F}(X))\}) \in (\mathrm{Gr}(m, V_{\bar{\kappa}}) \times \mathrm{Gr}(m-1, V_{\bar{\kappa}}))(R).$$

Lemma 3.15. *ϕ defines a bijection between $\mathcal{N}_\Lambda(\bar{\kappa})$ and $Y_V(\bar{\kappa})$.*

Proof. A point $z \in \mathcal{N}_\Lambda(\bar{\kappa})$ corresponds to a pair (M, M') as in Proposition 3.6. By definition we have

$$\phi(z) = (U, U') := (M/\breve{\Lambda}, \Phi(M'/\breve{\Lambda})).$$

By the definition of $\langle\, ,\, \rangle_V$, the condition $M = M^\sharp$ is equivalent to the fact that U is Lagrangian. The condition $M' \subset \tau^{-1}(M) \cap M$ is equivalent to $U' \subset U \cap \Phi(U)$. This shows that $\phi(z) \in Y_V(\bar{\kappa})$.

Conversely assume $(U, U') \in Y_V(\bar{\kappa})$ and let $M = \mathrm{Pr}^{-1}(U)$ and $M' = \mathrm{Pr}^{-1}(\Phi^{-1}(U'))$ where $\mathrm{Pr} : \breve{\Lambda}^\sharp \to \breve{\Lambda}^\sharp/\breve{\Lambda}$ is the natural quotient map. Then we have

$$\Pi\tau(M) \subset \Pi\tau(\breve{\Lambda}^\sharp) = \Pi\breve{\Lambda}^\sharp \subset \breve{\Lambda} \subset M,$$

and

$$M \subset \breve{\Lambda}^\sharp \subset \Pi^{-1}\breve{\Lambda} = \Pi^{-1}\tau\breve{\Lambda} \subset \Pi^{-1}\tau M. \tag{3.16}$$

This shows that (M, M') satisfies the conditions in Proposition 3.6 and Corollary 3.7. This defines the inverse of ϕ on the level of $\bar{\kappa}$-points. Hence ϕ defines a bijection between $\mathcal{N}_\Lambda(\bar{\kappa})$ and $Y_V(\bar{\kappa})$. □

Theorem 3.16. *Let Λ be a vertex lattice of type $2m$ ($m \geq 1$) in $\mathbb{V}$. The morphism ϕ defines an isomorphism $\mathcal{N}_\Lambda \to Y_{V,\bar{\kappa}}$. In particular $\mathcal{N}_\Lambda$ is smooth of dimension m over $\bar{\kappa}$.*

Proof. Let $\mathcal{N}_\Lambda^{\mathrm{red}}$ be the underlying reduced $\bar{\kappa}$-scheme of $\mathcal{N}_\Lambda$. Lemma 3.15 shows that ϕ induces a morphism $\phi^{\mathrm{red}} : \mathcal{N}_\Lambda^{\mathrm{red}} \to Y_{V,\bar{\kappa}}$. In fact the argument of Lemma 3.15 shows that ϕ^{red} is bijection on any κ'-valued points for any perfect extension κ' of $\bar{\kappa}$. Using the theory of relative display and windows developed in [ACZ16], we can show that ϕ^{red} induces a bijection on κ'-valued points for any field extension κ' of $\bar{\kappa}$. Hence ϕ^{red} is an isomorphism on fibers, in particular quasi-finite. Since it is a morphism between projective varieties, it is proper. Hence ϕ^{red} is finite by (the Grothendieck version of) Zariski's main theorem. As $\mathcal{N}_\Lambda^{\mathrm{red}}$ is reduced, [Bou, §3 $n^0 2$ Proposition 7] implies that ϕ^{red} is flat. As ϕ^{red} is finite, flat and an isomorphism on fibers, we can use Lemma [Krä03, Lemma 4.4] to conclude that ϕ^{red} is an isomorphism. In particular $\mathcal{N}_\Lambda^{\mathrm{red}}$ is a smooth $\bar{\kappa}$-variety of dimension m. Now Proposition 3.13 shows that $\mathcal{N}_\Lambda^{\mathrm{red}} = \mathcal{N}_\Lambda$, this finishes the proof of the theorem. □

Proposition 3.17. *Let Λ be a vertex lattice of type 0 in $\mathbb{V}$. Then $\mathcal{N}_\Lambda$ is the exceptional divisor Exc_Λ and is isomorphic to $\mathbb{P}^{n-1}_{\bar{\kappa}}$.*

Proof. Let R be any $\bar{\kappa}$-algebra and z be any point in $\mathcal{N}_\Lambda(R)$ and $(X, \iota, \lambda, \rho, \mathcal{F})$ be the pullback of the universal object of $\mathcal{N}$ to z. As Λ is a self-dual lattice, the quasi-isogeny ρ_- has height 0. Thus the isogeny

$$\rho_{X,-} = \rho^{-1} \circ (\rho_-)_R : (X_-)_R \to X$$

has height 0 and is an isomorphism, hence we can identify $(X, \ldots, \rho)$ with $((X_-)_R, \ldots, (\rho_-)_R)$. As $\pi|_\Lambda = \mathbf{V}_\Lambda$ for any vertex lattice Λ, and $\operatorname{Lie} X_- = M(X_-)/\mathbf{V}M(X_-)$, the action of $\iota(\pi)$ on $\operatorname{Lie} X_-$ is trivial. The point z is uniquely determined by the filtration $\mathcal{F} \subset \operatorname{Lie} X$. Hence $\mathcal{F}$ can be any rank $n-1$ locally free R-module on $\operatorname{Lie} X$. This shows that $\mathcal{N}_\Lambda$ is isomorphic to $\mathbb{P}^{n-1}_{\bar{\kappa}}$ and is in particular reduced. Moreover if $R = \bar{\kappa}$, then $\rho(M(X)) = \breve{\Lambda}$. This shows that $\mathcal{N}_\Lambda$ is a subscheme of $\operatorname{Exc}_\Lambda$ according Definition 2.4. By the proof of Lemma 2.5, we know that $\mathcal{N}_\Lambda$ and $\operatorname{Exc}_\Lambda$ have the same $\bar{\kappa}$-points. As they are both reduced subscheme of $\mathcal{N}$, they must be the same. This proves the proposition. $\square$

3.4. **Bruhat-Tits stratification.**

Lemma 3.18. *For any pair $(M, \mathcal{F})$ satisfying the condition in Proposition 3.6, there is a unique vertex lattice $\Lambda(M)$ such that $\Lambda(M) \subset M$ and $\Lambda(M)$ is maximal among all such vertex lattices.*

Proof. This is essentially [RTW14, Proposition 4.1] as such M satisfies the conditions in Proposition 2.4 of loc.cit.. $\square$

Theorem 3.19. *There is a stratification of $\mathcal{N}_{\mathrm{red}}$ by closed strata $\mathcal{N}_\Lambda$ given by*

$$\mathcal{N}_{\mathrm{red}} = \bigcup_{\Lambda} \mathcal{N}_\Lambda. \tag{3.17}$$

where the union is over all vertex lattices in $\mathbb{V}$. We call this the Bruhat-Tits stratification of $\mathcal{N}_{\mathrm{red}}$. In the following, assume that Λ and Λ' are vertex lattices of type greater or equal to 2, and Λ_0 and Λ_0' are vertex lattices of type 0.

1. *If $\Lambda \subset \Lambda'$, then $\mathcal{N}_{\Lambda'}$ is a subscheme of $\mathcal{N}_\Lambda$.*
2. *The intersection of $\mathcal{N}_{\Lambda'} \cap \mathcal{N}_\Lambda$ is nonempty if and only if $\Lambda'' = \Lambda + \Lambda'$ is a vertex lattice, in which case we have $\mathcal{N}_{\Lambda'} \cap \mathcal{N}_\Lambda = \mathcal{N}_{\Lambda''}$.*
3. *The intersection of $\mathcal{N}_{\Lambda_0'} \cap \mathcal{N}_{\Lambda_0}$ is always empty if $\Lambda_0 \neq \Lambda_0'$.*
4. *The intersection $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}$ is nonempty if and only if $\Lambda \subset \Lambda_0$ in which case $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}$ is isomorphic to $\mathbb{P}^{m-1}_{\bar{\kappa}}$ where $2m$ is the type of Λ.*

Proof. To prove (3.17), it suffices to check this on $\bar{\kappa}$-points. A point $z \in \mathcal{N}_{\mathrm{red}}(\bar{\kappa})$ corresponds to a pair (M, M') as in Proposition 3.6. Take $\Lambda = \Lambda(M)$ as in Lemma 3.18. If Λ has type 0, then both $\breve{\Lambda}$ and M are self-dual and $\breve{\Lambda} \subset M$, so they have to be equal. Hence $z \in \mathcal{N}_\Lambda$ by Corollary 3.7. If Λ is not of type 0, then M is not τ-invariant, hence $M' = M \cap \tau^{-1}(M)$ is uniquely determined. Since Λ is τ-invariant, $\breve{\Lambda} \subset M'$. Hence $z \in \mathcal{N}_\Lambda(\bar{\kappa})$ by Corollary 3.7. This proves (3.17).

(1) follows immediately from Definition 3.5.

(2). If Λ'' is a vertex lattice, then $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda'} = \mathcal{N}_{\Lambda''}$ by Definition 3.5. Conversely if $\mathcal{N}_{\Lambda'} \cap \mathcal{N}_\Lambda(\bar{\kappa})$ is nonempty, let $(M, M') \in \mathcal{N}_{\Lambda'} \cap \mathcal{N}_\Lambda(\bar{\kappa})$. Then $\Lambda(M) \supset \Lambda + \Lambda'$ by the maximality of $\Lambda(M)$. Then $\Lambda + \Lambda' \subset \Lambda(M) \subset \Lambda(M)^\sharp \subset \Lambda^\sharp \cap (\Lambda')^\sharp = (\Lambda + \Lambda')^\sharp$. Hence $\Lambda + \Lambda'$ is a vertex lattice.

(3) follows directly from Corollary 3.7.

(4). By Corollary 3.7, a point $(M, M') \in \mathcal{N}(\bar{\kappa})$ is in $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}$ if and only if $M = \Lambda_0 \otimes_{O_F} O_{\breve{F}}$ and $\Lambda \subset M' \subset M$. This show that $\Lambda \subset \Lambda_0$ and $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}(\bar{\kappa}) = \mathbb{P}(\Lambda_0/\Lambda)(\bar{\kappa})$. Similarly one can show that

$$\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda_0}(R) = \mathbb{P}(\Lambda_0/\Lambda)(R)$$

for any $\bar{\kappa}$-algebra R using relative display theory. This finishes the proof of (4). $\square$

Proposition 3.20. *For a rank r lattice $L \subset \mathbb{V}$, the reduced subscheme $\mathcal{Z}(L)_{\mathrm{red}}$ of $\mathcal{Z}(L)$ is a union of Bruhat-Tits strata:*

$$\mathcal{Z}(L)_{\mathrm{red}} = \bigcup_{L \subset \Lambda} \mathcal{N}_\Lambda, \tag{3.18}$$

where the union is taken over all vertex lattices Λ such that $L \subset \Lambda$. Moreover, the intersection of $\mathcal{Z}(L)$ with $\mathcal{N}_\Lambda$ is nonempty if and only if $L \subseteq \Lambda^\sharp$.

Proof. The proof of (3.18) is the same as that of [Shi18, Proposition 3.8].

If $L \subseteq \Lambda^\sharp$ and L is integral, define $\Lambda' := L + \Lambda$. Then Λ' is a vertex lattice and $\Lambda \subseteq \Lambda'$. By (3.18), $\mathcal{N}_{\Lambda'}$ is in the intersection of $\mathcal{Z}(L)$ and $\mathcal{N}_\Lambda$.

Conversely if the intersection of $\mathcal{Z}(L)$ and $\mathcal{N}_\Lambda$ is not empty, then by (3.18) and Theorem 3.19, there exists a vertex lattice Λ' such that $\Lambda \subseteq \Lambda'$ and $L \subseteq \Lambda'$. Since $\Lambda' \subseteq (\Lambda')^\sharp \subseteq \Lambda^\sharp$, we know that $L \subseteq \Lambda^\sharp$. This finishes the proof of the lemma. $\square$

4. Fourier transform: the geometric side

4.1. Horizontal and vertical part of ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)$.

Definition 4.1. Let $L^\flat$ be a rank $n-1$ integral lattice in $\mathbb{V}$. We say that $L^\flat$ is horizontal if one of the following conditions is satisfied

(1) $L^\flat$ is unimodular.

(2) $L^\flat$ is of the form $L^\flat = M \obot L'$ where M is a unimodular sublattice of rank $n-2$ such that $(M_F)^\perp$ (the perpendicular complement of M_F in $\mathbb{V}$) is nonsplit.

We denote the set of horizontal lattices by Hor.

Lemma 4.2. *Let $L^\flat$ be a rank $n-1$ lattice in $\mathbb{V}$. Then $L^\flat$ is horizontal if and only if there is a unique vertex lattice Λ which contains $L^\flat$. If this is the case, Λ is of type 0.*

Proof. We first prove the "only if" direction. Let Λ be any vertex lattice containing $L^\flat$. If $L^\flat$ is unimodular, then Λ has to be of the form $L^\flat \obot L'$ where L' is the unique unimodular lattice in $(L^\flat_F)^\perp$. If $L^\flat$ is of the form $M \obot L'$ such that M is of rank $n-2$ and $(M_F)^\perp$ is nonsplit, then the proof of [Shi18, Theorem 3.10] implies that there is a unique vertex lattice Λ' in $(M_F)^\perp$ which is of unimodular (this fact is the same as the fact that the Bruhat-Tits building of $(M_F)^\perp$ has a only one point). Then Λ must be of the form $M \obot \Lambda'$. In both cases, Λ is unique and is of type 0.

We now prove the "if" direction. If $t(L^\flat) \geq 2$, then there exist a type 2 vertex lattice Λ_2 containing $L^\flat$ and any type 0 vertex lattice containing Λ_2 (there are $q+1$ of them) also contains $L^\flat$. Hence $t(L^\flat) \leq 1$ and $L^\flat$ is of the form $M \obot L'$ such that M is of rank $n-2$. If $(M_F)^\perp$ is split and $\mathrm{val}(L') > 0$, then [HSY20, Corollary 3.11] implies that there are more than one type 0 vertex lattices Λ' in $(M_F)^\perp$ containing L'. For any such Λ', $M \oplus \Lambda'$ is a vertex lattice of type 0 containing $L^\flat$. This shows that in order for such Λ to be unique, $L^\flat$ must satisfies the conditions in Definition 4.1. The lemma is proved. $\square$

For a rank $n-1$ lattice $L^\flat$ in $\mathbb{V}$, define

$$\mathrm{Hor}(L^\flat) := \{M^\flat \in \mathrm{Hor} \mid L^\flat \subseteq M^\flat\}. \tag{4.1}$$

When $\dim(\mathbb{V}) = 2$ and $\chi(\mathbb{V}) = -1$, for $y \in \mathbb{V}$, define

$$\tilde{\mathcal{Z}}(y)^o := \begin{cases} \mathcal{Z}^+_{\mathrm{val}(y)} \sqcup \mathcal{Z}^-_{\mathrm{val}(y)} & \text{if } \mathrm{val}(y) > 0, \\ \mathcal{Z}_0 & \text{if } \mathrm{val}(y) = 0. \end{cases}$$

Here $\mathcal{Z}_0 \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ and $\mathcal{Z}^+_s \cong \mathcal{Z}^-_s \cong \mathrm{Spf}\, W_s$ are quasi-canonical lifting cycles defined in [Shi20, §3] where W_s is a totally ramified abelian extension of $\mathcal{O}_{\breve{F}}$ of degree q^s. When $\dim(\mathbb{V}) = 2$ and $\chi(\mathbb{V}) = 1$, for $y \in \mathbb{V}$, define $\tilde{\mathcal{Z}}(y)^o$ to be $\mathcal{Z}^h(y)$, where $\mathcal{Z}^h(y) \cong \mathrm{Spf}\, O_{\breve{F}}$ is as in [HSY20, Theorem 4.1]. In all cases, $\tilde{\mathcal{Z}}(y)^\circ$ is a closed subscheme of $\mathcal{N}_2$.

Now for a rank $n-1$ lattice $L^\flat$ in $\mathbb{V}$ and $M^\flat \in \mathrm{Hor}(L^\flat)$, we can decompose $M^\flat$ as $M \oplus \mathrm{Span}\{y\}$ where M is unimodular and $\mathrm{val}(y)$ has to be zero if $(M_F)^\perp$ is split. By [HSY21, Proposition 2.6], the unimodular lattice M induces a closed embedding $\mathcal{N}_2 \hookrightarrow \mathcal{N}_n$. We define $\tilde{\mathcal{Z}}(M^\flat)^\circ$ to be the image of the composed embedding $\tilde{\mathcal{Z}}(y)^\circ \hookrightarrow \mathcal{N}_2 \hookrightarrow \mathcal{N}_n$ where $\tilde{\mathcal{Z}}(y)^\circ$ is the closed formal subscheme of $\mathcal{N}_2$ defined above. The following is [HSY21, Theorem 4.1].

Theorem 4.3. *Let $L^\flat$ be a rank $n-1$ integral lattice in $\mathbb{V}$, then*

$$\mathcal{Z}(L^\flat)_{\mathscr{H}} = \bigcup_{M^\flat \in \mathrm{Hor}(L^\flat)} \tilde{\mathcal{Z}}(M^\flat)^\circ. \tag{4.2}$$

In particular, $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ is of pure dimension 1. We have the following identity in $\mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$:

$$[\mathcal{O}_{\mathcal{Z}(L^\flat)_{\mathscr{H}}}] = \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^\circ}]. \tag{4.3}$$

Lemma 4.4. *For any formal subscheme Z of $\mathcal{N}$ and $0 \le i \le n$, $F^i K_0^Z(\mathcal{N})$ is an ideal in $K_0(\mathcal{N})$.*

Proof. By definition $F^i K_0^Z(\mathcal{N})$ is generated by elements of the form $[\mathcal{F}^\bullet]$ where $\mathcal{F}^\bullet$ is a finite complex of locally free coherent $\mathcal{O}_{\mathcal{N}}$-modules acyclic outside a sub formal scheme Y of Z such that the codimension of Y in $\mathcal{N}$ is greater or equal to i. By Kunneth formula for chain complexes, the product complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_{\mathcal{N}}} \mathcal{K}^\bullet$ is acyclic outside Y as well for any finite complexes of locally free coherent $\mathcal{O}_{\mathcal{N}}$-modules $\mathcal{K}^\bullet$. This proves the lemma. $\square$

By Lemma 4.4, for any formal subscheme Z of $\mathcal{N}$, we can define a quotient ring (not necessary with identity)

$$\mathrm{Gr}' K_0^Z(\mathcal{N}) := K_0^Z(\mathcal{N}) / F^n K_0^Z(\mathcal{N}). \tag{4.4}$$

In particular $\mathrm{Gr}^{n-1} K_0^Z(\mathcal{N}) = F^{n-1} K_0^Z(\mathcal{N}) / F^n K_0^Z(\mathcal{N})$ is a subgroup of $\mathrm{Gr}' K_0^Z(\mathcal{N})$.

Since $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ is one-dimensional, the intersection $\mathcal{Z}(L^\flat)_{\mathscr{H}} \cap \mathcal{Z}(L^\flat)_{\mathscr{V}}$ must be 0-dimensional if nonempty. It follows that there is a decomposition

$$\mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}) = \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)_{\mathscr{H}}}(\mathcal{N}) \oplus \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N}). \tag{4.5}$$

Under this decomposition, we have

$$^{\mathbb{L}}\mathcal{Z}(L^\flat) = {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}} + {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}), \tag{4.6}$$

where we denote by the same notation the image of $^{\mathbb{L}}\mathcal{Z}(L^\flat)$ under the natural quotient map $K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}) \to \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$. It follows that the element $^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$ is canonically defined although $\mathcal{Z}(L^\flat)_{\mathscr{V}}$ depends on the choice of a large integer $m \gg 0$.

Since $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ has expected dimension, $^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}}$ is in fact in $\mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$ and is represented by the structure sheaf of $\mathcal{Z}(L^\flat)_{\mathscr{H}}$. In order to match the analytic side of our conjecture, we need to slightly modify $^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}}$.

Definition 4.5. Let $L^\flat$ be a horizontal lattice in $\mathbb{V}$. Define $^{\mathbb{L}}\mathcal{Z}(L^\flat)^\circ \in \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$ by

$$^{\mathbb{L}}\mathcal{Z}(L^\flat)^\circ = \begin{cases} [\mathcal{O}_{\tilde{\mathcal{Z}}(L^\flat)^\circ}] + \frac{1-(-1)^{n-1}}{2}[\mathcal{O}_{\mathbb{P}_\Lambda}] & \text{if } L^\flat \text{ is unimodular,} \\ [\mathcal{O}_{\tilde{\mathcal{Z}}(L^\flat)^\circ}] + [\mathcal{O}_{\mathbb{P}_\Lambda}] & \text{otherwise,} \end{cases}$$

where Λ is the unique type 0 vertex lattice containing $L^\flat$ as in Lemma 4.2 and $\mathbb{P}_\Lambda$ is a projective line in Exc_Λ.

Remark 4.6. *$^{\mathbb{L}}\mathcal{Z}(L^\flat)^\circ$ is the difference cycle $\mathcal{D}(L^\flat)$ defined in* [HSY21, Definition 2.15].

Definition 4.7. Let $L^\flat$ be a rank $n-1$ lattice in $\mathbb{V}$. Define $^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} \in \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N})$ by

$$^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} := \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ,$$

where $\mathcal{Z}(M^\flat)^\circ_{\mathscr{H}}$ is as in Definition 4.5. Define the modified vertical part of the derived special cycle $^{\mathbb{L}}\mathcal{Z}(L^\flat)$ by

$$^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}} := {}^{\mathbb{L}}\mathcal{Z}(L^\flat) - {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} \in \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}).$$

For any $x \in \mathbb{V} \setminus L^\flat_F$, define

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) := \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} \cdot [\mathcal{O}_{\mathcal{Z}(x)}]), \text{ and } \mathrm{Int}_{L^\flat,\mathscr{V}}(x) := \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}} \cdot [\mathcal{O}_{\mathcal{Z}(x)}]). \tag{4.7}$$

Lemma 4.8. *If $L^\flat$ is a horizontal lattice of rank $n-1$ in $\mathbb{V}$, then*

$$^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} = {}^{\mathbb{L}}\mathcal{Z}(L^\flat). \tag{4.8}$$

In particular for any $x \in \mathbb{V} \setminus L^\flat_F$ we have

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) = \mathrm{Int}_{L^\flat}(x).$$

Proof. Let Λ be the unique type 0 vertex lattice containing $L^\flat$ as indicated by Lemma 4.2. Then $\Lambda \cap L^\flat_F$ is the unique unimodular lattice in $\mathrm{Hor}(L^\flat)$. By Theorem 4.3, we have

$$^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{H}} - {}^{\mathbb{L}}\mathcal{Z}(L^\flat) = (m-1+\frac{1-(-1)^{n-1}}{2})[\mathcal{O}_{\mathbb{P}_\Lambda}] - {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}},$$

where $m := |\mathrm{Hor}(L^\flat)|$. By Proposition 3.20 and Lemma 4.2, we know that ${}^{\mathbb{L}}Z(L^\flat) \in \mathrm{Gr}' K_0^{\mathcal{N}_\Lambda}(\mathcal{N})$. [HSY21, Corollary 3.5] implies

$$ {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} = m'[\mathcal{O}_{\mathbb{P}_\Lambda}] $$

for some integer m'. Hence in order to prove (4.8), it suffices to show

$$ m' = m - \frac{1 + (-1)^{n-1}}{2}. \tag{4.9} $$

Now assume $L^\flat = M \oplus L'$ where M is unimodular and of rank $n-2$ and $\mathrm{val}(L') = a$. Then $m = a+1$. By [HSY21, Lemma 4.3], we know that

$$ \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}} \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = 2a + 1 = 2m - 1. $$

By [HSY21, Corollary 3.7], we know

$$ \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = m' \cdot \chi(\mathcal{N}, [\mathcal{O}_{\mathbb{P}_\Lambda}] \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = -2m'. $$

On the other hand, by [HSY21, Corollary 3.6],

$$ \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}} \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) + \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \cdot [\mathcal{O}_{\mathcal{N}_\Lambda}]) = (-1)^{n-1}. $$

Combine the above equations, we get (4.9). ☐

Lemma 4.9. *We have*

$$ {}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}} \in \mathrm{Gr}' K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N}). $$

Proof. It follows immediately from Theorem 4.3, Lemma 4.8, and the definition of ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}}$. ☐

4.2. **Hermitian lattices and Fourier transform.** We fix an additive character $\psi : F_0 \to \mathbb{C}^\times$ whose conductor is O_{F_0}. Recall that the Fourier transform with respect to ψ is defined by

$$ \widehat{\varphi}(x) = \int_{\mathbb{V}} \varphi(y) \cdot \psi(\mathrm{Tr}_{F/F_0}(x, y)) d\mu(y), \tag{4.10} $$

where $d\mu$ is the unique self-dual Haar measure on $\mathbb{V}$ with respect to this transform. For a lattice L in $\mathbb{V}$ we use $L^\vee$ to denote its dual under the quadratic form $\mathrm{Tr}_{F/F_0}((\ ,\ ))$. The following lemma is well-known and easy to check.

Lemma 4.10. *Let $L \subset \mathbb{V}$ be a lattice of rank n and $1_L \in \mathscr{S}(\mathbb{V})$ be its characteristic function. Then*

$$ \widehat{1_L} = \mathrm{vol}(L, d\mu) \cdot 1_{L^\vee}. $$

Lemma 4.11. *Let L be a rank n lattice in $\mathbb{V}$. A function $\varphi \in \mathscr{S}(\mathbb{V})$ is L-invariant (invariant under the translation of L) if and only if its Fourier transform $\widehat{\varphi}$ is supported on $L^\vee$.*

Proof. We first prove the "only if" direction. For any $\mu \in \mathbb{V}$,let

$$ L(\bar{\mu}) := \mu + L. $$

Any L-invariant function $\phi \in \mathscr{S}(\mathbb{V})$ is a linear combination of the characteristic functions $1_{L(\bar{\mu})}$. So it suffices to assume $\phi = 1_{L(\mu)}$. In this case,

$$ \widehat{\phi}(x) = \psi(\mathrm{Tr}_{F/F_0}(x, \mu)) \cdot \widehat{1_L}(x) $$

is supported on $L^\vee$ by Lemma 4.10. This proves the "only if" direction.

For the "if" direction, it suffices to show that if φ is supported on $L^\vee$, then $\widehat{\varphi}$ is L-invariant. For any $z \in L$ we have

$$\widehat{\varphi}(x+z) = \int_{\mathbb{V}} \varphi(y) \cdot \psi(\mathrm{Tr}_{F/F_0}(x,y)) \cdot \psi(\mathrm{Tr}_{F/F_0}(z,y)) d\mu(y),$$

Since $\psi(\mathrm{Tr}_{F/F_0}(z,y)) = 1$ for any $z \in L$ and $y \in L^\vee$ and φ is supported on $L^\vee$, the above is equal to $\widehat{\varphi}(x)$. This finishes the proof of the lemma. $\square$

For an integer m, recall that

$$\mathbb{V}^{\geq m} = \{x \in \mathbb{V} \mid \mathrm{val}(x) \geq m\}.$$

Definition 4.12. Define $\mathscr{S}(\mathbb{V})^{\geq m}$ to be the subspace of $\mathscr{S}(\mathbb{V})$ consisting of functions φ such that $\widehat{\varphi}$ is supported on $\mathbb{V}^{\geq m}$.

Lemma 4.13. *Let Λ be a vertex lattice. Any Λ-invariant function in $\mathscr{S}(\mathbb{V})$ is in $\mathscr{S}(\mathbb{V})^{\geq -1}$.*

Proof. By Lemma 4.11, it suffices to show that $\Lambda^\vee \subset \mathbb{V}^{\geq -1}$. Since Λ is a vertex lattice, we have

$$\Lambda^\sharp = H^t \obot I_{n-2t},$$

for some t. Simple calculation gives then

$$\Lambda^\vee = \frac{1}{\pi}\Lambda^\sharp = \frac{1}{\pi}H^t \obot \frac{1}{\pi}I_{n-2t} \subset \mathbb{V}^{\geq -1}.$$

$\square$

4.3. **Fourier transform of** $\mathrm{Int}_{L^\flat,\mathscr{V}}$.

Theorem 4.14. *Let Λ be a vertex lattice and $\mathcal{K} \in K_0^{\mathcal{N}_\Lambda}(\mathcal{N})$. For any $x \in \mathbb{V} \setminus \{0\}$, the class $\mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x)}] \in K_0^{\mathcal{N}_\Lambda}(\mathcal{N})$ is Λ-invariant. To be more precise for any $y \in \Lambda$ such that $x + y \neq 0$, we have*

$$\mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x)}] = \mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x+y)}]. \tag{4.11}$$

Moreover, the function

$$\mathrm{Int}_{\mathcal{K}}(x) := \chi(\mathcal{N}, \mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x)}])$$

extends to a Λ-invariant function in $\mathscr{S}(\mathbb{V})^{\geq -1}$.

Proof. Any element $\mathcal{K} \in K_0^{\mathcal{N}_\Lambda}(\mathcal{N}) \cong K_0'(\mathcal{N}_\Lambda)$ is a sum of elements of the form $[\mathcal{F}]$ where $[\mathcal{F}]$ is a coherent sheave of $\mathcal{O}_{\mathcal{N}_\Lambda}$-module. Hence it suffices to prove the theorem for $\mathcal{K} = [\mathcal{F}]$. By [How19, Corollary C], we know

$$[\mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}(x)}] = [\mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}(x+y)}].$$

For any $y \in \Lambda$ with $x + y \neq 0$, $\mathcal{N}_\Lambda$ is a subscheme of $\mathcal{Z}(y)$ by Proposition 3.20. Hence we have

$$\begin{aligned}
\mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x)}] =& [\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}(x)}] \\
=& [\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_\Lambda}} \mathcal{O}_{\mathcal{N}_\Lambda} \otimes_{\mathcal{O}_{\mathcal{Z}(y)}} \mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}(x)}] \\
=& [\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_\Lambda}} \mathcal{O}_{\mathcal{N}_\Lambda} \otimes_{\mathcal{O}_{\mathcal{Z}(y)}} \mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}(x+y)}] \\
=& \mathcal{K} \cdot [\mathcal{O}_{\mathcal{Z}(x+y)}]
\end{aligned}$$

We have proved the Λ-invariance of $\mathcal{K}\cdot[\mathcal{O}_{\mathcal{Z}(x)}]$. It follows that $\mathrm{Int}_{\mathcal{K}}(x)$ is also Λ-invariant. Hence we can define $\mathrm{Int}_{\mathcal{K}}(0)$ to be $\mathrm{Int}_{\mathcal{K}}(x)$ for any $0\neq x\in\Lambda$ and obtain a (unique) Λ-invariant function (still denoted by $\mathrm{Int}_{\mathcal{K}}(x)$) for all $x\in\mathbb{V}$. In particular $\mathrm{Int}_{\mathcal{K}}(x)$ is locally constant. If $x\notin\Lambda^\sharp$, by Proposition 3.20, the intersection of $\mathcal{Z}(x)$ with $\mathcal{N}_\Lambda$ is empty, which implies $\mathrm{Int}_{\mathcal{K}}(x)=0$. This shows that the function $\mathrm{Int}_{\mathcal{K}}(x)$ is compactly supported. Hence it is in $\mathscr{S}(\mathbb{V})$ and is in fact in $\mathscr{S}(\mathbb{V})^{\geq -1}$ by Lemma 4.13. This finishes the proof of the theorem. $\square$

Theorem 4.15. *For every non-degenerate lattice $L^\flat$ of $\mathbb{V}$ of rank $n-1$, the function $\mathrm{Int}_{L^\flat,\mathscr{V}}$ on $\mathbb{V}\setminus L^\flat_F$ can be extended to an element in $\mathscr{S}(\mathbb{V})^{\geq-1}$ which we denote by the same notation.*

Proof. Lemmas 4.9 and 2.8 imply that ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)^*_{\mathscr{V}}\in \mathrm{Gr}'K_0^{\mathcal{N}_{\mathrm{red}}}(\mathcal{N})\cap\mathrm{Gr}'K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N})$. Lemma 2.7 implies that there exist finitely many classes $\mathcal{K}_i\in\mathrm{Gr}'K_0^{\mathcal{N}_{\mathrm{red}}}(\mathcal{N})_{\mathbb{Q}}$ together with $C_i\in\mathbb{Q}$ such that

$$\mathrm{Int}_{L^\flat,\mathscr{V}}(x)=\sum_i C_i\cdot\chi(\mathcal{N},\mathcal{K}_i\cdot[\mathcal{O}_{\mathcal{Z}(x)}]).$$

By Theorem 3.19 we may assume that $\mathcal{K}_i$ is supported on some $\mathcal{N}_\Lambda$. Now we can apply Theorem 4.14 to conclude the proof. $\square$

4.4. **Partial Fourier transform.** Let $L^\flat$ be a rank $n-1$ non-degenerate lattice in $\mathbb{V}$. Let $\mathbb{W}=(L^\flat_F)^\perp$. For any function φ defined on $\mathbb{V}\setminus L^\flat_F$, we define its partial Fourier transform $\varphi^\perp$ as a function on $\mathbb{W}\setminus\{0\}$ by

$$\varphi^\perp(x):=\int_{L^\flat_F}\varphi(x+y)dy,\quad\forall\, x\in\mathbb{W}. \tag{4.12}$$

Theorem 4.16. *The partial Fourier transform $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}\in\mathscr{S}(\mathbb{W})^{\geq-1}$ and is $\mathbb{W}^{\geq0}$-invariant. In particular it is constant on $\mathbb{W}^{\geq 0}$.*

Proof. It is easy to see that partial Fourier transform maps $\mathscr{S}(\mathbb{V})$ to $\mathscr{S}(\mathbb{W})$. It remains to show that the partial Fourier transform of $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}\in\mathscr{S}(\mathbb{W})$ is supported on $\mathbb{W}^{\geq-1}$. For $x\in\mathbb{W}$, we have

$$\widehat{\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}}(x)=\widehat{\mathrm{Int}_{L^\flat,\mathscr{V}}}(x),$$

where $\widehat{\mathrm{Int}_{L^\flat,\mathscr{V}}}(x)$ is the Fourier transform of $\mathrm{Int}_{L^\flat,\mathscr{V}}\in\mathscr{S}(\mathbb{V})$. Since $\widehat{\mathrm{Int}_{L^\flat,\mathscr{V}}}$ is supported on $\mathbb{V}^{\geq-1}$ by Theorem 4.15, we know that $\widehat{\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}}(x)$ is supported on $\mathbb{W}^{\geq-1}$.

Since $\mathbb{W}$ is one-dimensional, $\mathbb{W}^{\geq m}$ is a full rank lattice in $\mathbb{W}$ for any $m\in\mathbb{Z}$. By Lemma 4.11 and what we just proved, $\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}$ is invariant under the translation of $(\mathbb{W}^{\geq-1})^\vee=\mathbb{W}^{\geq0}$. $\square$

5. Review of local densities and primitive local densities

In this section, we recall various explicit formulas of local density polynomials following Section 5 of [HSY21].

5.1. **Basic properties of local density and primitive local density polynomials.**

Definition 5.1. Let M and L be two hermitian O_F-lattices of rank m and n respectively. We use $\mathrm{Herm}_{L,M}$ to denote the scheme of hermitian O_F-module homomorphisms from L to M, which is a scheme of finite type over O_{F_0}. Moreover, we define

$$\mathrm{Den}(M,L) := \lim_{d\to\infty} \frac{\left|\mathrm{Herm}_{L,M}(O_{F_0}/(\pi_0^d))\right|}{q^{N\cdot d_{L,M}}},$$

where $d_{L,M}$ is the dimension of $\mathrm{Herm}_{L,M}\otimes_{O_{F_0}} F_0$. More specifically, for a hermitian O_F-lattice L with hermitian from $(\, ,\,)$, we define $L_{(d)} := L\otimes_{O_{F_0}} O_{F_0}/(\pi_0^d)$ and $(x,y)_{(d)} = \pi(\tilde{x},\tilde{y}) \in O_{F_0}/(\pi_0^d)$ where $x,y\in L_{(d)}$ and $\tilde{x},\tilde{y}\in L$ are any lifts of x,y. Then

$$\mathrm{Herm}_{L,M}(O_{F_0}/(\pi_0^d)) = \{\phi\in \mathrm{Hom}_{O_F}(L_{(d)}, M_{(d)}) \mid (\phi(x),\phi(y))_{(d)} \equiv (x,y)_{(d)} \text{ for all } x,y\in L_{(d)}\}.$$

It is well-known that there is a local density polynomial $\mathrm{Den}(M,L,X)\in \mathbb{Q}[X]$ such that

$$\mathrm{Den}(M,L,q^{-2k}) = \mathrm{Den}(M\oplus H^k, L). \tag{5.1}$$

Moreover, we denote $\mathrm{Den}(M,L) = \mathrm{Den}(M,L,1)$ and

$$\mathrm{Den}'(M,L) := -2\cdot \frac{\partial}{\partial X}\mathrm{Den}(M,L,X)|_{X=1}. \tag{5.2}$$

Similarly, the primitive local density polynomial $\mathrm{Pden}(M,L,X)$ is defined to be the polynomial in $\mathbb{Q}[X]$ such that

$$\mathrm{Pden}(M,L,q^{-2k}) = \lim_{d\to\infty} q^{-dn(2(m+2k)-n)}|\mathrm{Pherm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))|, \tag{5.3}$$

where

$$\mathrm{Pherm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) := \{\phi\in \mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \ \phi \text{ is primitive}\}.$$

Here we recall that $\phi\in \mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$ is primitive if $\dim_{\mathbb{F}_q}((\phi(L)+\pi(M\oplus H^k))/\pi(M\oplus H^k) = n$. In particular, we have $\mathrm{Pden}(M,M) = \mathrm{Den}(M,M)$ since any $\phi\in\mathrm{Herm}_{M,M}(O_{F_0}/(\pi_0^d)$ is primitive.

Recall that without explicit mentioning, we assume $\epsilon = \chi(L)$. As an analogue of (1.3) and (1.4), we define

$$\mathrm{Pden}'(L) := -2\cdot \frac{\frac{\mathrm{d}}{\mathrm{d}X}\big|_{X=1}\mathrm{Pden}(I_n,L,X)}{\mathrm{Den}(I_n,I_n)} \text{ and } \mathrm{Pden}_t(L) := \frac{\mathrm{Pden}(\Lambda_t^\sharp, L)}{\mathrm{Den}(\Lambda_t^\sharp,\Lambda_t^\sharp)}. \tag{5.4}$$

To save notation, we simply denote $\mathrm{Pden}_0(L)$ (resp. $\mathrm{Pden}_0'(L)$) by $\mathrm{Pden}(L)$ (resp. $\mathrm{Pden}'(L)$). We define

$$\partial\mathrm{Pden}(L) := \mathrm{Pden}'(L) + \sum_{j=1}^{\frac{t_{\max}}{2}} c_{2j}\cdot \mathrm{Pden}_{2j}(L), \tag{5.5}$$

where c_{2j} is as in (1.5).

Lemma 5.2. *Let L be a lattice. If there exists $x\in L$ such* $\mathrm{val}(x)\le -1$, *then* $\partial\mathrm{Den}(L) = \mathrm{Den}'(L) = \partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) = 0$.

Proof. Assume $M \cong I_n$ or $M \cong \Lambda_{2t}$ for some t. Then $\mathrm{Den}(M \oplus H^k, L) = 0$ and $\mathrm{Pden}(M \oplus H^k, L) = 0$ since there is no vector in M with valuation less than or equal to -1. $\square$

Now we record several results that describe the relation between local density and primitive local density polynomials.

Lemma 5.3. [HSY21, Lemma 5.1] *Let M and L be lattices of rank m and n. Then we have*

$$\mathrm{Den}(M, L, X) = \sum_{L \subset L' \subset L_F} (q^{n-m} X)^{\ell(L'/L)} \mathrm{Pden}(M, L', X),$$

where $\ell(L'/L) = \mathrm{length}_{O_F} L'/L$. Here $\mathrm{Pden}(M, L', X) = 0$ for L' with fundamental invariant less than the smallest fundamental invariant of M. In particular, the summation is finite.

Corollary 5.4. *Let L be a lattice. We have the following identity:*

$$\partial \mathrm{Den}(L) = \sum_{L \subset L' \subset L_F} \partial \mathrm{Pden}(L').$$

Proof. Since $\mathrm{Pden}(I_n, L', 1) = \mathrm{Pden}(I_n, L') = 0$, we have by Lemma 5.3

$$-2 \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}(I_n, L, X) = -2 \sum_{L \subset L' \subset L_F} \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Pden}(I_n, L', X) = \sum_{L \subset L' \subset L_F} \mathrm{Pden}'(I_n, L').$$

Similarly, according to Lemma 5.3, we have

$$\mathrm{Den}(\Lambda_{2j}, L) = \sum_{L \subset L' \subset L_F} \mathrm{Pden}(\Lambda_{2j}, L')$$

for $0 \le j \le t_{\max}/2$. Now the corollary follows from (1.5) and (5.5). $\square$

Conversely, the primitive local density polynomial is a linear combination of local density polynomials.

Theorem 5.5. [HSY21, Theorem 5.2] *Let M and L be lattices of rank m and n. We have*

$$\mathrm{Pden}(M, L, X) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2 + i(n-m)} X^i \sum_{\substack{L \subset L' \subset \pi^{-1} L \\ \ell(L'/L) = i}} \mathrm{Den}(M, L', X).$$

Corollary 5.6. *Let L be a lattice of rank n. Then*

$$\partial \mathrm{Pden}(L) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} \sum_{\substack{L \subset L' \subset \pi^{-1} L \\ \ell(L'/L) = i}} \partial \mathrm{Den}(L').$$

Recall that for two lattices $L, L' \subset \mathbb{V}$ of rank n, $n(L', L) = |\{L'' \subset L_F \mid L \subset L'', L'' \cong L'\}|$.

Lemma 5.7. *For two lattices L and M of the same rank n, we have*

$$\mathrm{Pden}(M, L) = \begin{cases} \mathrm{Den}(M, L) & \text{if } M \cong L, \\ 0 & \text{if } M \not\cong L. \end{cases} \tag{5.6}$$

Moreover,

$$\mathrm{Den}(M,L) = n(M,L)\cdot \mathrm{Den}(M,M).$$

In particular, if $\chi(M)\neq\chi(L)$*, then* $\mathrm{Den}(M,L)=0$.

Proof. First of all, for $M\cong L$, $\mathrm{Pden}(M,L)=\mathrm{Den}(M,L)$ by the definition of primitive local density. Now we show that if $\mathrm{Pden}(M,L)\neq 0$, then $M\cong L$. If $\mathrm{Pden}(M,L)\neq 0$, then for any large enough d we have

$$\mathrm{Pherm}_{L,M}(O_{F_0}/(\pi_0^d))\neq 0.$$

Let $\phi\in\mathrm{Pherm}_{L,M}(O_{F_0}/(\pi_0^d))$ be a primitive embedding. Let $\overline{\phi(L_{(d)}}$ be the image of $\phi(L_{(d)})$ in $\overline{M_{(d)}}$. Since ϕ is primitive, we have $\overline{\phi(L_{(d)})}=\overline{M_{(d)}}$. Then by Nakayama's lemma, we know $\phi(L_{(d)})=M_{(d)}$. Hence ϕ is an isometry between $L_{(d)}\cong M_{(d)}$. Since this holds for any large enough d, we have $L\cong M$.

Now the formula of $\mathrm{Den}(M,L)$ follows from (5.6) and Lemma 5.3. □

Corollary 5.8. *Let L be a lattice. Then for any even integer t such that $0\le t\le t_{\max}$, we have*

$$\mathrm{Den}_t(L) := \frac{\mathrm{Den}(\Lambda_t^\sharp, L)}{\mathrm{Den}(\Lambda_t^\sharp,\Lambda_t^\sharp)}\in\mathbb{Z}.$$

Corollary 5.9. *Assume $L\ncong\Lambda_t^\sharp$ for any vertex lattice Λ_t with $t>0$. Then*

$$\partial\mathrm{Pden}(L)=\mathrm{Pden}'(L).$$

Corollary 5.10. *Let c_t be the coefficients in* (1.6) *with even t and $0<t\le t_{\max}$. Then*

$$c_t = -\mathrm{Pden}'(\Lambda_t^\sharp).$$

Proof. On the one hand, combining Corollary 5.6 with (1.6), we obtain

$$\partial\mathrm{Pden}(\Lambda_t^\sharp)=\begin{cases}1 & \text{if } t=0 \text{ and } n \text{ is odd},\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, by Lemma 5.7 and (1.5),

$$\partial\mathrm{Pden}(\Lambda_t^\sharp)=\mathrm{Pden}'(\Lambda_t^\sharp)+c_t.$$

□

Write $\Lambda_t^\sharp = H^t\oplus L_1$ where L_1 is unimodular of rank n_1. Then by Corollary 6.2 and Corollary 5.10, we have (see the following subsections for the relevant notations)

$$c_t = -2\frac{\prod_{\ell=1}^{t-1}(1-q^{2\ell})}{\mathrm{Den}(I_n,I_n)}\cdot\sum_{i=0}^{n_1}\prod_{\ell=0}^{n_1-i-1}(1-q^{2(\ell+t)})\cdot\sum_{V_1\in\mathrm{Gr}(i,\overline{L_1})(\mathbb{F}_q)}|\mathrm{O}(V_1,\overline{I_n})|.$$

Combining this formula with Lemma 5.16 and Lemma 5.18, we can compute c_t explicitly. We give some examples here.

Example 5.11. If n is odd, we have

$$c_{t_{\max}} = -\frac{\mathrm{Pden}'(I_n, \Lambda^\sharp_{t_{\max}})}{\mathrm{Den}(I_n, I_n)} = \frac{(-1)^{\frac{n+1}{2}}}{q^{(\frac{n-1}{2})^2}(q^{\frac{n-1}{2}}+1)}. \tag{5.7}$$

If n is even and $\epsilon = 1$, we have

$$c_{t_{\max}} = -\frac{\mathrm{Pden}'(I_n, \Lambda^\sharp_{t_{\max}})}{\mathrm{Den}(I_n, I_n)} = \frac{(-1)^{\frac{n}{2}}}{q^{\frac{n}{2}(\frac{n}{2}-1)}(q^{\frac{n}{2}}+1)}.$$

We also give a list of c_t for small n, t and $\epsilon = 1$ in the following table:

t \ n	2	3	4	5	6
2	$\frac{-1}{q+1}$	$\frac{1}{q(q+1)}$	$\frac{-1}{q^2(q+1)}$	$\frac{1}{q^3(q+1)}$	$\frac{-1}{q^4(q+1)}$
4	0	0	$\frac{1}{q^2(q^2+1)}$	$\frac{-1}{q^4(q^2+1)}$	$\frac{1}{q^6(q^2+1)}$
6	0	0	0	0	$-\frac{1}{q^6(q^3+1)}$

5.2. **Explicit formulas for some simple primitive local density polynomials.**

Lemma 5.12. *(*[LL22, Lemma 2.15]*) Assume L is an integral lattice of rank n. Then*

$$\mathrm{Pden}(H^k, L) = \prod_{\ell=0}^{n-1}(1 - q^{-2k+2\ell}).$$

Lemma 5.13. [HSY21, Corollary 5.8] *Assume $L = H^j \oplus L_1$ where $j > 0$ and L_1 is an integral lattice of rank n_1. Then*

$$\mathrm{Den}(I_m, L, X) = \Big(\prod_{\ell=0}^{j-1}(1 - q^{2\ell}X)\Big)\mathrm{Den}(I_m, L_1, q^{2j}X),$$
$$\mathrm{Pden}(I_m, L, X) = \Big(\prod_{\ell=0}^{j-1}(1 - q^{2\ell}X)\Big)\mathrm{Pden}(I_m, L_1, q^{2j}X).$$

In particular,

$$\mathrm{Pden}'(I_m, L) = 2\Big(\prod_{\ell=1}^{j-1}(1 - q^{2\ell})\Big)\mathrm{Pden}(I_m, L_1, q^{2j}). \tag{5.8}$$

Proof. First, by [HSY21, Corollary 5.8] and Lemma 5.12,

$$\mathrm{Den}(I_m, L, X) = \Big(\prod_{\ell=0}^{j-1}(1 - q^{2\ell}X)\Big)\mathrm{Den}(I_m, L_1, q^{2j}X). \tag{5.9}$$

Notice that if $L \subset L'$ and L' is not of the form $H^j \oplus L_1'$, then there exists $v \in L' \setminus L$ such that $\mathrm{Pr}_{H^j_F}(v) \neq 0$ and $\mathrm{Pr}_{H^j_F}(v) \notin H_j$. Hence some fundamental invariant of L' is smaller or equal to

-2. Hence $\mathrm{Den}(I_m, L', X) = 0$ by Lemma 5.2. Now Theorem 5.5 and (5.9) implies

$$\begin{aligned}
&\mathrm{Pden}(I_m, L, X)\\
&= \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2+i(n-m)} X^i \sum_{\substack{L_1 \subset L_1' \subset \pi^{-1} L_1 \\ \ell(L_1'/L_1)=i}} \mathrm{Den}(I_m, H^j \oplus L_1', X)\\
&= \Big(\prod_{\ell=0}^{j-1} (1 - q^{2\ell} X)\Big) \sum_{i=0}^{n_1} (-1)^i q^{i(i-1)/2+i(n-m-2j)} (q^{2j} X)^i \sum_{\substack{L_1 \subset L_1' \subset \pi^{-1} L_1 \\ \ell(L'/L)=i}} \mathrm{Den}(I_m, L_1', q^{2j} X)\\
&= \Big(\prod_{\ell=0}^{j-1} (1 - q^{2\ell} X)\Big) \mathrm{Pden}(I_m, L_1, q^{2j} X)
\end{aligned}$$

as expected. □

Definition 5.14. Assume U and V are quadratic spaces over F_q. We define $\mathrm{O}(U, V)$ to be the set of isometries from U into V, and $M(U, V)$ to be the set of subspaces $V_1 \subset V$ such that $V_1 \cong U$. Moreover, we define $m(U, V) = |M(U, V)|$.

Definition 5.15. We define U_i^ϵ to be the i-dimensional non-degenerate quadratic space over $\mathbb{F}_q$ with $\chi(U_n^\epsilon) = \epsilon$. Moreover, we define 0_i to be the i-dimensional totally isotropic space.

Lemma 5.16. [HSY21, Lemma A.11] *Assume $L = I_{n-t}^{\epsilon_1} \oplus L_2$ where L_2 is a lattice of full type t and $n \le m$. Then*

$$\mathrm{Pden}(I_m^{\epsilon_2}, L) = q^{-mn+n^2} \cdot |\mathrm{O}(0_t \oplus U_{n-t}^{\epsilon_1}, U_m^{\epsilon_2})|.$$

Specifically, we have by [LZ22b, Lemma 3.2.1],

$$\begin{aligned}
|\mathrm{O}(0_j \oplus U_k^{\epsilon_1}, U_m^\epsilon)| &= q^{(k+j)(2m-k-j-1)/2} \prod_{\lfloor \frac{m-k}{2} \rfloor + 1 - j \le l \le \lfloor \frac{m-1}{2} \rfloor} (1 - q^{-2l})\\
&\quad \cdot \begin{cases} (1 + \epsilon\epsilon_1 q^{-\frac{m-k}{2}+j}) & \text{if } m \equiv k \equiv 1 \pmod 2,\\ 1 & \text{if } m \equiv k-1 \equiv 1 \pmod 2,\\ (1 - \epsilon q^{-\frac{m}{2}}) & \text{if } m \equiv k-1 \equiv 0 \pmod 2,\\ (1 - \epsilon q^{-\frac{m}{2}})(1 + \epsilon\epsilon_1 q^{-\frac{m-k}{2}+j}) & \text{if } m \equiv k \equiv 0 \pmod 2. \end{cases}
\end{aligned}$$

Corollary 5.17. *Let I_n be the unimodular lattice of rank n and sign $-\epsilon$. Then*

$$\mathrm{Den}(I_n, I_n) = \begin{cases} 2q^{\frac{n(n-1)}{2}} \prod_{s=1}^{\frac{n-1}{2}} (1 - q^{-2s}) & \text{if } n \text{ is odd},\\ 2q^{\frac{n(n-1)}{2}} (1 + \epsilon q^{-\frac{n}{2}}) \prod_{s=1}^{\frac{n}{2}-1} (1 - q^{-2s}) & \text{if } n \text{ is even.} \end{cases}$$

5.3. **Counting formulas for subspaces of a quadratic space over $\mathbb{F}_q$.** The main results of §6 transforms the calculation of primitive local density polynomial into a sum over the subspaces of a quadratic space over $\mathbb{F}_q$. In this subsection, we count the number of such subspaces with a fixed quadratic form.

Lemma 5.18. *Given quadratic spaces U and V over F_q, let $M(U,V)$ be the set of subspaces $V_1 \subset V$ such that $V_1 \cong U$, and let $m(U,V) = |M(U,V)|$. Then*

$$m(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon}) = \frac{|\mathrm{O}(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon})|}{q^{jk}|\mathrm{O}(U_k^{\epsilon_2}, U_k^{\epsilon_2})| \cdot |\mathrm{GL}_j(\mathbb{F}_q)|}.$$

In particular,

$$m(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon}) = q^{-jk} m(0_j, U_{n-k}^{\delta(n,k,\epsilon,\epsilon_2)}) m(U_k^{\epsilon_2}, U_n^{\epsilon}), \tag{5.10}$$

where

$$\delta(n,k,\epsilon,\epsilon_2) = \begin{cases} \epsilon & \text{if } k = 0, \\ -\epsilon\epsilon_2 & \text{if both } k \text{ and } n-k \text{ are odd,} \\ \epsilon\epsilon_2 & \text{otherwise.} \end{cases} \tag{5.11}$$

Proof. We prove the first identity first. Fix a subspace V_1 of U_n^{ϵ} such that $V_1 \cong 0_j \oplus U_k^{\epsilon_2}$. Then by Witt's theorem we have a surjection

$$\mathrm{O}(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon}) \to M(0_j \oplus U_k^{\epsilon_2}, U_n^{\epsilon}), \quad \phi \to \phi(V_1).$$

Moreover, each fiber of this surjection is in bijection with $\mathrm{O}(0_j \oplus U_k^{\epsilon_2})$. Any $\phi \in \mathrm{O}(0_j \oplus U_k^{\epsilon_2})$ is determined uniquely by $\phi_1 = \phi|_{0_j}$ and $\phi_2 = \phi|_{U_k^{\epsilon_2}}$. The number of different choices of ϕ_1 is $|\mathrm{GL}_j(\mathbb{F}_q)|$. The number of different choices of ϕ_2 is $q^{jk}|\mathrm{O}(U_k^{\epsilon_2}, U_k^{\epsilon_2})|$. $\square$

Lemma 5.19. *For any $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, we have*

$$m(0_j \oplus U_k^{\epsilon_2}, 0_t \oplus U_{n-t}^{\epsilon_1}) = \sum_{\ell=0}^{\min\{t,j\}} \binom{t}{\ell}_q \cdot q^{(t-\ell)(j+k-\ell)} m(0_{j-\ell} \oplus U_k^{\epsilon_2}, U_{n-t}^{\epsilon_1}).$$

Proof. Let V and U be quadratic spaces over $\mathbb{F}_q$ such that $V \cong 0_t \oplus U_{n-t}^{\epsilon}$, and $U \cong 0_j \oplus U_k^{\epsilon_2}$. Let $R \cong 0_t$ be the radical of V. First, we consider a partition of

$$M(U,V) = \bigsqcup_{\ell=0}^{\min\{t,j\}} M_\ell(U,V),$$

where

$$V_1 \in M_\ell(U,V) \text{ if and only if } \dim_{\mathbb{F}_q}(V_1 \cap R) = \ell.$$

The number of choices of ℓ-dimensional subspace of R is $\binom{t}{\ell}_q$. Now we fix an ℓ-dimensional subspace W of R. Let $\overline{R} \cong 0_{t-\ell}$ be the radical of the quotient space of V/W. Then a choice of $V_1 \in M_\ell(U,V)$ such that $V_1 \cap R = W$ corresponds to an element of

$$S = \{\overline{V_1} \subset V/W \mid \overline{V_1} \cap \overline{R} = \{0\} \text{ and } \overline{V_1} \cong 0_{j-\ell} \oplus U_k^{\epsilon_2}\}.$$

Write $V/W = \overline{R} \oplus \overline{V_2} \cong 0_{t-\ell} \oplus U_{n-t}^{\epsilon_1}$. Let $\mathrm{Pr} : V/W \to \bar{V}_2$ be the natural quotient map. For $\overline{V_1} \in S$, the condition $\overline{V_1} \cap \overline{R} = \{0\}$ implies that $\mathrm{Pr}(\overline{V_1}) \cong 0_{j-\ell} \oplus U_k^{\epsilon_2}$ by the rank-nullity theorem. Moreover, the following map

$$S \to M(0_{j-\ell} \oplus U_k^{\epsilon_2}, \overline{V_2}), \quad \overline{V_1} \mapsto \mathrm{Pr}(\overline{V_1})$$

is a surjection with each fiber in a bijection with $\overline{R}^{j+k-\ell}$. $\square$

Corollary 5.20.

$$
\begin{aligned}
|\mathrm{O}(0_j \oplus U_k^{\epsilon_2}, 0_t \oplus U_{n-t}^{\epsilon})| &= q^{jk} |\mathrm{O}(U_k^{\epsilon_2}, U_k^{\epsilon_2})| \cdot |\mathrm{GL}_j(\mathbb{F}_q)| \cdot m(0_j \oplus U_k^{\epsilon_2}, 0_t \oplus U_{n-t}^{\epsilon}) \\
&= |\mathrm{GL}_j(\mathbb{F}_q)| \cdot \sum_{\ell=0}^{\min\{t,j\}} \binom{t}{\ell}_q \cdot q^{(t-\ell)(j+k-\ell)+\ell k} \cdot \frac{|\mathrm{O}(0_{j-\ell} \oplus U_k^{\epsilon_2}, U_{n-t}^{\epsilon})|}{|\mathrm{GL}_{j-\ell}(\mathbb{F}_q)|}.
\end{aligned}
$$

We will also need the following lemma later (Section 7), which follows from Lemmas 5.18 and 5.16 .

Lemma 5.21. *For integers $0 \le r \le n$ and $\epsilon_1, \epsilon = \pm 1$, we have*

$$
\frac{m(U_r^{-\epsilon_1}, U_n^{\epsilon})}{m(U_r^{\epsilon_1}, U_n^{\epsilon})} = \begin{cases} 1 & \text{if } r \equiv n-1 \equiv 1 \pmod 2, \\ \frac{1-\epsilon\epsilon_1 q^{-\frac{n-r}{2}}}{1+\epsilon\epsilon_1 q^{-\frac{n-r}{2}}} & \text{if } r \equiv n \equiv 1 \pmod 2, \\ \frac{1-\epsilon_1 q^{-\frac{r}{2}}}{1+\epsilon_1 q^{-\frac{r}{2}}} & \text{if } r \equiv n-1 \equiv 0 \pmod 2, \\ \frac{1-\epsilon\epsilon_1 q^{-\frac{n-r}{2}}}{1+\epsilon\epsilon_1 q^{-\frac{n-r}{2}}} \cdot \frac{1-\epsilon_1 q^{-\frac{r}{2}}}{1+\epsilon_1 q^{-\frac{r}{2}}} & \text{if } r \equiv n \equiv 0 \pmod 2, \end{cases}
$$

and

$$
\frac{m(U_{r+1}^{\epsilon_1}, U_n^{\epsilon})}{m(U_r^{\epsilon_1}, U_n^{\epsilon})} = \frac{q^{n-2r-1}(1-(-1)^{n-r}\epsilon\epsilon_1 q^{-\lfloor \frac{n-r}{2} \rfloor})}{1-(-1)^{r+1}\epsilon_1 q^{-\lfloor \frac{r+1}{2} \rfloor}}.
$$

Lemma 5.22. *Assume $i \le r \le n$ and $\epsilon, \sigma, \delta' \in \{\pm 1\}$. Let $\delta(r,i,\delta',\sigma)$ or $\delta(n,i,\epsilon,\sigma)$ be as in (5.11). Then*

$$
m(U_i^{\sigma}, U_n^{\epsilon}) m(U_{r-i}^{\delta(r,i,\delta',\sigma)}, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}) = m(U_r^{\delta'}, U_n^{\epsilon}) m(U_i^{\sigma}, U_r^{\delta'}).
$$

Proof. Let $V = U_n^{\epsilon}$ and S be the following set of flags in V,

$$
S = \{0 \subset F_1 \subset F_2 \subset V \mid F_1 \cong U_i^{\sigma}, F_2 \cong U_r^{\delta'}\}.
$$

We can count the cardinality of S in two ways. One way is to first count the number of $F_1 \in M(U_i^{\sigma}, V)$, then for a fixed F_1 count the number of $F_2' \in M(U_{r-i}^{\delta(r,i,\delta',\sigma)}, (F_1)^{\perp})$ which has a one-to-one correspondence with $F_2 = F_2' \oplus F_1 \in M(U_r^{\delta'}, V)$. This way we get

$$
\#|S| = m(U_i^{\sigma}, U_n^{\epsilon}) m(U_{r-i}^{\delta(r,i,\delta',\sigma)}, U_{n-i}^{\delta(n,i,\epsilon,\sigma)}).
$$

On the other hand, we can first count the number of $F_2 \in M(U_r^{\delta'}, V)$, then count the number of $F_1 \in M(U_i^{\sigma}, F_2)$ and get

$$
\#|S| = m(U_r^{\delta'}, U_n^{\epsilon}) m(U_i^{\sigma}, U_r^{\delta'}).
$$

This finishes the proof of the lemma. □

5.4. **q-binomial theorem.** In this subsection we discuss the q-binomial theorem and related results, which are used repeatedly in §7 to obtain certain vanishing results and transform complicated linear combinations into simple formulas.

Definition 5.23. The q-analogue of $\binom{n}{i}$ is defined to be

$$
\binom{n}{i}_q := \frac{(q^n-1)\cdots(q^{n-i+1}-1)}{(q^i-1)\cdots(q-1)}.
$$

In fact, $\binom{n}{i}_q$ is the number of i-dimensional subspaces of a n-dimensional vector space over $\mathbb{F}_q$. Now we recall the q-binomial theorem.

Lemma 5.24 (q-binomial theorem)**.** *The following identity between polynomials of X holds:*

$$\prod_{i=0}^{n-1}(1-q^iX)=\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q X^i. \tag{5.12}$$

Corollary 5.25. *Let f be a polynomial of degree $\leq n-1$. Then*

$$\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q\cdot f(q^{-i})=0.$$

Proof. Let $f=a_{n-1}x^{n-1}+\cdots+a_0$. For $0\leq s\leq n-1$, by evaluating (5.12) at $t=q^{-s}$, we have

$$\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q\cdot a_s q^{-si}=0.$$

Hence

$$\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q\cdot f(q^{-i})=\sum_{s=0}^{n-1}\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\binom{n}{i}_q\cdot a_s q^{-si}=0.$$

□

The following is in some sense an inverse of q-binomial theorem that will be used in §7.

Lemma 5.26.

$$\sum_{i=0}^{n}(-1)^i q^{\frac{i(i-1)}{2}}\cdot\binom{n}{i}_q\cdot\prod_{\ell=0}^{i-1}(1+q^{-\ell}X)=(-X)^n.$$

Proof. Let $g_i(X)=\prod_{\ell=0}^{i-1}(q^{-\ell}X+1)$. Since $\{g_i(X)\}$ forms a basis of the vector space of polynomials of degree $\leq n$, there exist $a_{n,i}\in\mathbb{C}$ such that

$$(-X)^n=\sum_{i=0}^{n}a_{n,i}\cdot g_i(X).$$

Notice that $g_{i+1}(X)=(1+q^{-i}X)g_i(X)$, hence $Xg_i(X)=q^i(g_{i+1}(X)-g_i(X))$. Then we have

$$\begin{aligned}\sum_{i=0}^{n+1}a_{n+1,i}\cdot g_i(X)=(-X)^{n+1}=(-X)\cdot(-X)^n&=\sum_{i=0}^{n}(-a_{n,i})\cdot Xg_i(X)\\&=\sum_{i=0}^{n}(-a_{n,i})q^i\cdot(g_{i+1}(X)-g_i(X)).\end{aligned}$$

As a result, we have

$$a_{n+1,i}=\begin{cases}a_{n,0} & \text{if } i=0,\\ q^ia_{n,i}-q^{i-1}a_{n,i-1} & \text{if } 0<i<n+1,\\ -a_{n,n}q^n & \text{if } i=n+1.\end{cases} \tag{5.13}$$

It is easy to check that $b_{n,i}=(-1)^iq^{\frac{i(i-1)}{2}}\cdot\binom{n}{i}_q$ satisfies (5.13) and that $a_{1,i}=b_{1,i}$. So we have $a_{n,i}=b_{n,i}$. □

6. Decomposition of primitive local densities

This section is devoted to prove the following decomposition of the primitive local density polynomial, which is a vast generalization of [HSY21, Proposition A.14] and one of the main tools we use to prove Theorem 7.1.

Theorem 6.1. *Assume that L is an integral lattice of rank n. For any $m \geq 0$ we have*

$$\mathrm{Pden}(I_m, L, X) = \sum_{i=0}^{n} \mathrm{Pden}^{n-i}(I_m, L, X),$$

where $\mathrm{Pden}^{n-i}(I_m, L, X)$ is a polynomial characterized by

$$\mathrm{Pden}^{n-i}(I_m, L, q^{-2k}) = q^{-2ki}\mathrm{Pden}(H^k, 0_{n-i}) \sum_{V_1 \in \mathrm{Gr}(i, \overline{L})(\mathbb{F}_q)} q^{(n-i)i}\mathrm{Pden}(I_m, L_{V_1}), \tag{6.1}$$

where 0_{n-i} is a totally isotropic lattice of rank $n-i$ and $L_{V_1} \subset L$ is a sublattice of rank i such that $\overline{L}_{V_1} = V_1$.

Here, an important special case is when $m = n$. In this case, $\mathrm{Pden}^0(I_n^{-\epsilon}, L, X) = 0$ since $\chi(L) \neq \chi(I_n^{-\epsilon})$. Hence,

$$\mathrm{Pden}(I_n, L, X) = \sum_{i=0}^{n-1} \mathrm{Pden}^{n-i}(I_n, L, X).$$

Applying the formulas of $\mathrm{Pden}(H^k, 0_{n-i})$ and $\mathrm{Pden}(I_m, L_{V_1})$ given in Lemmas 5.12 and 5.16, we obtain the following corollary.

Corollary 6.2. *Let L be an integral lattice of rank n. We have*

$$\mathrm{Pden}(I_m, L, X) = \sum_{i=0}^{n} \left((q^{n-m}X)^i \prod_{\ell=0}^{n-i-1} (1 - q^{2\ell}X) \right) \cdot \sum_{V_1 \in \mathrm{Gr}(i, \overline{L})(\mathbb{F}_q)} |\mathrm{O}(V_1, \overline{I_m})|. \tag{6.2}$$

In particular,

$$\mathrm{Pden}'(I_n, L) = 2 \sum_{i=0}^{n} \left(\prod_{\ell=1}^{n-i-1} (1 - q^{2\ell}) \right) \cdot \sum_{V_1 \in \mathrm{Gr}(i, \overline{L})(\mathbb{F}_q)} |\mathrm{O}(V_1, \overline{I_n})|.$$

When L is a full type lattice of rank n, $\overline{L}$ is totally isotropic. So we obtain

Corollary 6.3. *Let L be a full type lattice of rank n. Then*

$$\mathrm{Pden}(I_m, L, X) = \sum_{i=0}^{n} \left((q^{n-m}X)^i \prod_{\ell=0}^{n-i-1} (1 - q^{2\ell}X) \right) \cdot \binom{n}{i}_q |\mathrm{O}(0_i, \overline{I_m})|.$$

In particular,

$$\mathrm{Pden}'(I_n, L) = 2 \sum_{i=0}^{n} \left(\prod_{\ell=1}^{n-i-1} (1 - q^{2\ell}) \right) \cdot \binom{n}{i}_q |\mathrm{O}(0_i, \overline{I_n})|.$$

Here by Lemma 5.16, we have

$$|\mathrm{O}(0_i,\overline{I_m})| = q^{\frac{i(i-1)}{2}}\cdot\begin{cases}\prod_{1\le \ell\le i}\left(q^{m+1-2\ell}\right) & \text{if } m \text{ is odd,}\\ \left(q^{m/2}-\chi(I_m)\right)\left(q^{m/2-i}+\chi(I_m)\right)\cdot\prod_{1\le\ell<i}\left(q^{m-2\ell}-1\right) & \text{if } m \text{ is even.}\end{cases}$$

Proof of Theorem 6.1. To save notation, we use M to denote I_m in this proof. Recall that by (5.3),

$$\mathrm{Pden}(M,L,q^{-2k}) = \lim_{d\to\infty} q^{-dn(2(m+2k)-n)}|\mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))|.$$

First, we define a partition

$$\mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) = \bigsqcup_{0\le i\le n}\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)),$$

where

$$\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) \coloneqq \{\phi\in\mathrm{Herm}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \dim_{\mathbb{F}_q}\overline{\mathrm{Pr}_{H^k}(\phi(L))} = i\}. \tag{6.3}$$

Here Pr_{H^k} denote the projection map to H^k, and $\overline{\mathrm{Pr}_{H^k}(\phi(L))}$ denote the image of $\mathrm{Pr}_{H^k}(\phi(L))$ in $\overline{H}$. As a result, we have

$$\mathrm{Pden}(M,L,X) = \sum_{i=0}^{n}\mathrm{Pden}^i(M,L,X),$$

where $\mathrm{Pden}^i(M,L,X)$ is the function such that

$$\mathrm{Pden}^i(M,L,q^{-2k}) \coloneqq \lim_{d\to\infty} q^{-(2n(2k+m)-n^2)d}|\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))|.$$

We need to count $|\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))|$. For $\phi\in\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$, it induces

$$\overline{\phi}: V=\overline{L}\longrightarrow M\oplus H^k/\pi(M\oplus H^k), \text{ and } \overline{\phi}_{H^k}\coloneqq \mathrm{Pr}_{\overline{H}^k}\circ\overline{\phi}.$$

By the definition of $\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$, For a $(n-i)$-dimensional subspace $V_1\subset\overline{L}$, let

$$\mathrm{Herm}^{V_1}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) = \{\phi\in\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \mathrm{Ker}(\overline{\phi}_{H^k}) = V_1\subset\overline{L}\}.$$

Since $\mathrm{Ker}(\overline{\phi}_{H^k})\subset\overline{L}$ has dimension $n-i$ for any $\phi\in\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))$, we have

$$\mathrm{Herm}^i_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)) = \bigsqcup_{V_1\in\mathrm{Gr}(n-i,V)(\mathbb{F}_q)}\mathrm{Herm}^{V_1}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d)). \tag{6.4}$$

We need to show

$$q^{-(2(n+2k)n-n^2)d}|\mathrm{Herm}^{V_1}_{L,M\oplus H^k}(O_{F_0}/(\pi_0^d))| = q^{(n-i)i}X^{n-i}\mathrm{Pden}(H^k,L_{V_2})\cdot\mathrm{Pden}(M,L_{V_1}). \tag{6.5}$$

Let V_2 be a subspace of V such that $V=V_1\oplus V_2$. Let $L_{V_1}\subset L$ be a sublattice of rank $n-i$ such that the image of L_{V_1} in V is V_1. Similarly, let $L_{V_2}\subset L$ be a sublattice of rank i such that the image of L_{V_2} in V is V_2. Let $\phi_i=\phi|_{L_{V_i}}$ for $i\in\{1,2\}$. According to Lemma 6.4, the number of different choices of ϕ_2 is given by

$$|\mathrm{Herm}^i_{L_{V_2},M\oplus H^k}(O_{F_0}/(\pi_0^d))| = q^{i(2(n+2k)-i)d}\mathrm{Pden}(H^k,L_{V_2}). \tag{6.6}$$

Now for a fixed $\phi_2 \in \mathrm{Perm}^i_{L_{V_2}, M\oplus H^k}(O_{F_0}/(\pi_0^d))$, let

$$\begin{aligned}&\mathrm{Perm}^{\phi_2}_{L_{V_1}, M\oplus H^k}(O_{F_0}/(\pi_0^d))\\ &:= \{\phi_1 \in \mathrm{Perm}_{L_{V_1}, M\oplus H^k}(O_{F_0}/(\pi_0^d)) \mid (\phi_1,\phi_2) \in \mathrm{Perm}^{V_1}_{L, M\oplus H^k}(O_{F_0}/(\pi_0^d))\}.\end{aligned}$$

Claim: For any $\phi_2 \in \mathrm{Perm}^i_{L_{V_2}, M\oplus H^k}(O_{F_0}/(\pi_0^d))$,

$$\begin{aligned}|\mathrm{Perm}^{\phi_2}_{L_{V_1}, M\oplus H^k}(O_{F_0}/(\pi_0^d))| &= q^{(2d-1)(2k-i)(n-i)}|\mathrm{Perm}_{L_{V_1}, M}(O_{F_0}/(\pi_0^d))|\\ &= q^{(2d-1)(2k-i)(n-i)+(2n(n-i)-(n-i)^2)d}\mathrm{Pden}(M, L_{V_1}).\end{aligned} \tag{6.7}$$

Assuming the claim holds, we obtain (6.5) by (6.6) and (6.7).
Proof of the claim: For $\phi_1 \in \mathrm{Perm}^{\phi_2}_{L_{V_1}, M\oplus H^k}(O_{F_0}/(\pi_0^d))$, write $\phi_1 = \phi_{1,H^k} + \phi_{1,M}$, where $\phi_{1,H^k} = \mathrm{Pr}_{H^k} \circ \phi_1$ and $\phi_{1,M} = \mathrm{Pr}_M \circ \phi_1$. First, for any $g \in \mathrm{U}(M \oplus H^k)$, one can directly check the map

$$\begin{aligned}\mathrm{Perm}^{\phi_2}_{L_{V_1}, M\oplus H^k}(O_{F_0}/(\pi_0^d)) &\to \mathrm{Perm}^{g\circ\phi_2}_{L_{V_1}, g(M\oplus H^k)}(O_{F_0}/(\pi_0^d)),\\ \phi_1 &\mapsto g \circ \phi_1\end{aligned} \tag{6.8}$$

is well-defined and is in fact a bijection. Then according to Lemma 6.5, we may assume $\phi_2(L_{V_2}) \subset H^k$.

Now finding ϕ_1 such that $(\phi_1,\phi_2) \in \mathrm{Perm}^{V_1}_{L, M\oplus H^k}(O_{F_0}/(\pi_0^d))$ is equivalent to finding ϕ_1 such that $\phi_{1,M}$ is primitive,

$$(\phi_{1,H^k}(v), \phi_2(w)) \equiv (\phi_1(v), \phi_2(w)) \equiv (v, w) \mod (\pi^{2d-1}) \text{ for any } v \in L_{V_1}, w \in L_{V_2}, \tag{6.9}$$

and

$$(\phi_1(v), \phi_2(w)) \equiv (v, w) \equiv 0 \mod (\pi) \text{ for any } v \in L_{V_1}, w \in L_{V_2}. \tag{6.10}$$

Notice that when k is large enough, we can always find and fix a ϕ'_{1,H^k} that satisfies (6.9). Then finding ϕ_{1,H^k} that satisfies (6.9) is equivalent to find $\Phi_{1,H^k} = \phi_{1,H^k} - \phi'_{1,H^k}$, which satisfies

$$(\Phi_{1,H^k}(v), \phi_2(w)) \equiv 0 \mod (\pi^{2d-1}) \text{ for any } v \in L_{V_1}, w \in L_{V_2},$$

Then according to Lemma 6.6, the number of different choices for ϕ_{1,H^k} is $q^{(2d-1)(2k-i)(n-i)}$.

Since $\phi_{1,H^k}(v) \in \pi H^k$ for any $v \in L_{V_1}$, (6.10) is equivalent to

$$(\phi_{1,M}(v), \phi_{1,M}(w)) \equiv (\phi_1(v), \phi_2(w)) \equiv (v, w) \equiv 0 \mod (\pi).$$

Hence, the number of different choices of primitive $\phi_{1,M}$ is given by

$$q^{(2n(n-i)-(n-i)^2)d}\mathrm{Pden}(M, L_{V_1}).$$

Therefore, we have

$$|\mathrm{Perm}^{\phi_2}_{L_{V_1}, M\oplus H^k}(O_{F_0}/(\pi_0^d))| = q^{(2d-1)(2k-i)(n-i)} \cdot q^{(2n(n-i)-(n-i)^2)d}\mathrm{Pden}(M, L_{V_1}).$$

This finishes the proof of the claim. □

Lemma 6.4. *Assume that L is an integral lattice of rank n and $k \geq n$. Then*

$$|\mathrm{Perm}^n_{L, I_m\oplus H^k}(O_{F_0}/(\pi_0^d))| = q^{2mnd}|\mathrm{Perm}_{L, H^k}(O_{F_0}/(\pi_0^d))|.$$

Proof. For $\phi \in \mathrm{Herm}^n_{L,I_m \oplus H^k}(O_{F_0}/(\pi_0^d))$, we may identify ϕ with (ϕ_{H^k}, ϕ_{I_m}) where $\phi_{H^k} = \mathrm{Pr}_{H^k} \circ \phi$ and $\phi_{I_m} = \mathrm{Pr}_{I_m} \circ \phi$. As

$$|\,\mathrm{Hom}_{O_F}(L, I_m)(O_{F_0}/(\pi_0^d))| = q^{2mnd},$$

it suffices to show that for any fixed $\varphi \in \mathrm{Hom}_{O_F}(L, I_m)$, we have

$$|\{\phi \in \mathrm{Herm}^n_{L,I_m \oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \phi_{I_m} = \varphi\}| = |\mathrm{Herm}_{L,H^k}(O_{F_0}/(\pi_0^d))|. \tag{6.11}$$

Let L_φ be the lattice L equipped with the hermitian form $(x, y)_{L_\varphi} := (\phi_{H^k}(x), \phi_{H^k}(y))$ where ϕ is any element in $\mathrm{Herm}^n_{L,I_m \oplus H^k}(O_{F_0}/(\pi_0^d))$ such that $\phi_{I_m} = \varphi$. Since each such ϕ is an isometry and ϕ_{I_m} is fixed, $(\,,\,)_{L_\varphi}$ is independent of the choice of ϕ. Then we have a bijection

$$\begin{aligned} \{\phi \in \mathrm{Herm}^n_{L,I_m \oplus H^k}(O_{F_0}/(\pi_0^d)) \mid \phi_{I_m} = \varphi\} &\to \mathrm{Herm}_{L_\varphi,H^k}(O_{F_0}/(\pi_0^d)), \\ \phi &\mapsto \phi_{H^k}. \end{aligned} \tag{6.12}$$

Since L is integral and ϕ is an isometric embedding, $L_{\phi_{H^k}}$ is also integral. Then according to [LL22, Lemma 2.16],

$$|\mathrm{Herm}_{L_\varphi,H^k}(O_{F_0}/(\pi_0^d))| = |\mathrm{Herm}_{L,H^k}(O_{F_0}/(\pi_0^d))|.$$

Combining with the bijection in (6.12), this proves (6.11), hence finishes the proof of the lemma. □

Lemma 6.5. *Assume that M is a unimodular lattice, L is an integral lattice of rank n, and $\phi : L \to M \oplus H^k$ is a primitive isometric embedding such that $\mathrm{Pr}_{H^k}(\phi(L))$ is primitive in H^k. Then there exists a $g \in \mathrm{U}(M \oplus H^k)$ such that $g(\phi(L)) \subset H^k$.*

Proof. Consider the non-degenerate symplectic space over $\mathbb{F}_q$: $(\overline{H}^k, \langle\,,\,\rangle) = (\overline{H}^k, \overline{\pi(\,,\,)})$. Let $\overline{v}$ denote the image of $v \in H^k$ in $\overline{H}^k$. Since $\mathrm{Pr}_{H^k}(\phi(L))$ is primitive in H^k, $V(L) := \overline{\mathrm{Pr}_{H^k}(\phi(L))}$ is a n-dimensional subspace. Since M and L are integral, $\mathrm{Pr}_{H^k}(\phi(L))$ is integral. Hence $V(L)$ is an isotropic subspace. Let $\{\ell_1, \cdots, \ell_n\}$ be a basis of L, $\ell_{s,H^k} = \mathrm{Pr}_{H^k}(\ell_s)$ and $e_s = \overline{\ell}_{s,H^k}$. Since $V(L)$ is an n-dimensional isotropic space, we have $k \geq n$ and we can extend $\{e_1, \cdots, e_n\}$ to a standard symplectic basis $\{e_1, f_1, \cdots, e_k, f_k\}$ of $\overline{H}^k$, where $(e_s, f_t) = \delta_{st}$, and $(e_s, e_t) = (f_s, f_t) = 0$ for $1 \leq s, t \leq k$.

Now let $\{\tilde{f}_1, \cdots, \tilde{f}_n\}$ be a lifting of $\{f_1, \cdots, f_n\}$. In particular, for $1 \leq s \leq n$, we have $(\ell_s, \tilde{f}_s) = \pi^{-1} + x$ for some $x \in O_F$. Therefore, $L \oplus \langle \tilde{f}_1 \cdots, \tilde{f}_n\rangle \cong H^n$. Hence, there exists $g \in \mathrm{U}(M \oplus H^k)$ such that $g(L \oplus \langle \tilde{f}_1 \cdots, \tilde{f}_n\rangle) \subset H^k$. □

Lemma 6.6. *Let $N \subset H^k$ be a primitive integral lattice of rank i. Then*

$$\#\{w \in \pi H^k/\pi_0^d H^k \mid (N, w) = 0 \mod (\pi^{2d-1})\} = q^{(2d-1)(2k-i)}.$$

Proof. Through this proof, we use $\overline{L}$ to denote the image of L in $\overline{H}^k$ for any sublattice L of H^k. Let $N^\perp$ be the perpendicular lattice of N in H^k. First we show $N^\perp$ is primitive of rank $2k - i$. If $N^\perp$ is not primitive, then there exists $v \in N^\perp$ such that $\pi^{-1}v \in H^k \setminus N^\perp$. However, $(\pi^{-1}v, N) = 0$, hence $\pi^{-1}v \in N^\perp$, which is a contradiction.

We claim that for any $w \in \pi H^k$ and $\pi(N, w) = 0 \mod \pi^a$ with $a \geq 0$, there exists a $x \in \pi^a H^k$ such that $w - x \in N^\perp$. We prove the lemma by assuming the claim, and give the proof of the claim in the last paragraph. Taking $a = 2d$, the claim implies that

$$\begin{aligned}
&\#\{w \in \pi H^k/\pi_0^d H^k \mid (N, w) = 0 \mod (\pi^{2d-1})\} \\
&= \#\{w \in \pi(N^\perp + \pi_0^d H^k)/\pi_0^d H^k \mid (N, w) = 0 \mod (\pi^{2d-1})\}.
\end{aligned}$$

Since $N^\perp$ is primitive of rank $2k - i$, we have

$$\#\{w \in \pi N^\perp/\pi_0^d H^k \mid (N, w) = 0 \mod (\pi^{2d-1})\} = q^{(2d-1)(2k-i)}.$$

This proves the lemma assuming the claim.

Now we prove the claim. Consider the symplectic space over $\mathbb{F}_q$: $(\overline{H}^k, \overline{\pi(\,,\,)})$. Since N is integral, $\overline{N}$ is isotropic in $\overline{H}^k$. Let $\overline{N}$ be spanned by $\{e_1, \cdots, e_i\}$. Then we may extend $\{e_1, \cdots, e_i\}$ to a standard symplectic basis $\{e_1, f_1, \cdots, e_k, f_k\}$ of $\overline{H}^k$, where $(e_s, f_t) = \delta_{st}$, and $(e_s, e_t) = (f_s, f_t) = 0$ for $1 \leq s, t \leq k$. Let $\{\tilde{e}_1, \tilde{f}_1, \cdots, \tilde{e}_k, \tilde{f}_k\}$ be a lifting of $\{e_1, f_1, \cdots, e_k, f_k\}$. By our choice of $\tilde{e}_s$, we can find a basis $\{w_1, \cdots, w_i\}$ of N such that $w_s - \tilde{e}_s \in \pi H^k$ for any $1 \leq s \leq i$. Consider $x = a_1\tilde{f}_1 + \cdots + a_i\tilde{f}_i \in \langle \tilde{f}_1, \cdots, \tilde{f}_k\rangle$. In order to have $w - x \in N^\perp$, we need to solve the following system of equations:

$$\pi(w_s, x) = \pi(w_s, w) \text{ for } 1 \leq s \leq i. \tag{6.13}$$

Let A denote the $i \times i$ matrix corresponding to this system of linear equations. Since $w_s - \tilde{e}_s \in \pi H^k$, we have $A \equiv \mathrm{Id}_i \mod (\pi)$. Therefore, there exists a unique solution x of (6.13). Moreover, since $\pi(N, w) = 0 \mod \pi^a$, we have $\pi(w_s, w) \in (\pi^a)$ for $1 \leq s \leq i$. Then (6.13) implies that $a_s \in (\pi^a)$, hence $x \in \pi^a H^k$. The claim is proved. □

7. Explicit formulas for $\mathrm{Pden}'(L)$

7.1. **Explicit formulas and consequences.** The goal of this section is to establish the following formulas for

$$\partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) + \sum_{j=0}^{t_{\max}/2} c_{2j}\mathrm{Pden}_{2j}(L). \tag{7.1}$$

Here

$$\mathrm{Pden}'(L) = \frac{\mathrm{Pden}'(I_n, L)}{\mathrm{Den}(I_n, I_n)}$$

is normalized as in (5.4). Recall, from Lemma 5.9, that

$$\partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) \tag{7.2}$$

when L is not dual to some vertex lattice Λ_t of positive type $t > 0$.

Theorem 7.1. *Let $L \subset \mathbb{V}$ be a lattice of rank n.*

(1) If L is not integral, then $\partial\mathrm{Pden}(L) = 0$.

(2) If L is unimodular, then

$$\partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

(3) If $L = I_{n-t} \oplus L_2$ where L_2 is of full type t, then

$$\partial\mathrm{Pden}(L) = \mathrm{Pden}'(L) = \begin{cases} \prod_{\ell=1}^{\frac{t-1}{2}}(1-q^{2\ell}), & \text{if } t \text{ is odd,} \\ (1-\chi(L_2)q^{\frac{t}{2}})\prod_{\ell=1}^{\frac{t}{2}-1}(1-q^{2\ell}), & \text{if } t \text{ is even.} \end{cases}$$

Corollary 7.2. *Let L be a lattice. Then $\partial\mathrm{Den}(L) \in \mathbb{Z}$. Moreover, $\partial\mathrm{Den}(L) = 0$ for non-integral L.*

Proof. According to Corollary 5.4, we have

$$\partial\mathrm{Den}(L) = \sum_{L \subset L' \subset L_F} \partial\mathrm{Pden}(L').$$

Now Theorem 7.1 implies that $\partial\mathrm{Pden}(L') \in \mathbb{Z}$, hence $\partial\mathrm{Den}(L) \in \mathbb{Z}$. If L is non-integral, then $\partial\mathrm{Pden}(L') = 0$ for each L' such that $L \subset L'$ by (1) of Theorem 7.1. □

As another corollary, we prove the following cancellation law for $\partial\mathrm{Den}(L)$. Recall that for Hermitian lattices L and L' of the same rank, $n(L', L) = \#\{L'' \subset L_F \mid L \subset L'', L'' \cong L'\}$.

Corollary 7.3. *Let $L = L_1 \oplus L_2 \subset \mathbb{V}$ be a rank n lattice, with L_1 being unimodular and L_i of rank n_i. Then*

$$\partial\mathrm{Den}(L) - \partial\mathrm{Den}(L_2) = n(I_{n_2}, L_2) \cdot (\delta_{\mathrm{odd}}(n) - \delta_{\mathrm{odd}}(n_2)). \tag{7.3}$$

Proof. By Corollary 5.4 and Lemma 7.4, we have

$$\partial\mathrm{Den}(L) = \sum_{L \subset L' \subset L_F} \partial\mathrm{Pden}(L') = \sum_{L_2 \subset L_2' \subset L_{2,F}} \partial\mathrm{Pden}(L_1 \oplus L_2').$$

Similarly,

$$\partial\mathrm{Den}(L_2) = \sum_{L_2 \subset L_2' \subset L_{2,F}} \partial\mathrm{Pden}(L_2').$$

Hence

$$\partial\mathrm{Den}(L) - \partial\mathrm{Den}(L_2) = \sum_{L_2 \subset L_2' \subset L_{2,F}} (\partial\mathrm{Pden}(L_1 \oplus L_2') - \partial\mathrm{Pden}(L_2')).$$

If L_2' is not integral, then both $\partial\mathrm{Pden}(L_1 \oplus L_2')$ and $\partial\mathrm{Pden}(L_2')$ vanishes by (1) of Theorem 7.1. If L_2' is integral but not unimodular, then (3) of Theorem 7.1 implies $\partial\mathrm{Pden}(L_1 \oplus L_2') - \partial\mathrm{Pden}(L_2') = 0$. Hence

$$\partial\mathrm{Den}(L) - \partial\mathrm{Den}(L_2) = \sum_{\substack{L_2 \subset L_2' \subset L_{2,F} \\ L_1 \oplus L_2' \cong \Lambda_0}} (\partial\mathrm{Pden}(L_1 \oplus L_2') - \partial\mathrm{Pden}(L_2')). \tag{7.4}$$

Combining (7.4) with (2) of Theorem 7.1, we have

$$\partial\mathrm{Den}(L) - \partial\mathrm{Den}(L_2) = n(I_{n_2}, L_2) \cdot (\delta_{\mathrm{odd}}(n) - \delta_{\mathrm{odd}}(n_2)).$$

This proves the corollary. $\square$

Lemma 7.4. *Assume $L = L_1 \obot L_2$ is a lattice where L_1 is unimodular. If $L \subsetneq L' \subset \pi^{-1}L$ and L' is not of the form $L_1 \obot L_2'$, then L' is not integral and $\partial\mathrm{Pden}(L') = 0$.*

Proof. Consider the $\mathbb{F}_q$-vector space $\pi^{-1}L/L$. Since we assume L' is not of the form $L_1 \obot L_2'$, there exists $v \in L' \setminus L$ such that $\mathrm{Pr}_{\pi^{-1}L_1}(v) \neq 0$, which in turn implies that L' is not integral. Hence $\partial\mathrm{Pden}(L') = 0$ by (1) of Theorem 7.1. . $\square$

7.2. **Proof strategy.** The proof of Theorem 7.1 occupies the rest of this section. Since the proof is rather long and technical, we summarize the main idea of the proof first. When there is some $x \in L$ with $\mathrm{val}(x) \leq -1$, $\partial\mathrm{Pden}(L) = 0$ by Lemma 5.2. Otherwise, write

$$L = H^j \obot I_{n_1 - t} \obot L_2,$$

where L_2 is of full type t. There are four cases.

(a) The case $n_1 - t = 0$ (i.e., $L = H^j \oplus L_2$) is significantly simpler than the general case, and we will deal with it in next subsection although it is part of the general case. For example, when L is of full type, the reduction $\overline{L}$ of L modulo π is a totally isotropic quadratic space over $\mathbb{F}_q$. Hence, the summation in Corollary 6.2 is simply:

$$\sum_{V_1 \in \mathrm{Gr}(i,\overline{L})(\mathbb{F}_q)} |\mathrm{O}(V_1, \overline{I_m})| = \binom{n}{i}_q |\mathrm{O}(0_i, \overline{I_m})|.$$

An application of q-binomial theorem settles this case.

(b) The case $j = 0$, i.e., L is integral.

(c) The case $j > 0$ and $t > 0$.

(d) The case that $t = 0$ and $j > 0$ is part of the modification assumption.

In general, the problem becomes harder when $n_1 - t$ is larger. In fact, when $t > 0$, i.e., $n_1 - t < n_1$, (b) and (c) can be proved via Corollary 6.2 and an involved application of the induction formulas of $\mathrm{Den}(I_n, L)$ established in [HSY21]. However, when $t = 0$, i.e., L is unimodular, this method fails. To overcome this difficulty and give a uniform proof of (b) and (c), we introduce a new method which is different from [HSY21] even in the case $n \leq 3$.

To illustrate the idea, we stick to case (b) for now. The first key step is to discover a finer structure of $\partial\mathrm{Pden}(L)$ and prove the following formula (Lemma 7.11):

$$\partial\mathrm{Pden}(L) = (*) \sum_{r=0}^{\min\{n-t,n-1\}} \sum_{i=0}^{n-1} \binom{n-r}{i-r}_q (-1)^{n-1+i+r} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g(n, n_1, r, q^{-i}), \tag{7.5}$$

where $(*)$ is some constant number and $g(n, n_1, r, X)$ is a linear combination of polynomials of degree $n-1$. The second key observation is that there is a lot of cancellation underlying this linear combination. Indeed, We show for $r < n$ that $g(n, n_1, r, X)$ is actually of degree $\leq n - r - 1$ and is essentially a simple multiple of some simple polynomial (denoted by $h(n, r, X)$) (Lemmas 7.13

and 7.15). This enables us to apply q-binomial theorem (Corollary 5.25) to the inner sums in (7.5). Consequently, we obtain

$$\partial\mathrm{Pden}(L) = (*) \sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g(n, n-t, r, q^{-n}). \tag{7.6}$$

The last step is to evaluate this sum and the result is given by Lemma 7.16. It is in this step that the case $n_1 = n$ (L is unimodular) becomes different: the sum above is a sum from $r = 0$ only to $n-1$, not to $n_1 = n$. To make it worse, the 'missing' term $g(n, n, n, X)$ is in fact ill-behaved.

One common strategy in proving Lemmas 7.15 and 7.16 is to express both sides of the identity as (uniquely) linear combinations of certain basis of polynomials, and prove that the coefficients satisfy the same recursion formulas and the same initial conditions. Here we use crucially the combinatorical properties of $m(U, V)$ (Lemma 5.18) for U and V quadratic spaces over $\mathbb{F}_q$.

In Case (c), the derivative becomes the value of some primitive local density polynomials at some non-central point q^j by Lemma 5.13. Strikingly, the formula for this value (see (7.39)) is very similar to the formula (see (7.40) for the derivative $\mathrm{Pden}(L_1 \oplus L_2)$). Proof of Cases (b) and (c) will be given in Subsection 7.6 after long preparation in Subsections 7.4 and 7.5.

7.3. **The case $n_1 - t = 0$.** In this subsection we assume that $n_1 - t = 0$ and divide it further into two subcases: $j = 0$ or $j > 0$.

Proposition 7.5. *Assume that $L = H^j \oplus L_2$ where $j > 0$ and L_2 is of full type and has rank $n_2 = n - 2j$. Then*

$$\partial\mathrm{Pden}(L) = 0.$$

Proof. By Lemma 5.13, we have

$$\mathrm{Pden}'(I_n, L) = 2\Big(\prod_{\ell=1}^{j-1}(1 - q^{2\ell})\Big)\mathrm{Pden}(I_n, L_2, q^{2j}).$$

Hence it suffice to show

$$\mathrm{Pden}(I_n, L, q^{n-n_2}) = 0.$$

We prove the odd n case and leave the even n case to the reader. According to Corollary 6.3, we have

$$\begin{aligned}\mathrm{Pden}(I_n, L, q^{n-n_2}) &= \sum_{i=0}^{n_2}(-1)^{n_2-i} q^{\frac{i(i-1)}{2}} \binom{n_2}{i}_q \cdot \prod_{\ell=0}^{n_2-i-1} (q^{2\ell+n-n_2} - 1) \cdot \prod_{\ell=1}^{i}(q^{n+1-2\ell} - 1) \\ &= \sum_{i=0}^{n_2}(-1)^{n_2-i} q^{\frac{i(i-1)}{2}} \binom{n_2}{i}_q \cdot \prod_{\ell=\frac{n-n_2}{2}}^{\frac{n+n_2}{2}-i-1} (q^{2\ell} - 1) \cdot \prod_{\ell=\frac{n+1}{2}-i}^{\frac{n-1}{2}} (q^{2\ell} - 1).\end{aligned}$$

We can factor out $\displaystyle\prod_{\ell=\frac{n-n_2}{2}}^{\frac{n-1}{2}}(q^{2\ell}-1)$ so that

$$\mathrm{Pden}(I_n,L,q^{n-n_2})\prod_{\ell=\frac{n-1}{2}}^{\frac{n-n_2}{2}}(q^{2\ell}-1)^{-1}=\sum_{i=0}^{n_2}(-1)^{n_2-i}q^{\frac{i(i-1)}{2}}\binom{n_2}{i}_q g(n_2,q^{-i}),$$

where

$$g(n_2,X)=\prod_{\ell=\frac{n+1}{2}}^{\frac{n_2+n}{2}-1}(q^{2\ell}X^2-1) \tag{7.7}$$

is a polynomial of X of degree n_2-1. Now $\mathrm{Pden}(I_n,L,q^{n-n_2})=0$ by Corollary 5.25. $\square$

Proposition 7.6. *Assume that L is a full type lattice of rank n. We have*

$$\mathrm{Pden}'(L)=\begin{cases}\prod_{\ell=1}^{\frac{n-1}{2}}(1-q^{2\ell}), & \text{if } n \text{ is odd},\\ (1-\epsilon q^{\frac{n}{2}})\prod_{\ell=1}^{\frac{n}{2}-1}(1-q^{2\ell}), & \text{if } n \text{ is even.}\end{cases}$$

Proof. First of all, recall that

$$\mathrm{Pden}'(L)=\frac{\mathrm{Pden}'(I_n,L)}{\mathrm{Den}(I_n,I_n)},$$

where

$$\mathrm{Den}(I_n,I_n)=\begin{cases}2q^{(\frac{n-1}{2})^2}\prod_{\ell=1}^{\frac{n-1}{2}}(q^{2\ell}-1) & \text{if } n \text{ is odd},\\ 2q^{(\frac{n}{2})(\frac{n}{2}-1)}(q^{\frac{n}{2}}+\epsilon)\prod_{\ell=1}^{\frac{n}{2}-1}(q^{2\ell}-1) & \text{if } n \text{ is even.}\end{cases} \tag{7.8}$$

We verify the even n case and leave the odd n case to the reader. Direct calculation using Corollary 6.3 gives

$$\begin{aligned}\mathrm{Pden}'(I_n,L)\Big((q^{\frac{n}{2}}+\epsilon)\prod_{\ell=1}^{\frac{n}{2}-1}(q^{2\ell}-1)\Big)^{-1}&=2\sum_{i=0}^{\frac{n}{2}-1}(-1)^{n-i-1}q^{\frac{i(i-1)}{2}}\binom{n}{i}_q(q^{\frac{n}{2}-i}-\epsilon)\prod_{\ell=\frac{n}{2}-i+1}^{n-i-1}(q^{2\ell}-1)\\&=2\sum_{i=0}^{n-1}(-1)^{n-i-1}q^{\frac{i(i-1)}{2}}\binom{n}{i}_q(q^{\frac{n}{2}-i}-\epsilon)\prod_{\ell=\frac{n}{2}+1}^{n-1}(q^{2\ell-2i}-1).\end{aligned}$$

According to Corollary 5.25,

$$\begin{aligned}\sum_{i=0}^{n-1}(-1)^{n-i-1}q^{\frac{i(i-1)}{2}}\binom{n}{i}_q(q^{\frac{n}{2}-i}-\epsilon)\prod_{\ell=\frac{n}{2}+1}^{n-1}(q^{2\ell-2i}-1)&=q^{\frac{n(n-1)}{2}}(q^{-\frac{n}{2}}-\epsilon)\prod_{\ell=\frac{n}{2}+1}^{n-1}(q^{2\ell-2n}-1)\\&=q^{(\frac{n}{2})(\frac{n}{2}-1)}(1-\epsilon q^{\frac{n}{2}})\prod_{\ell=1}^{\frac{n}{2}-1}(1-q^{2\ell}).\end{aligned}$$

Now by (7.8), we conclude that

$$\mathrm{Pden}'(L) = (1-\epsilon q^{\frac{n}{2}}) \prod_{\ell=1}^{\frac{n}{2}-1} (1-q^{2\ell}),$$

as claimed. □

7.4. **Preparation.** In this subsection, we rewrite $\mathrm{Pden}'(I_n, L)$ as a linear combination of special values of certain polynomial $g_{\epsilon_1}(n,m,r,X)$ as in Lemma 7.11. We then express general $g_{\epsilon_1}(n,m,r,X)$ as a simple combination of $g_{\epsilon_3}(n,r,r,X)$, see Lemma 7.13.

Let $L = I_{n-t}^{\epsilon_1} \oplus L_2$ where L_2 is of full type t. By Corollary 6.2, we have

(7.9)

$$\mathrm{Pden}'(I_n, L) = 2\sum_{i=0}^{n-1} \prod_{\ell=1}^{n-i-1} (1-q^{2\ell}) \sum_{0\le j\le i} \sum_{\epsilon_2\in\{\pm1\}} m\left(0_j \oplus U_{i-j}^{\epsilon_2}, 0_t \oplus U_{n-t}^{\epsilon_1}\right) \left|\mathrm{O}\left(0_j \oplus U_{i-j}^{\epsilon_1}, U_n^{-\epsilon}\right)\right|.$$

Here and in the following, we interpret $\sum_{\epsilon_2\in\{\pm1\}} f(U_0^{\epsilon_2})$ as $f(U_0^1)$ for a function e f on $U_i^{\epsilon_2}$.

Let s and n be integers such that $0 \le s < n$, and let $\epsilon_2 = \pm1$. For odd n, we define

(7.10)

$$f_{\epsilon_2}(n,s,X) = \begin{cases} \prod_{\ell=\frac{n+1}{2}}^{\frac{n+s-2}{2}} (q^{2\ell}X^2) \cdot q^{\frac{n+s}{2}} X (q^{\frac{n+s}{2}} X - \epsilon\epsilon_2) \cdot \prod_{\ell=\frac{n+s+2}{2}}^{n-1} (q^{2\ell}X^2 - 1) & \text{if } s \text{ is odd,} \\ \prod_{\ell=\frac{n+1}{2}}^{\frac{n-1}{2}+\frac{s}{2}} (q^{2\ell}X^2) \cdot \prod_{\ell=\frac{n+1+s}{2}}^{n-1} (q^{2\ell}X^2 - 1) & \text{if } s \text{ is even.} \end{cases}$$

Similarly, for an even integer $n > 0$ and $0 \le s < n$, we define

$$(7.11)\quad f_{\epsilon_2}(n,s,X) = \begin{cases} \prod_{\ell=\frac{n}{2}}^{\frac{n+s-3}{2}} (q^{2\ell}X^2) q^{\frac{n}{2}+s-1} X \cdot \prod_{\ell=\frac{n+s+1}{2}}^{n-1} (q^{2\ell}X^2 - 1) & \text{if } s \text{ is odd,} \\ \prod_{\ell=\frac{n}{2}}^{\frac{n+s-2}{2}} (q^{2\ell}X^2) \cdot q^{\frac{s}{2}} (q^{\frac{n+s}{2}} X - \epsilon\epsilon_2) \cdot \prod_{\ell=\frac{n+s+2}{2}}^{n-1} (q^{2\ell}X^2 - 1) & \text{if } s \text{ is even.} \end{cases}$$

Here when $s = 0$, we always take $\epsilon_2 = 1$. Notice that $f_{\epsilon_2}(n,s,X)$ is a polynomial of degree $n-1$.

Lemma 7.7.

(1) Assume $0 \le i < n$. We have

$$\prod_{\ell=1}^{n-i-1} (1-q^{2\ell}) |\mathrm{O}(0_{i-s} \oplus U_s^{\epsilon_2}, U_n^{-\epsilon})| = (-1)^{n-i-1} q^{\frac{i(i-1)}{2}} I_\epsilon(n) f_{\epsilon_2}(n,s,q^{-i}),$$

where

$$I_\epsilon(n) := \begin{cases} \prod_{\ell=1}^{\frac{n-1}{2}} (q^{2\ell}-1) & \text{if } n \text{ is odd,} \\ (q^{\frac{n}{2}}+\epsilon) \prod_{\ell=1}^{\frac{n}{2}-1} (q^{2\ell}-1) & \text{if } n \text{ is even.} \end{cases}$$

(2) Assume $0 \le i \le n$, $s < n$ and that $n' - n > 0$ is even. We have

$$\prod_{\ell=0}^{n-i-1} (1-q^{2\ell+n'-n}) |\mathrm{O}(0_{i-s} \oplus U_s^{\epsilon_2}, U_{n'}^{-\epsilon})| = (-1)^{n-i} q^{\frac{i(i-1)}{2}} I_\epsilon(n', n) f_{\epsilon_2}(n, s, q^{\frac{n'-n}{2}-i}),$$

where

$$I_\epsilon(n',n) := \begin{cases} \prod_{\ell=\frac{n'-n}{2}}^{\frac{n'-1}{2}} (q^{2\ell}-1) & \text{if } n \text{ is odd,} \\ (q^{\frac{n'}{2}}+\epsilon) \prod_{\ell=\frac{n'-n}{2}}^{\frac{n'}{2}-1} (q^{2\ell}-1) & \text{if } n \text{ is even.} \end{cases}$$

Proof. This follows from the formula of $|\mathrm{O}(0_{i-s} \oplus U_s^{\epsilon_2}, U_n^{-\epsilon})|$ given in Lemma 5.16 and a straightforward computation. $\square$

Lemma 7.8. *For integers $0 \le i, t \le n$, we have*

$$\binom{t}{i}_q = \sum_{a=0}^{n-t} (-1)^a \cdot q^{a(t+1-i)} q^{\frac{a(a-1)}{2}} \cdot \binom{n-t}{a}_q \cdot \binom{n-a}{i-a}_q. \tag{7.12}$$

Proof. The identity is automatically true for $i > t$ as both sides are zero. Recall the following analogue of Pascal's identity for q-binomial coefficients:

$$\binom{t}{i}_q = \binom{t+1}{i}_q - q^{t-i+1}\binom{t}{i-1}_q. \tag{7.13}$$

By this identity, we obtain 2 terms, one with the t-index raised, another with the i-index lowered. Applying again (7.13) to $\binom{t+1}{i}_q$ and $\binom{t}{i-1}_q$ respectively, we obtain

$$\binom{t}{i}_q = \binom{t+2}{i}_q - q^{t-i+2}\binom{t+1}{i-1}_q - q^{t-i+1}\Big(\binom{t+1}{i-1}_q - q^{t-i+2}\binom{t-1}{i-2}_q\Big).$$

We may continue this process and after $n-t$ steps, we obtain 2^{n-t} many terms. Each term corresponds to a lattice path starting from the origin, going to north and east as follows. If the ℓ-th step raises the index of t (resp. reduces the index of i), we define the lattice path goes towards north for the ℓ-th step (resp. east). We use $I = (i_1, \cdots, i_{n-t})$ where $i_\ell \in \{0,1\}$ to denote the path whose ℓ-th step goes towards north (east) if $i_\ell = 0$ ($i_\ell = 1$) and let $|I| = i_1 + \cdots + i_{n-t}$. We use P_I to denote the term corresponding to I. Now the lemma follows from the following claim.
Claim:

$$\sum_{I,|I|=a} P_I = (-1)^a \cdot q^{a(t+1-i)} q^{\frac{a(a-1)}{2}} \cdot \binom{n-t}{a}_q \cdot \binom{n-a}{i-a}_q.$$

Indeed a direct calculation shows that

$$P_{(1^a,0^{n-t-a})} = (-1)^a \cdot q^{a(t+1-i)} q^{\frac{a(a-1)}{2}} \cdot \binom{n-a}{i-a}_q.$$

Let A_I denote the area bounded by the lattice path I, the horizontal axis, and the vertical line given by $x = |I|$. Then a direct computation shows that

$$P_I = q^{A_I} \cdot P_{(1^a,0^{n-t-a})}.$$

Now the claim follows from the well-known formula of q-binomial coefficient (see [Cam, Theorem 6.9] for example):

$$\sum_{I,|I|=a} q^{A_I} = \binom{n-t}{a}_q.$$

This proves the claim and the lemma. $\square$

Lemma 7.9. *For an integer $n \geq 0$ and $\epsilon = \pm 1$, let*

$$\alpha(n) = q^{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor} = \begin{cases} q^{(\frac{n-1}{2})^2} & \textit{if } n \textit{ is odd,} \\ q^{\frac{n}{2}(\frac{n}{2}-1)} & \textit{if } n \textit{ is even,} \end{cases} \quad \textit{and} \quad \beta_\epsilon(n) = \begin{cases} (-1)^{\frac{n-1}{2}} & \textit{if } n \textit{ is odd,} \\ \epsilon(-1)^{\frac{n}{2}} & \textit{if } n \textit{ is even,} \\ 1 & \textit{if } n = 0. \end{cases}$$

Then

$$\sum_{j=0}^{n} (-1)^j q^{\frac{j(j-1)}{2}} m(0_j, U_n^\epsilon) = \alpha(n)\beta_\epsilon(n).$$

Proof. If $n = 0$, the statement both sides are 1 by definition. From now on we assume $n > 0$. By [LZ22b, Lemma 3.2.2.], we have

$$|m(0_j, U_n^\epsilon)| = \binom{d}{j}_q \cdot \prod_{l=1}^{j} (q^{d+e-l} + 1),$$

with

$$d = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} - \frac{1-\epsilon}{2} & \text{if } n \text{ is even,} \end{cases} \quad \text{and} \quad e = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1-\epsilon & \text{if } n \text{ is even.} \end{cases}$$

So we have

$$\sum_{j=0}^{n} (-1)^j q^{\frac{j(j-1)}{2}} \cdot m(0_j, U_n^\epsilon) = \sum_{j=0}^{d} (-1)^j q^{\frac{j(j-1)}{2}} \binom{d}{j}_q \prod_{l=1}^{j} (q^{d+e-l} + 1),$$

which by Lemma 5.26 equals to $(-q^{d+e-1})^d$. A direct calculation checks that $(-q^{d+e-1})^d = \alpha(n)\beta_\epsilon(n)$. $\square$

Definition 7.10. For $0 \leq r \leq m \leq n$, we define

$$\begin{aligned} & g_{\epsilon_1}(n, m, r, X) \\ := & \sum_{k=0}^{r} (-1)^k q^{\frac{k(k-1)}{2}} \sum_{\epsilon_2 \in \{\pm 1\}} m(U_k^{\epsilon_2}, U_m^{\epsilon_1}) \sum_{j=0}^{r-k} (-1)^j q^{\frac{j(j-1)}{2}} \cdot \binom{m-j-k}{r-j-k}_q \cdot m(0_j, U_{m-k}^{\delta}) f_{\epsilon_2}(n, k, X) \end{aligned}$$

with $\delta = \delta(m, k, \epsilon_1, \epsilon_2)$ as defined in (5.11). In the following, when n is clear in the context, we simply write $g_{\epsilon_1}(m, r, X)$ for $g_{\epsilon_1}(n, m, r, X)$.

In particular, $g_{\epsilon_1}(n, m, 0, X) = f_1(n, 0, X)$ and by Lemma 7.9,

$$g_{\epsilon_1}(n, r, r, X) = \sum_{k=0}^{r} (-1)^k q^{\frac{k(k-1)}{2}} \cdot \sum_{\epsilon_2 \in \{\pm 1\}} m(U_k^{\epsilon_2}, U_r^{\epsilon_1}) \cdot \alpha(r-k)\beta_\delta(r-k) \cdot f_{\epsilon_2}(n, k, X). \tag{7.14}$$

In the rest of this section, we let $m = n - t$ without explicit mentioning.

Lemma 7.11. *Assume L is a lattice of rank n and type t.*

(1) Let $\mathrm{m}(n,t) := \min\{n-t, n-1\}$. Then for $0 \leq i \leq n-1$, we have

$$(\mathrm{Pden}^{n-i})'(I_n, L) = 2I_\epsilon(n) \cdot \sum_{r=0}^{\mathrm{m}(n,t)} \binom{n-r}{i-r}_q (-1)^{i-r+n-1} q^{\frac{(i-r)(i-r-1)}{2}} q^{r(n-m)} \cdot g_{\epsilon_1}(n, m, r, q^{-i}).$$

(2) Assume that $n'-n$ is a positive even integer and $m<n$. Then for $0\le i\le n$, we have

$$\mathrm{Pden}^{n-i}(I_{n'}^{-\epsilon},L,q^{n'-n})=I_\epsilon(n',n)\sum_{r=0}^{m}\binom{n-r}{i-r}_q(-1)^{i-r+n}q^{\frac{(i-r)(i-r-1)}{2}}q^{r(n-m)}\cdot g_{\epsilon_1}(n,m,r,q^{\frac{n'-n}{2}-i}).$$

Proof. We prove (1) first. By (7.9), Lemma 7.7 and Lemma 5.19, we have

$$\begin{aligned}
&(\mathrm{Pden}^{n-i})'(I_n,L)\cdot\Big(2(-1)^{n-i-1}I_\epsilon(n)\Big)^{-1}\\
&=\sum_{\ell=0}^{t}\binom{t}{\ell}_q\sum_{k=0}^{i-\ell}\sum_{\epsilon_2\in\pm1}q^{(t-\ell)(i-\ell)}q^{\frac{i(i-1)}{2}}m(0_{i-k-\ell}\oplus U_k^{\epsilon_2},U_{n-t}^{\epsilon_1})f_{\epsilon_2}(n,k,q^{-i})\\
&=\sum_{s=\max\{i-t,0\}}^{i}\binom{t}{i-s}_q\sum_{k=0}^{s}\sum_{\epsilon_2\in\pm1}q^{(t-(i-s))s}q^{\frac{i(i-1)}{2}}m(0_{s-k}\oplus U_k^{\epsilon_2},U_{n-t}^{\epsilon_1})f_{\epsilon_2}(n,k,q^{-i}),
\end{aligned}$$

where the last identity is obtained by setting $s=i-\ell$. Notice that if $s>n-t$, then $m(0_{s-k}\oplus U_k^{\epsilon_2},U_{n-t}^{\epsilon_1})=0$. Hence we may assume $s\le n-t$, or equivalently $t\le n-s$ in the above summation. Now applying (7.12) to $i-s\le t\le n-s$ and let $m=n-t$, we may rewrite the above summation as

$$\begin{aligned}
&\sum_{s=\max\{i-t,0\}}^{i}\sum_{a=0}^{m-s}(-1)^a\binom{n-s-a}{i-s-a}_q\cdot\binom{m-s}{a}_q\cdot\\
&\qquad\sum_{k=0}^{s}\sum_{\epsilon_2\in\pm1}q^{a(n-m+1-(i-s))+\frac{a(a-1)}{2}+(n-m+s-i)s}q^{\frac{i(i-1)}{2}}m(0_{s-k}\oplus U_k^{\epsilon_2},U_m^{\epsilon_1})f_{\epsilon_2}(n,k,q^{-i})\\
&=\sum_{s=\max\{i-t,0\}}^{i}\sum_{a=0}^{m-s}\binom{n-s-a}{i-s-a}_q q^{\frac{(i-(s+a))(i-(s+a+1))}{2}}q^{(s+a)(n-m)}\\
&\qquad\cdot(-1)^a\binom{m-s}{a}_q\sum_{k=0}^{s}\sum_{\epsilon_2\in\pm1}q^{(s+a)(s-\frac{s+a+1}{2})+\frac{a(a+1)}{2}}m(0_{s-k}\oplus U_k^{\epsilon_2},U_m^{\epsilon_1})f_{\epsilon_2}(n,k,q^{-i}).
\end{aligned}$$

Now let $r=s+a$. Notice that $r\le m$ and $\binom{n-r}{i-r}_q=0$ for $r>i$. Rearranging the summation index, we obtain

$$(\mathrm{Pden}^{n-i})'(I_n,L)\cdot\Big((-1)^{n-i-1}I_\epsilon(n)\Big)^{-1}=\sum_{r=0}^{\mathrm{m}\{n,t\}}\binom{n-r}{i-r}_q q^{\frac{(i-r)(i-r-1)}{2}}q^{r(n-m)}\cdot g_{\epsilon_1}(n,m,r,q^{-i}),$$

where

$$\begin{aligned}
&g_{\epsilon_1}(n,m,r,X)\\
&=\sum_{s=0}^{r}(-1)^{r-s}\cdot\binom{m-s}{r-s}_q\cdot\sum_{k=0}^{s}\sum_{\epsilon_2\in\pm1}q^{\frac{s(s-1)}{2}}m(0_{s-k}\oplus U_k^{\epsilon_2},U_m^{\epsilon_1})f_{\epsilon_2}(n,k,X)\\
&=\sum_{k=0}^{r}(-1)^kq^{\frac{k(k-1)}{2}}\sum_{\epsilon_2\in\{\pm1\}}m(U_k^{\epsilon_2},U_m^{\epsilon_1})\sum_{j=0}^{r-k}(-1)^jq^{\frac{j(j-1)}{2}}\cdot\binom{m-j-k}{r-j-k}_q\cdot m(0_j,U_{m-k}^{\delta})f_{\epsilon_2}(n,k,X).
\end{aligned}$$

Here, we use Lemma 5.18 to obtain the last identity.

Using (2) of Lemma 7.7, the same proof of (1) proves (2). $\square$

We conclude this subsection by establishing a formula to express $g_{\epsilon_1}(m,r,X)$ in terms of $g_{\epsilon_3}(r,r,X)$, which, as we will see, has a particular simple form (Lemma 7.15). First, we need the following identity which might have independent interest.

Lemma 7.12. *For any integers $0 \le r \le n$, we have*

$$\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q m(0_j,U_n^\epsilon) = \sum_{\tau\in\{\pm1\}} m(U_r^\tau,U_n^\epsilon)\alpha(r)\beta_\tau(r), \tag{7.15}$$

where $\alpha(r)$ and $\beta_\tau(r)$ are defined in Lemma 7.9.

Proof. We proceed by induction on n. The case $n=1$ is obvious. Now recall the identities

$$\begin{aligned} m(0_j,U_n^\epsilon) &= \binom{n}{j}_q - \sum_{i=1}^{j}\sum_{\sigma\in\{\pm1\}} m(0_{j-i}\oplus U_i^\sigma,U_n^\epsilon)\\ &= \binom{n}{j}_q - \sum_{i=1}^{j}\sum_{\sigma\in\{\pm1\}} q^{-(j-i)i} m(U_i^\sigma,U_n^\epsilon)m(0_{j-i},U_{n-i}^{\delta(n,i,\epsilon,\sigma)}), \end{aligned}$$

by (5.10), and

$$\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q\binom{n}{j}_q = \sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n}{r}_q\binom{r}{j}_q = 0. \tag{7.16}$$

These imply that

$$\begin{aligned} &\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q m(0_j,U_n^\epsilon)\\ &= -\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q \sum_{i=1}^{j}\sum_{\sigma\in\{\pm1\}} q^{-(j-i)i} m(U_i^\sigma,U_n^\epsilon)m(0_{j-i},U_{n-i}^{\delta(n,i,\epsilon,\sigma)})\\ &= \sum_{i=1}^{r}\sum_{\sigma\in\{\pm1\}}(-1)^{i+1}q^{i(i-1)/2}m(U_i^\sigma,U_n^\epsilon)\sum_{j=0}^{r-i}(-1)^j q^{j(j-1)/2}\binom{n-i-j}{r-i-j}_q m(0_j,U_{n-i}^{\delta(n,i,\epsilon,\sigma)}). \end{aligned}$$

where in the last step we switch the order of summation and substitute j by $j+i$.

We can now use the induction hypothesis

$$\sum_{j=0}^{r-i}(-1)^j q^{j(j-1)/2}\binom{n-i-j}{r-i-j}_q m(0_j,U_{n-i}^{\delta(n,i,\epsilon,\sigma)}) = \sum_{\tau\in\{\pm1\}} m(U_{r-i}^\tau,U_{n-i}^{\delta(n,i,\epsilon,\sigma)})\alpha(r-i)\beta_\tau(r-i)$$

and

$$m(U_i^\sigma,U_n^\epsilon)m(U_{r-i}^\tau,U_{n-i}^{\delta(n,i,\epsilon,\sigma)}) = m(U_r^{\delta'},U_n^\epsilon)m(U_i^\sigma,U_r^{\delta'}),$$

where $\delta'\in\{\pm1\}$ such that $\tau=\delta(r,i,\delta',\sigma)$ (see Lemma 5.22) to obtain

$$\begin{aligned} &\sum_{j=0}^{r}(-1)^j q^{j(j-1)/2}\binom{n-j}{r-j}_q m(0_j,U_n^\epsilon)\\ &= \sum_{i=1}^{r}\sum_{\sigma\in\{\pm1\}}(-1)^{i+1}q^{i(i-1)/2}\sum_{\tau\in\{\pm1\}} m(U_r^{\delta'},U_n^\epsilon)m(U_i^\sigma,U_r^{\delta'})\alpha(r-i)\beta_\tau(r-i) \end{aligned} \tag{7.17}$$

Hence (7.15) is equivalent to

$$\sum_{i=0}^{r}\sum_{\sigma\in\{\pm1\}}(-1)^i q^{i(i-1)/2}\sum_{\tau\in\{\pm1\}} m(U_r^{\delta'},U_n^{\epsilon})m(U_i^{\sigma},U_r^{\delta'})\alpha(r-i)\beta_\tau(r-i)=0 \tag{7.18}$$

Now applying Lemma 7.9, the left hand side of (7.18) is equal to

$$\begin{aligned}
&\sum_{i=0}^{r}\sum_{\sigma\in\{\pm1\}}(-1)^i q^{i(i-1)/2}\sum_{\tau\in\{\pm1\}} m(U_r^{\delta'},U_n^{\epsilon})m(U_i^{\sigma},U_r^{\delta'})\sum_{j=0}^{r-i}(-1)^j q^{j(j-1)/2}m(0_j,U_{r-i}^{\tau})\\
=&\sum_{\tau\in\{\pm1\}} m(U_r^{\delta'},U_n^{\epsilon})\sum_{i=0}^{r}\sum_{j=0}^{r-i}\sum_{\sigma\in\{\pm1\}}(-1)^{i+j}q^{(i+j)(i+j-1)/2}m(0_j\oplus U_i^{\sigma},U_r^{\delta'})\\
=&\sum_{\delta'\in\{\pm1\}} m(U_r^{\delta'},U_n^{\epsilon})\sum_{k=0}^{r}(-1)^k q^{k(k-1)/2}\binom{r}{k}_q=0.
\end{aligned}$$

The lemma is proved. □

Lemma 7.13. *For $0\le r\le m<n$, we have*

$$g_{\epsilon_1}(m,r,X)=\sum_{\epsilon_3\in\{\pm1\}} m(U_r^{\epsilon_3},U_m^{\epsilon_1})g_{\epsilon_3}(r,r,X).$$

Proof. When $r=0$, we have by definition

$$g(m,0,X)=f_1(n,0,X)=\sum_{\epsilon_3\in\{\pm1\}} m(U_0^{\epsilon_3},U_m^{\epsilon_1})g_{\epsilon_3}(0,0,X).$$

Now we assume $r>0$, $\delta=\delta(m,k,\epsilon_1,\epsilon_2)$ and $\delta'=\delta(r,k,\epsilon_3,\epsilon_2)$. On the one hand, by definition

$$\begin{aligned}
&g_{\epsilon_1}(m,r,X)\\
=&\sum_{k=0}^{r}(-1)^k q^{\frac{k(k-1)}{2}}\sum_{\epsilon_2\in\{\pm1\}} m(U_k^{\epsilon_2},U_m^{\epsilon_1})\sum_{j=0}^{r-k}(-1)^j q^{\frac{j(j-1)}{2}}\cdot\binom{m-j-k}{r-j-k}_q\cdot m(0_j,U_{m-k}^{\delta})f_{\epsilon_2}(n,k,X).
\end{aligned}$$

On the other hand, we have by (7.14),

$$\begin{aligned}
&\sum_{\epsilon_3\in\{\pm1\}} m(U_r^{\epsilon_3},U_m^{\epsilon_1})g_{\epsilon_3}(r,r,X)\\
=&\sum_{k=0}^{r}(-1)^k q^{\frac{k(k-1)}{2}}\cdot\sum_{\epsilon_2\in\{\pm1\}}\sum_{\epsilon_3\in\{\pm1\}} m(U_k^{\epsilon_2},U_r^{\epsilon_3})m(U_r^{\epsilon_3},U_m^{\epsilon_1})\cdot\alpha(r-k)\beta_{\delta'}(r-k)\cdot f_{\epsilon_2}(n,k,X).
\end{aligned}$$

By Lemma 5.22,

$$m(U_k^{\epsilon_2},U_r^{\epsilon_3})m(U_r^{\epsilon_3},U_m^{\epsilon_1})=m(U_k^{\epsilon_2},U_m^{\epsilon_1})m(U_{r-k}^{\delta'},U_{m-k}^{\delta}).$$

Hence, in order to prove the theorem, it suffices to show for any k and ϵ_2

$$\sum_{j=0}^{r-k}(-1)^j q^{\frac{j(j-1)}{2}}\cdot\binom{m-j-k}{r-j-k}_q\cdot m(0_j,U_{m-k}^{\delta})=\sum_{\delta'\in\{\pm1\}} m(U_{r-k}^{\delta'},U_{m-k}^{\delta})\cdot\alpha(r-k)\beta_{\delta'}(r-k),$$

which is exactly the content of Lemma 7.12. □

7.5. **Some identities between polynomials.** Although $g_{\epsilon_1}(r,r,X)$ by definition is a complicated linear combination of $f_{\epsilon_2}(n,k,X)$. We show in this subsection that in fact $g_{\epsilon_1}(r,r,X)$ has a very simple form (Lemma 7.15). Similarly, although $\mathrm{Pden}'(I_n,L)$ is a complicated linear combination of the special values of $g_{\epsilon_1}(n-t,r,X)$, certain linear combination of $g_{\epsilon_1}(n-t,r,X)$ is of a simple form (Lemma 7.16). By a direct computation, we can check the following lemma.

Lemma 7.14. *For $0 \le s \le n-1$, let*

$$h_{\epsilon_1}(n,s,X) := \begin{cases} \prod_{l=\frac{n+s+1}{2}}^{n-1}(q^{2l}X^2-1) & \text{if } n-s \text{ is odd,} \\ (q^{\frac{n+s}{2}}X-\epsilon\epsilon_1)\prod_{l=\frac{n+s+2}{2}}^{n-1}(q^{2l}X^2-1) & \text{if } n-s \text{ is even.} \end{cases} \tag{7.19}$$

Then

$$h_{\epsilon_1}(n,j,qX) = h_{\epsilon_1}(n+1,j+1,X), \tag{7.20}$$

$$q^{\lfloor\frac{n+j+2}{2}\rfloor}Xh_{\epsilon_1}(n+1,j+1,X) = h_{\epsilon_1}(n+1,j,X) + (-1)^{n+j+1}\epsilon\epsilon_1 h_{\epsilon_1}(n+1,j+1,X). \tag{7.21}$$

Lemma 7.15. *For integers $0 < r \le n-1$ and $\epsilon, \epsilon_1 = \pm 1$, we have*

$$g_{\epsilon_1}(r,r,X) = (-1)^{r(n-1)}\epsilon_1^n\epsilon^r\alpha(r)h_{\epsilon_1}(n,r,X).$$

In particular, $g_{\epsilon_1}(r,r,X)$ is a polynomial of degree $n-r-1$.

Proof. We prove the case when n case is odd and r is even, and leave the other three cases to the reader. The idea is the same (a little bit more complicated). In this case, we need to show

(7.22)
$$\frac{g_{\epsilon_1}(r,r,X)}{h_{\epsilon_1}(n,r,X)} = \sum_{k=0}^{r}(-1)^k q^{\frac{k(k-1)}{2}} \cdot \sum_{\epsilon_2\in\{\pm1\}} m(U_k^{\epsilon_2},U_r^{\epsilon_1})\cdot\alpha(r-k)\beta_\delta(r-k)\cdot\frac{f_{\epsilon_2}(n,k,X)}{h_{\epsilon_1}(n,r,X)} = \epsilon_1\alpha(r),$$

where $\delta = \delta(r,k,\epsilon_1,\epsilon_2)$. Since n is odd and r is even, we have

$$\frac{f_{\epsilon_2}(n,k,X)}{h_{\epsilon_1}(n,r,X)} = \begin{cases} \prod_{\ell=\frac{n+1}{2}}^{\frac{n+k-2}{2}}(q^{2\ell}X^2)\cdot q^{\frac{n+k}{2}}X(q^{\frac{n+k}{2}}X-\epsilon\epsilon_2)\prod_{\ell=\frac{n+2+k}{2}}^{\frac{n+r-1}{2}}(q^{2\ell}X^2-1) & \text{if } k \text{ is odd,} \\ \prod_{\ell=\frac{n+1}{2}}^{\frac{n-1+k}{2}}(q^{2\ell}X^2)\cdot\prod_{\ell=\frac{n+1+k}{2}}^{\frac{n+r-1}{2}}(q^{2\ell}X^2-1) & \text{if } k \text{ is even.} \end{cases}$$

As a result, dividing (7.22) by $q^{(n+\frac{r-1}{2})(\frac{r-1}{2})+\frac{n+r}{2}}\cdot X^r$ and setting $Y=(q^{\frac{n+1}{2}}X)^{-1}, j=r-k$, (7.22) is equivalent to

$$\tilde{g}_{\epsilon_1}(r,Y) := \sum_{j=0}^{r}(-1)^{r-j}q^{\frac{(r-j)(r-j-1)}{2}}\cdot\alpha(j)\cdot\sum_{\epsilon_2\in\{\pm1\}} m(U_{r-j}^{\epsilon_2},U_r^{\epsilon_1})\beta_\delta(j)\cdot\tilde{f}_{\epsilon_2}(j,Y) = \epsilon_1 Y^r, \tag{7.23}$$

where $\tilde{f}_{\epsilon_2}(j,Y)$ is a polynomial of degree j defined as follows:

$$\tilde{f}_{\epsilon_2}(j,Y) := \begin{cases} \prod_{\ell=\frac{r-j}{2}}^{\frac{r-2}{2}}(1-q^{-2\ell}Y^2) & \text{if } j \text{ is even,} \\ (1-\epsilon\epsilon_2 q^{-\frac{r-j-1}{2}}Y)\cdot\prod_{\ell=\frac{r-j+1}{2}}^{\frac{r-2}{2}}(1-q^{-2\ell}Y^2) & \text{if } j \text{ is odd.} \end{cases}$$

Since $\{\tilde{f}_1(j,Y), 0 \le j \le r\}$ forms a basis of the space of polynomials with degree $\le r$, there exists unique tuples (a_j) and (b_j) such that

$$f(Y) = \epsilon_1 Y^r = \sum_{j=0}^{r} a_j \tilde{f}_1(j,Y), \quad \text{and} \quad \tilde{g}_{\epsilon_1}(r,Y) = \sum_{j=0}^{r} b_j \tilde{f}_1(j,Y).$$

We need to show $(a_j) = (b_j)$. It is easy to check

$$a_r = b_r = (-1)^{\frac{r}{2}} \epsilon_1 \alpha(r).$$

Now to prove $a_j = b_j$ for all j, it suffices to prove that both a_j and b_j satisfy the recursion formula for $j < r$

$$a_j = \begin{cases} -q^j \frac{1-q^{-j-2}}{1-q^{j-r}} a_{j+2} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases} \tag{7.24}$$

We start with a_j. First of all, we have

$$\sum_{j=0}^{r} q^{-r} a_j \tilde{f}_1(j,Y) = q^{-r} f(Y) = f(q^{-1}Y) = \sum_{j=0}^{r} a_j \tilde{f}_1(j, q^{-1}Y). \tag{7.25}$$

Notice that

$$\tilde{f}_1(j, q^{-1}Y) = (1 - q^{-r}Y^2)\tilde{f}_1(j-2, Y).$$

Since

$$\tilde{f}_1(j+1,Y)/\tilde{f}_1(j,Y) = \begin{cases} 1 - \epsilon q^{-\frac{r-j-2}{2}} Y & \text{if } j \text{ is even,} \\ 1 + \epsilon q^{-\frac{r-j-1}{2}} Y & \text{if } j \text{ is odd,} \end{cases}$$

we have

$$Y\tilde{f}_1(j,Y) = \begin{cases} -\epsilon q^{\frac{r-j-2}{2}} (\tilde{f}_1(j+1,Y) - \tilde{f}_1(j,Y)) & \text{if } j \text{ is even,} \\ \epsilon q^{\frac{r-j-1}{2}} (\tilde{f}_1(j+1,Y) - \tilde{f}_1(j,Y)) & \text{if } j \text{ is odd,} \end{cases}$$

$$Y^2\tilde{f}_1(j,Y) = \begin{cases} -q^{r-j-2}(\tilde{f}_1(j+2,Y) - \tilde{f}_1(j,Y)) & \text{if } j \text{ is even,} \\ -q^{r-j-2}(\tilde{f}_1(j+2,Y) + (q-1)\tilde{f}_1(j+1,Y) - q\tilde{f}_1(j,Y)) & \text{if } j \text{ is odd.} \end{cases}$$

Therefore,

$$\tilde{f}_1(j, q^{-1}Y) = \begin{cases} q^{-j}\tilde{f}_1(j,Y) + (1-q^{-j})\tilde{f}_1(j-2,Y) & \text{if } j \text{ is even,} \\ q^{-j}\tilde{f}_1(j,Y) + (q-1)q^{-j}\tilde{f}_1(j-1,Y) + (1-q^{1-j})\tilde{f}_1(j-2,Y) & \text{if } j \text{ is odd.} \end{cases}$$

Plugging this into (7.25), we obtain

$$(q^{-r} - q^{-j})a_j = \begin{cases} (q-1)q^{-j-1}a_{j+1} + (1-q^{-j-2})a_{j+2} & \text{if } j \text{ is even,} \\ (1-q^{-j-1})a_{j+2} & \text{if } j \text{ is odd,} \end{cases}$$

with $a_{r+1} = a_{r+2} = 0$. So we have (7.24). We remark that in other cases, we have similar recursion formula as above but could not be simplified like (7.24).

Now we compute b_j for $j < r$. Recall that r is even. First, if $j = 0$, we have

$$\sum_{\epsilon_2 \in \{\pm 1\}} m(U^{\epsilon_2}_{r-j}, U^{\epsilon_1}_r)\beta_\delta(j) \cdot \tilde{f}_{\epsilon_2}(j, Y) = \tilde{f}_{\epsilon_1}(0, Y).$$

It is easy to check that

$$\tilde{f}_{\epsilon_2}(j, Y) = \begin{cases} \tilde{f}_1(j, Y) & \text{if } j \text{ is even,} \\ (1 - \epsilon\epsilon_2 q^{-\frac{r-j-1}{2}})\tilde{f}_1(j-1, Y) & \text{if } j \text{ is odd,} \end{cases} \tag{7.26}$$

Now Lemmas 5.21 and 7.9 imply for $j \neq 0$ and $\delta = \delta(r, j, \epsilon_1, \epsilon_2)$, we have

$$\sum_{\epsilon_2 \in \{\pm 1\}} m(U^{\epsilon_2}_{r-j}, U^{\epsilon_1}_r) \cdot \beta_\delta(j) \cdot \tilde{f}_{\epsilon_2}(j, Y) = \begin{cases} \frac{2\epsilon_1(-1)^{\frac{j}{2}}(\epsilon_1 q^{-\frac{j}{2}} + q^{-\frac{r-j}{2}})}{(1+\epsilon_1 q^{-\frac{j}{2}})(1+q^{-\frac{r-j}{2}})} m(U^1_{r-j}, U^{\epsilon_1}_r)\tilde{f}_1(j, Y) & \text{if } j \text{ is even,} \\ 2(-1)^{\frac{j-1}{2}} m(U^1_{r-j}, U^{\epsilon_1}_r)\tilde{f}_1(j-1, Y) & \text{if } j \text{ is odd.} \end{cases}$$

Plugging this into the definition of $\tilde{g}_{\epsilon_1}(r, X)$ as in (7.23), we obtain

$$\tilde{g}_{\epsilon_1}(r, X) = \sum_{j=0}^{r} b_j \tilde{f}_1(j, Y)$$

with $b_j = 0$ for odd j, $b_0 = q^{\frac{r(r-1)}{2}}(1 - 2q^{-(r-1)}m(U^1_1, U^{\epsilon_1}_r))$, and

$$\begin{aligned} b_j =& 2(-1)^{r-\frac{j}{2}} q^{\frac{(r-j)(r-j-1)}{2}} \frac{(q^{-\frac{j}{2}} + \epsilon_1 q^{-\frac{r-j}{2}})}{(1+\epsilon_1 q^{-\frac{j}{2}})(1+q^{-\frac{r-j}{2}})} m(U^1_{r-j}, U^{\epsilon_1}_r) \\ &- 2(-1)^{r-\frac{j}{2}} q^{\frac{(r-j-1)(r-j-2)}{2}} \alpha(j+1) m(U^1_{r-j-1}, U^{\epsilon_1}_r) \end{aligned}$$

for even $j \neq 0$. Applying Lemma 5.21, for even $j \neq 0$, we have

$$b_j = 2(-1)^{r-\frac{j}{2}} q^{\frac{(r-j)(r-j-1)}{2}} \frac{q^{-\frac{r-j}{2}}(\epsilon_1 + q^{-\frac{r}{2}})}{(1+\epsilon_1 q^{-\frac{j}{2}})(1+q^{-\frac{r-j}{2}})} \alpha(j) m(U^1_{r-j}, U^{\epsilon_1}_r).$$

Now applying Lemma 5.21 twice, we can check that b_j satisfy (7.24). $\square$

Recall that $\mathrm{m}(n, t) := \min\{n - t, n - 1\}$.

Lemma 7.16. *For $0 \leq t \leq n$, and $m = n - t$, we have*

$$\sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(m, r, X) = F_{\epsilon_1}(n, m, X), \tag{7.27}$$

where

$$F_{\epsilon_1}(n, m, X) = \begin{cases} q^{\frac{n(n-1)}{2}} f_{\epsilon_1}(n, m, X) & \text{if } t \neq 0, \\ (-1)^{n-1}\alpha(n) \sum_{\ell=0}^{n-1} (-q^n X)^\ell & \text{if } t = 0. \end{cases}$$

Proof. We treat the case $t = 0$ first. In this case, $\epsilon_1 = \epsilon$. Before we give the details of the proof, we summarize the main idea. Since $\{h_\epsilon(n, s, X), 0 \le s \le n-1\}$ forms a basis of the space of polynomials with degree $\le n-1$, there exists unique tuples $(a_{n,j})$ and $(b_{n,j}) \in \mathbb{Q}^n$ such that

(7.28)
$$F_\epsilon(n, n, X) = \sum_{j=0}^{n-1} a_{n,j} h_\epsilon(n, j, X), \quad \text{and} \quad \sum_{r=0}^{n-1} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} \cdot g_\epsilon(n, r, X) = \sum_{j=0}^{n-1} b_{n,j} h_\epsilon(n, j, X).$$

We need to show $a_{n,j} = b_{n,j}$ for all n and j. We first show that $a_{n,j}$ satisfy the recursion relations (7.32), which gives a description of $a_{n+1,j}$ in terms of $a_{n,j}$ and $a_{n,j-1}$. We can directly check $a_{1,0} = b_{1,0} = 1$. Then by an induction on n, it suffices to show $b_{n,j}$ also satisfies (7.32).

Now we derive (7.32). It is easy to check that

$$F_\epsilon(n+1, n+1, X) = q^{\lfloor \frac{n}{2} \rfloor}(q^{n+1}X) F_\epsilon(n, n, qX) + (-1)^n \alpha(n+1). \tag{7.29}$$

Plugging (7.28) into the above formula and applying Lemma 7.14, we obtain

(7.30)
$$\sum_{j=0}^{n} a_{n+1,j} h_\epsilon(n+1, j, X) = q^{\lfloor \frac{n}{2} \rfloor}(q^{n+1}X) \sum_{j=0}^{n-1} a_{n,j} h_\epsilon(n+1, j+1, X) + (-1)^n \alpha(n+1) h_\epsilon(n, n-1, X).$$

Let

$$\gamma(n, j) = \lfloor \frac{n}{2} \rfloor + n + 1 - \lfloor \frac{n+j+2}{2} \rfloor = \begin{cases} n - \frac{j}{2} & \text{if } j \text{ is even,} \\ n - \lfloor \frac{j-(-1)^n}{2} \rfloor & \text{if } j \text{ is odd.} \end{cases} \tag{7.31}$$

Then Lemma 7.14 and (7.30) imply

$$\begin{aligned} \sum_{j=0}^{n} a_{n+1,j} h_\epsilon(n+1, j, X) = & \sum_{j=0}^{n-1} a_{n,j} q^{\gamma(n,j)} h_\epsilon(n+1, j, X) + \sum_{j=1}^{n} (-1)^{n+j} a_{n,j-1} q^{\gamma(n,j-1)} h_\epsilon(n+1, j, X) \\ & + (-1)^n \alpha(n+1) h_\epsilon(n+1, n, X). \end{aligned}$$

That is

$$\begin{cases} a_{n+1,0} = q^n a_{n,0}, \\ a_{n+1,j} = q^{\gamma(n,j)} a_{n,j} + (-1)^{n+j} q^{\gamma(n,j-1)} a_{n,j-1}, \quad 0 < j < n, \\ a_{n+1,n} = q^{\gamma(n,n-1)} a_{n,n-1} + (-1)^n \alpha(n+1). \end{cases} \tag{7.32}$$

Now we compute $b_{n,j}$. A direct computation shows that $b_{n,0} = q^{\frac{n(n-1)}{2}}$. In the following, we compute b_j for $j \neq 0$.

For $r = 0$, we have

$$g_\epsilon(n, 0, X) = f_1(n, 0, X) = \begin{cases} h_\epsilon(n, 0, X) & \text{if } n \text{ is odd,} \\ h_\epsilon(n, 0, X) + (1-\epsilon) h_\epsilon(n, 1, X) & \text{if } n \text{ is even.} \end{cases}$$

Now we assume $r \neq 0$. Recall that by Lemma 7.13, we have

$$g_\epsilon(n, r, X) = \sum_{\epsilon_3 \in \{\pm 1\}} m(U_r^{\epsilon_3}, U_n^\epsilon) g_{\epsilon_3}(r, r, X).$$

Notice that when $n-r$ is odd, $h_{\epsilon_1}(n,r,X)$ is independent of ϵ_1. Then a direct calculation using Lemma 7.15 and the formula for $m(U_r^{\epsilon_1}, U_n^{\epsilon})$ gives

$$g_\epsilon(n,r,X) = \alpha(r) h_\epsilon(n,r,X) m(U_r^\epsilon, U_n^\epsilon) \begin{cases} \frac{2q^{-\frac{r}{2}}}{1+\epsilon q^{-\frac{r}{2}}} & \text{if } n \equiv r-1 \equiv 1 \pmod 2, \\ -2\epsilon & \text{if } n \equiv r-1 \equiv 0 \pmod 2. \end{cases} \tag{7.33}$$

When $n-r$ is even, we have

$$h_{\epsilon_3}(n,r,X) = (q^{\frac{n+r}{2}} X - \epsilon\epsilon_3) h_\epsilon(n, r+1, X). \tag{7.34}$$

So a direct calculation gives

$$g_\epsilon(n,r,X) = \alpha(r) h_\epsilon(n,r+1,X) m(U_r^\epsilon, U_n^\epsilon) \left(q^{\frac{n+r}{2}} X - 1 + (-1)^n (q^{\frac{n+r}{2}} X + 1) \frac{m(U_r^{-\epsilon}, U_n^\epsilon)}{m(U_r^\epsilon, U_n^\epsilon)} \right). \tag{7.35}$$

We have by Lemma 5.18

$$\frac{m(U_r^{-\epsilon}, U_n^\epsilon)}{m(U_r^\epsilon, U_n^\epsilon)} = \frac{1-q^{-\frac{n-r}{2}}}{1+q^{-\frac{n-r}{2}}} \begin{cases} 1 & \text{if } n \equiv r \equiv 1 \pmod 2, \\ \frac{1-\epsilon q^{-\frac{r}{2}}}{1+\epsilon q^{-\frac{r}{2}}} & \text{if } n \equiv r \equiv 0 \pmod 2. \end{cases}$$

The equation (7.34) gives

$$X h_\epsilon(n, r+1, X) = q^{-\frac{n+r}{2}} (h_\epsilon(n,r,X) + h_\epsilon(n,r+1,X)).$$

So when $n \equiv r \equiv 1 \pmod 2$, we have

$$g_\epsilon(n,r,X) = 2\alpha(r) m(U_r^\epsilon, U_n^\epsilon) \left(\frac{q^{-\frac{n-r}{2}}}{1+q^{-\frac{n-r}{2}}} h_\epsilon(n,r,X) + \frac{q^{-\frac{n-r}{2}} - 1}{1+q^{-\frac{n-r}{2}}} h_\epsilon(n,r+1,X) \right),$$

and when $n \equiv r \equiv 0 \pmod 2$ and $r \neq 0$, we have

$$\begin{aligned} &g_\epsilon(n,r,X) \\ &= \frac{2\alpha(r) m(U_r^\epsilon, U_n^\epsilon)}{(1+q^{-\frac{n-r}{2}})(1+\epsilon q^{-\frac{r}{2}})} \left((1+\epsilon q^{-\frac{n}{2}}) h_\epsilon(n,r,X) + (1-q^{-\frac{n-r}{2}})(1-\epsilon q^{-\frac{r}{2}}) h_\epsilon(n,r+1,X) \right). \end{aligned}$$

In summary, we have the numbers $b_{n,j}$ for $j \neq 0$ are given by the following.
If n and j are odd, then

$$b_{n,j} = 2(-1)^j q^{\frac{(n-j)(n-j-1)}{2}} \alpha(j) \frac{m(U_j^\epsilon, U_n^\epsilon)}{1+q^{\frac{n-j}{2}}}.$$

If n is odd and j is even, then

$$b_{n,j} = 2(-1)^j q^{\frac{(n-j)(n-j-1)}{2}} \left(\frac{\alpha(j) q^{-\frac{j}{2}}}{1+\epsilon q^{-\frac{j}{2}}} m(U_j^\epsilon, U_n^\epsilon) - \frac{q^{n-j} \alpha(j-1)(q^{-\frac{n-j+1}{2}} - 1)}{1+q^{-\frac{n-j+1}{2}}} m(U_{j-1}^\epsilon, U_n^\epsilon) \right).$$

If n and j are even, then

$$b_{n,j} = \frac{2(-1)^j q^{\frac{(n-j)(n-j-1)}{2}} \alpha(j)(1+\epsilon q^{-\frac{n}{2}})}{(1+q^{-\frac{n-j}{2}})(1+\epsilon q^{-\frac{j}{2}})} m(U_j^\epsilon, U_n^\epsilon).$$

If n is even, then for $j=1$ we have

$$b_{n,1} = q^{\frac{n(n-1)}{2}} (1 - \epsilon + 2\epsilon q^{-(n-1)}) m(U_1^1, U_n^\epsilon),$$

and for odd $j > 1$, $b_{n,j}$ is equal to

$$2(-1)^{j+1}q^{\frac{(n-j)(n-j-1)}{2}}\left(\epsilon\alpha(j)m(U_j^\epsilon, U_n^\epsilon) + \frac{q^{n-j}\alpha(j-1)(1-q^{-\frac{n-j+1}{2}})(1-\epsilon q^{-\frac{j-1}{2}})}{(1+q^{-\frac{n-j+1}{2}})(1+\epsilon q^{-\frac{j-1}{2}})}m(U_{j-1}^\epsilon, U_n^\epsilon)\right).$$

Using the explicit formulas, direct calculation shows that $b_{n,j}$ satisfies (7.32).

From now, we assume $t \neq 0$ and let $m = n - t$. The proof is essentially the same as the proof of Lemma 7.15 and we only prove the case that n is odd and m is even in detail. According to Lemma 7.13, we have

$$\begin{aligned}
&\sum_{r=0}^{\mathrm{m}(n,t)}(-1)^r q^{\frac{(n-r)(n-r-1)}{2}}q^{rt}\cdot g_{\epsilon_1}(m,r,X)\\
&=\sum_{r=0}^{\mathrm{m}(n,t)}(-1)^r q^{\frac{(n-r)(n-r-1)}{2}}q^{rt}\cdot\sum_{\epsilon_3\in\{\pm1\}}m(U_r^{\epsilon_3},U_m^{\epsilon_1})g_{\epsilon_3}(r,r,X)\\
&=q^{\frac{(n-m)(n+m-1)}{2}}\sum_{r=0}^{\mathrm{m}(n,t)}(-1)^r q^{\frac{(m-r)(m-r-1)}{2}}\cdot\sum_{\epsilon_3\in\{\pm1\}}m(U_r^{\epsilon_3},U_m^{\epsilon_1})g_{\epsilon_3}(r,r,X).
\end{aligned}$$

Assume that n is odd and m is even. Factoring out $h_{\epsilon_1}(n,m,X)$, replacing X by $q^{\frac{n-1}{2}}X$, and apply Lemma 7.15, we have that (7.27) is equivalent to

$$q^{\frac{(n-m)(n+m-1)}{2}}\sum_{r=0}^{m}(-1)^r q^{\frac{(m-r)(m-r-1)}{2}}\cdot\sum_{\epsilon_3\in\{\pm1\}}m(U_r^{\epsilon_3},U_m^{\epsilon_1})\alpha(r)g'_{\epsilon_3}(m,r,X)=F'(n,m,X), \tag{7.36}$$

where

$$g'_{\epsilon_3}(m,r,X)=\begin{cases}\epsilon\epsilon_3(q^{\frac{r+1}{2}}X-\epsilon\epsilon_3)\cdot\prod_{\ell=\frac{r+3}{2}}^{\frac{m}{2}}(q^{2\ell}X^2-1) & \text{if } r \text{ is odd,}\\ \epsilon_3\prod_{\ell=\frac{r+2}{2}}^{\frac{m}{2}}(q^{2\ell}X^2-1) & \text{if } r \text{ is even,}\end{cases}$$

and

$$F'(n,m,X)=q^{\frac{n(n-1)}{2}}\prod_{\ell=1}^{\frac{m}{2}}(q^{2\ell}X^2).$$

Since $g'_\epsilon(r,m,X)$ forms a basis of the space of polynomials with degree $\leq m$, there exists unique tuples (a_j) and (b_j) such that

$$\text{LHS of (7.36)}=\sum_{j=0}^{m}a_j g'_\epsilon(m,j,X),\quad \text{RHS of (7.36)}=\sum_{j=0}^{m}b_j g'_\epsilon(m,j,X).$$

It suffices to show that $a_0 = b_0$ and both (a_j) and (b_j) satisfy the following recursive relation for $0 < r \leq m$:

$$a_r=\begin{cases}0 & \text{if } r \text{ is odd,}\\ \frac{q^{m-r+2}-1}{q^m-q^{m-r}}a_{r-2} & \text{if } r \text{ is even.}\end{cases} \tag{7.37}$$

We derive the recursive relation for a_j first. Notice that

$$g'_\epsilon(m,r,qX)=(q^{m+2}X^2-1)g'_\epsilon(m,r+2,X).$$

Moreover,

$$Xg'_\epsilon(m,r,X) = \begin{cases} q^{-\frac{r+1}{2}}(\epsilon g'_\epsilon(m,r-1,X) - g'_\epsilon(m,r,X)) & \text{if } r \text{ is odd,} \\ q^{-\frac{r}{2}}(\epsilon g'_\epsilon(m,r-1,X) + g'_\epsilon(m,r,X)) & \text{if } r \text{ is even.} \end{cases}$$

$$X^2 g'_\epsilon(m,r,X) = \begin{cases} q^{-(r+1)}(qg'_\epsilon(m,r-2,X) + \epsilon(q-1)g'_\epsilon(m,r-1,X) + g'_\epsilon(m,r,X)) & \text{if } r \text{ is odd,} \\ q^{-r}(g'_\epsilon(m,r-2,X) + g'_\epsilon(m,r,X)) & \text{if } r \text{ is even.} \end{cases}$$

Hence if m is even, then

$$g'_\epsilon(m,r,qX)$$
$$= \begin{cases} q^{m-r}g'_\epsilon(m,r,X) + \epsilon(q-1)q^{m-r-1}g'_\epsilon(m,r+1,X) + (q^{m-r-1}-1)g'_\epsilon(m,r+2,X) & \text{if } r \text{ is odd,} \\ q^{m-r}g'_\epsilon(m,r,X) + (q^{m-r}-1)g'_\epsilon(m,r+2,X) & \text{if } r \text{ is even.} \end{cases}$$

Hence for $1 < r \leq m+1$

$$(q^m - q^{m-r})a_r = \begin{cases} (q^{m-r+1}-1)a_{r-2} & \text{if } r \text{ is odd,} \\ \epsilon(q-1)q^{m-r}a_{r-1} + (q^{m-r+2}-1)a_{r-2} & \text{if } r \text{ is even,} \end{cases} \tag{7.38}$$

with $a_{m+1} = 0$, which implies (7.37).

Now we compute b_j. First, when $r = 0$,

$$\sum_{\epsilon_3 \in \{\pm 1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1})g'_{\epsilon_3}(m,0,X) = \epsilon g'_\epsilon(m,0,X).$$

For $r \neq 0$,

$$\sum_{\epsilon_3 \in \{\pm 1\}} m(U_r^{\epsilon_3}, U_m^{\epsilon_1})g'_{\epsilon_3}(m,r,X) = \begin{cases} -2\epsilon m(U_r^{\epsilon}, U_m^{\epsilon_1})g'_\epsilon(m,r+1,X) & \text{if } r \text{ is odd,} \\ \epsilon(m(U_r^1, U_m^{\epsilon_1}) - m(U_r^{-1}, U_m^{\epsilon_1}))g'_\epsilon(m,r,X) & \text{if } r \text{ is even.} \end{cases}$$

Then

$$b_r = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ a'_{r-1} + a'_r & \text{if } r \text{ is even.} \end{cases}$$

where

$$a'_r = q^{\frac{(n-m)(n+m-1)}{2}} \begin{cases} -2(-1)^r q^{\frac{(m-r)(m-r-1)}{2}}\alpha(r)\epsilon m(U_{r-1}^{\epsilon}, U_m^{\epsilon_1}) & \text{if } r \text{ is odd,} \\ (-1)^r\alpha(r)q^{\frac{(m-r)(m-r-1)}{2}}\epsilon(m(U_r^1, U_m^{\epsilon_1}) - m(U_r^{-1}, U_m^{\epsilon_1})) & \text{if } r \text{ is even.} \end{cases}$$

Finally, a direct calculation shows that $a_0 = b_0 = \epsilon q^{\frac{n(n-1)}{2}}$ and b_r satisfies (7.38). $\square$

7.6. **Proof of Theorem 7.1.** Now we are ready to prove Theorem 7.1.

Recall that

$$\mathrm{Pden}'(L) = \frac{\mathrm{Pden}'(I_n, L)}{\mathrm{Pden}(I_n, I_n)},$$

and

$$\mathrm{Pden}(I_n, I_n) = 2\alpha(n)I_\epsilon(n),$$

where $I_\epsilon(n)$ is defined in Lemma 7.7.

We first assume that $L = H^j \oplus L_1$ with $j > 0$ where $L_1 = I_{n_1-t} \oplus L_2$ is an integral lattice of rank n_1 and type $t > 0$ (the other non-integral cases were taken care of in the summary of the proof at the beginning of this section). By Lemma 5.13, we have

$$\mathrm{Pden}'(I_n, L) = 2\Big(\prod_{\ell=1}^{j-1}(1-q^{2\ell})\Big)\mathrm{Pden}(I_n, L_1, q^{2j}).$$

It suffices to show $\mathrm{Pden}(I_n, L_1, q^{2j}) = 0$. By Theorem 6.1 and Lemma 7.11, we have

$$\begin{aligned}\mathrm{Pden}(I_n, L, q^{2j}) &= \sum_{i=0}^{n_1} \mathrm{Pden}^{n_1-i}(I_n^{-\epsilon}, L, q^{2j}) \\ &= I_\epsilon(n, n_1) \sum_{r=0}^{n_1-t} \sum_{i=0}^{n_1} \binom{n_1-r}{i-r}_q (-1)^{n_1+i-r} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n_1, n_1-t, r, q^{j-i}).\end{aligned} \tag{7.39}$$

Notice that the assumption $t > 0$ implies that $r \leq n_1 - t \leq n_1 - 1$. Hence $g_{\epsilon_1}(n_1, n-t, r, X)$ is a polynomial of degree $n_1 - r - 1$ by Lemma 7.15. Then we may apply Corollary 5.25 to conclude

$$\sum_{i=0}^{n_1} \binom{n_1-r}{i-r}_q (-1)^{n_1+i-r} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n_1, n_1-t, r, q^{\frac{n-n_1}{2}-i}) = 0.$$

Hence

$$\mathrm{Pden}'(L) = \frac{2\Big(\prod_{\ell=1}^{j-1}(1-q^{2\ell})\Big)}{\mathrm{Pden}(I_n, I_n)} \mathrm{Pden}(I_n, L, q^{n-n_1}) = 0.$$

Next, we assume $L = I_{n-t} \oplus L_2$ is integral of rank n and type t (Cases (2) and (3) or equivalently Case (b) in the summary of the proof at the beginning of this section). Similarly, by Theorem 6.1 and Lemma 7.11, we have

$$\begin{aligned}\mathrm{Pden}'(I_n, L) &= \sum_{i=0}^{n-1} (\mathrm{Pden}^{n-i})'(I_n, L) \\ &= 2I_\epsilon(n) \sum_{r=0}^{\mathrm{m}(n,t)} \sum_{i=0}^{n-1} \binom{n-r}{i-r}_q (-1)^{n-1+i+r} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n, n-t, r, q^{-i}).\end{aligned} \tag{7.40}$$

Here, recall that $\mathrm{m}(n,t) \coloneqq \min\{n-t, n-1\}$. Applying Corollary 5.25 as before, we have

$$\sum_{i=0}^{n} \binom{n-r}{i-r}_q (-1)^r (-1)^{n-1+i} q^{\frac{(i-r)(i-r-1)}{2}} q^{rt} \cdot g_{\epsilon_1}(n, n-t, r, q^{-i}) = 0.$$

Hence,

$$\mathrm{Pden}'(I_n, L) = 2I_\epsilon(n) \sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g_\epsilon(n, n-t, r, q^{-n}).$$

By Lemma 7.16, if $t \neq 0$, then

$$\sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g_\epsilon(n, n-t, r, q^{-n}) = F_{\epsilon_1}(n, n-t, q^{-n})$$
$$= \alpha(n) \cdot \begin{cases} \prod_{\ell=1}^{\frac{t-1}{2}} (1-q^{2\ell}) & \text{if } t \text{ is odd,} \\ (1-\epsilon\epsilon_1 q^{\frac{t}{2}}) \prod_{\ell=1}^{\frac{t}{2}-1} (1-q^{2\ell}) & \text{if } t \text{ is even.} \end{cases}$$

Notice that if t is even, then $\chi(I_{n-t}^{\epsilon_1})\chi(L_2) = \epsilon$. Hence $\epsilon\epsilon_1 = \chi(L_2)$.

If $t = 0$, then by Lemma 7.16,

$$\sum_{r=0}^{\mathrm{m}(n,t)} (-1)^r q^{\frac{(n-r)(n-r-1)}{2}} q^{rt} \cdot g_\epsilon(n, n, r, q^{-n}) = F_{\epsilon_1}(n, n, q^{-n}) = \alpha(n) \cdot \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This proves the theorem.

8. Fourier transform: the analytic side

In this section, we study the partial Fourier transform of the vertical part of the analytic side following Section 8 of [LZ22b]. The main result is Theorem 8.2.

Definition 8.1. For a non-degenerate lattice $L^\flat \subset \mathbb{V}$ of rank $n-1$, and $x \in \mathbb{V} \setminus L_F^\flat$, we define

$$\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x) = \sum_{\substack{L^\flat \subset L' \subset L'^\sharp \\ L'^\flat \notin \mathrm{Hor}(L^\flat)}} \partial\mathrm{Pden}(L') 1_{L'}(x),$$

where $L'^\flat = L' \cap L_F^\flat$.

Theorem 8.2. *Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$, and let $\mathbb{W} = (L_F^\flat)^\perp$ be the perpendicular space of $L_F^\flat$ in $\mathbb{V}$. Recall the partial Fourier transform*

$$\partial\mathrm{Den}_{L^\flat,\mathscr{V}}^\perp(x) := \int_{L_F^\flat} \partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x+y)\, dy, \quad x \in \mathbb{W} \setminus \{0\}.$$

Then $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}^\perp(x)$ is constant on $\mathbb{W}^{\geq 0} \setminus \{0\}$ and is zero for $x \in \mathbb{W}^{<0}$.

Proof. It suffices to show that if $\mathrm{val}(x) > 0$, then

$$\partial\mathrm{Den}_{L^\flat,\mathscr{V}}^\perp(x) - \partial\mathrm{Den}_{L^\flat,\mathscr{V}}^\perp(\pi^{-1}x) = 0.$$

By definition, we have

$$\partial\mathrm{Den}_{L^\flat,\mathscr{V}}^\perp(x) = \int_{L_F^\flat} \sum_{\substack{L^\flat \subset L' \subset L'^\sharp \\ L'^\flat \notin \mathrm{Hor}(L^\flat)}} \partial\mathrm{Pden}(L') 1_{L'}(x+y)\, dy,$$

where L' runs over lattices of rank n in $L_F^\flat + \langle x \rangle$.

Recall that $\mathrm{Pr}_{L_F^\flat}$ denotes the projection to $L_F^\flat$. We rewrite the summation based on $L' \cap L_F^\flat$ and $\mathrm{Pr}_{L_F^\flat}(L')$. For lattices $L'^\flat \subset \widetilde{L}'^\flat$ in $L_F^\flat$ of rank $n-1$, let

$$\mathrm{Lat}(L'^\flat, \widetilde{L}'^\flat) := \{L' \subset \mathbb{V} \mid L' \cap L_F^\flat = L'^\flat, \quad \mathrm{Pr}_{L_F^\flat}(L') = \widetilde{L}'^\flat\}.$$

Then by Lemmas 7.2.1 and 7.2.2 of [LZ22a], we have

$$\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}(x)=\sum_{\substack{L^\flat\subset L'^\flat\\ L'^\flat\notin\mathrm{Hor}(L^\flat)}}\ \sum_{\substack{L'^\flat\subset\widetilde{L}'^\flat\\ \widetilde{L}'^\flat/L'^\flat\text{ cyclic}}}\ \sum_{L'\in\mathrm{Lat}(L'^\flat,\widetilde{L}'^\flat)}\partial\mathrm{Pden}(L')\int_{L^\flat_F}1_{L'}(x+y)\,dy.$$

Here we can switch the order of the sum and integral because there are only finitely many nonzero terms in the sum for a fixed x. Since $\tilde{L}'^\flat/L'^\flat$ is cyclic, it has a generator $u^\flat\in L^\flat_F$. Moreover, for $L'\in\mathrm{Lat}(L'^\flat,\widetilde{L}'^\flat)$, we can write $L'=L'^\flat+\langle u\rangle$ with $u=u^\flat+u^\perp\in\mathbb{V}$ where $0\neq u^\perp\in\mathbb{W}$. Moreover, write $x=\alpha u^\perp$ with $\alpha\in F^\times$, then

$$x+y=\alpha u+(y-\alpha u^\flat)\in L'$$

if and only if $\alpha\in O_F$ and $y-\alpha u^\flat\in L'^\flat$. As a result, we have

$$\int_{L^\flat_F}1_{L'}(x+y)-1_{L'}(\pi^{-1}x+y)dy=\begin{cases}\mathrm{vol}(L'^\flat), & \text{if }\langle x\rangle=\langle u^\perp\rangle,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, we have

$$\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}(x)-\partial\mathrm{Den}^{\perp}_{L^\flat,\mathscr{V}}(\pi^{-1}x)=\sum_{\substack{L^\flat\subset L'^\flat\\ L'^\flat\notin\mathrm{Hor}(L^\flat)}}\mathrm{vol}(L'^\flat)D(L'^\flat)(x),\tag{8.1}$$

where

$$D(L'^\flat)(x)=\sum_{\substack{L'^\flat\subset\widetilde{L}'^\flat\\ \widetilde{L}'^\flat/L'^\flat\text{ cyclic}}}\ \sum_{\substack{u^\perp\in\langle x\rangle\text{ generator}\\ L'=L'^\flat+\langle u^\flat+u^\perp\rangle}}\partial\mathrm{Pden}(L')=\sum_{\substack{u^\flat\in(L'^\flat)^\sharp/L'^\flat\\ \mathrm{val}(u^\flat)\geq 0}}\partial\mathrm{Pden}(L'^\flat+\langle u^\flat+x\rangle).\tag{8.2}$$

Here the last step uses the fact that $L'=L'^\flat+\langle u^\flat+x\rangle$ is integral if and only if $u^\flat\in(L'^\flat)^\sharp/L'^\flat$ and $\mathrm{val}(u^\flat)\geq 0$ (since $\mathrm{val}(x)>0$).

It suffices to show $D(L'^\flat)(x)=0$ for any $L'^\flat$ such that $L^\flat\subset L'^\flat$ and $L'^\flat\notin\mathrm{Hor}(L^\flat)$. To show this, we write $L'^\flat=I^{\epsilon_1}_{n_1}\obot L_2$ where L_2 is of full type (of rank $n-1-n_1$). Let u_2 be the projection of $u^\flat$ to L_2. Then

$$(L'^\flat)^\sharp/L'^\flat=(I^{\epsilon_1}_{n_1}\obot(L_2)^\sharp)/L'^\flat\cong L_2^\sharp/L_2\quad\text{and}\quad L'^\flat+\langle u^\flat+x\rangle=L'^\flat+\langle u_2+x\rangle.$$

We consider a partition of

$$L_2^\sharp/L_2=S^+(L_2)\sqcup S^0(L_2)\sqcup S^-(L_2)$$

with

$$S^+(L_2)=(\pi L_2^\sharp)^{\circ\circ}/L_2,\quad S^0(L_2)=((\pi L_2^\sharp)^\circ-(\pi L_2^\sharp)^{\circ\circ})/L_2,\quad S^-(L_2)=((L_2^\sharp)^\circ-(\pi L_2^\sharp)^\circ)/L_2.$$

Here, for a lattice L,

$$L^\circ:=\{x\in L\mid\mathrm{val}(x)\geq 0\}\text{ and }L^{\circ\circ}:=\{x\in L\mid\mathrm{val}(x)>0\}.$$

In general, for a full type lattice L_2, we also define

$$\mu^+(L):=|(\pi L^\sharp)^{\circ\circ}/L|,\quad\mu^0(L):=|((\pi L^\sharp)^\circ-(\pi L^\sharp)^{\circ\circ})/L|,\quad\mu^-(L):=|((L^\sharp)^\circ-(\pi L^\sharp)^\circ)/L|.\tag{8.3}$$

For $\nu \in \{\pm 1\}$, let

$$\mu^{0,\nu}(L) := |\{u \in (\pi L^{\sharp})^{\circ} - (\pi L^{\sharp})^{\circ\circ} : \chi(q(u)) = \nu\}/L|. \tag{8.4}$$

Since $L' = L'^{\flat} + \langle u_2 + x\rangle$ with u_2 integral and $\mathrm{val}(x) > 0$, it is not hard to check that

$$t(L') = \begin{cases} t(L'^{\flat}) + 1 & \text{if } u_2 \in S^+(L_2), \\ t(L'^{\flat}) & \text{if } u_2 \in S^0(L_2), \\ t(L'^{\flat}) - 1 & \text{if } u_2 \in S^-(L_2). \end{cases}$$

Set $t = t(L'^{\flat})$. There are two cases.

When t is odd, we can write

$$L' = L'^{\flat} + \langle u_2 + x\rangle = \begin{cases} I_{n_1}^{\epsilon_1} \oplus L_2' & \text{if } u \in S_2^+(L_2), \\ I_{n_1}^{\epsilon_1} \oplus I_2^1 \oplus L_2' & \text{if } u \in S_2^-(L_2). \end{cases}$$

In both cases, a simple calculation gives

$$\chi(L_2') = \epsilon\epsilon_1.$$

For $t > 1$, by Theorem 7.1,

$$\partial\mathrm{Pden}(L') = (1 - \epsilon\epsilon_1 q^{\frac{t-1}{2}}) \prod_{\ell=1}^{\frac{t-1}{2}-1} (1 - q^{2\ell}) \cdot \begin{cases} (1 - \epsilon\epsilon_1 q^{\frac{t+1}{2}})(1 + \epsilon\epsilon_1 q^{\frac{t-1}{2}}) & \text{if } u_2 \in S^+(L_2), \\ 1 + \epsilon\epsilon_1 q^{\frac{t-1}{2}} & \text{if } u_2 \in S^0(L_2), \\ 1 & \text{if } u_2 \in S^-(L_2). \end{cases}$$

For $t = 1$, $S^-(L_2)$ is empty and $L'^{\flat} \notin \mathrm{Hor}(L^{\flat})$ implies that $L' = L_{n_1}^{\epsilon_1} \oplus L_2'$ with $\chi(L_2') = 1$, i.e. $\epsilon_1 = \epsilon$. In this case, by Theorem 7.1,

$$\partial\mathrm{Pden}(L') = \begin{cases} 1 - q & \text{if } u \in S^+(L_2), \\ 1 & \text{if } u \in S^0(L_2). \end{cases}$$

Hence by (8.2), we have $D(L'^{\flat}) = 0$ if

$$(1 - \epsilon\epsilon_1 q^{\frac{t+1}{2}})(1 + \epsilon\epsilon_1 q^{\frac{t-1}{2}})\mu^+(L_2) + (1 + \epsilon\epsilon_1 q^{\frac{t-1}{2}})\mu^0(L_2) + \mu^-(L_2) = 0. \tag{8.5}$$

When $t = t(L'^{\flat})$ is even, $L'^{\flat} \notin \mathrm{Hor}(L^{\flat})$ implies that $t > 0$. Moreover, if $u_2 \in S^0(L_2)$, we have a decomposition (since $u_2 \in L_2$ is perpendicular to $I_{n_1}^{\epsilon_1}$)

$$L' = L'^{\flat} + \langle u_2 + x\rangle = I_{n_1}^{\epsilon_1} \oplus \langle u_2 + x\rangle \oplus L_2'$$

for some full type lattice L_2' of rank t. Then a direct calculation gives

$$\chi(L_2') = (-1)^{n_1}\epsilon_1\epsilon\chi(u_2).$$

So we have by Theorem 7.1,

$$\partial\mathrm{Pden}(L') = \prod_{\ell=1}^{\frac{t}{2}-1} (1 - q^{2\ell}) \cdot \begin{cases} (1 - q^t) & \text{if } u_2 \in S^+(L_2), \\ 1 - (-1)^{n_1}\epsilon_1\epsilon\chi(u_2) q^{\frac{t}{2}} & \text{if } u_2 \in S^0(L_2), \\ 1 & \text{if } u_2 \in S^-(L_2). \end{cases}$$

Hence by (8.2), we have $D(L'^{\flat}) = 0$ if

$$(1-q^t)\mu^+(L_2) + (1-(-1)^{n_1}\epsilon_1\epsilon q^{\frac{t}{2}})\mu^{0,1}(L_2) + (1+(-1)^{n_1}\epsilon_1\epsilon q^{\frac{t}{2}})\mu^{0,-1}(L_2) + \mu^-(L_2) = 0. \tag{8.6}$$

Now (8.5) follows from Proposition 8.7 and (8.6) follows from Proposition 8.9. Hence we have $D(L'^{\flat}) = 0$ for $L'^{\flat}$ such that $L^{\flat} \subset L'^{\flat}$ and $L'^{\flat} \notin \mathrm{Hor}(L^{\flat})$. Now the theorem follows from (8.1). □

The remaining of this section is devoted to prove Propositions 8.7 and 8.9.

Definition 8.3. Let L and L' be lattices of full type such that $L \subset L' \subset \pi^{-1}L$. For $? \in \{+, 0, -, \{0,+1\}, \{0,-1\}\}$, define

$$\mu^?(L, L') := \mu^?(L) - [L' : L]\mu^?(L'),$$

where $\mu^?(L)$ is defined in (8.3) and (8.4).

Lemma 8.4. *Let L be a full type lattice of rank t. Then*

$$\mu^+(L) + \mu^0(L) + \mu^-(L) = q^t \cdot \mu^+(L).$$

Let L and L' be full type lattices of rank t such that $L \subset L' \subset \pi^{-1}L$. Then

$$\mu^+(L, L') + \mu^0(L, L') + \mu^-(L, L') = q^t \cdot \mu^+(L, L').$$

Proof. It suffices to show that the following map

$$(L^{\sharp})^{\circ}/L \to (\pi L^{\sharp})^{\circ\circ}/L, \quad x \mapsto \pi x$$

is surjective and every fiber of this map has size q^t. For $x \in (\pi L^{\sharp})^{\circ\circ} = \pi(L^{\sharp})^{\circ}$, the fiber at x is

$$\{\pi^{-1}(y+x) \in (L^{\sharp})^{\circ} : y \in \overline{L}\}.$$

Since $x \in \pi(L^{\sharp})^{\circ}$,

$$\pi^{-1}(y+x) \in (L^{\sharp})^{\circ} \iff (y+x) \in \pi(L^{\sharp})^{\circ} \iff y \in \pi(L^{\sharp})^{\circ}.$$

Moreover, the assumption that L is a full type lattice implies that $L \subset \pi(L^{\sharp})^{\circ}$. Hence

$$|\{\pi^{-1}(y+x) \in (L^{\sharp})^{\circ} : y \in \overline{L}\}| = |\overline{L}| = q^t.$$

This proves the first statement. The second statement follows from the first and the definition of $\mu^?(L, L')$. □

Definition 8.5. Let L be a full type lattice of rank t. We call L maximal of type t if $t(L') < t$ for any L' such that $L \subsetneq L' \subset L_F$.

Lemma 8.6. *If L is non-maximal full type lattice of rank t, then there exists a L' such that $L \subset L' \subset \pi^{-1}L$ and*

$$\mu^+(L, L') + \mu^0(L, L') = q \cdot \mu^+(L, L').$$

Proof. We need to find a L' such that $L \subset L' \subset \pi^{-1}L$ and

$$|((\pi L^\sharp)^\circ - (\pi L'^\sharp)^\circ)/L| = q \cdot |((\pi L^\sharp)^{\circ\circ} - (\pi L'^\sharp)^{\circ\circ})/L|.$$

Let $(a_1, \cdots, a_t)$ be the fundamental invariants of L. We consider two cases seperately.
(i) If a_t is even and $a_t \geq 4$, then we may choose a normal basis $\{\ell_1, \cdots, \ell_t\}$ of L such that $\langle \ell_1, \cdots, \ell_{t-1} \rangle \perp \ell_t$. Write $(\ell_t, \ell_t) = u_t(-\pi_0)^{\frac{a_t}{2}}$. In this case, we choose $L' = \langle \ell_1, \cdots, \ell_{t-1}, \pi^{-1}\ell_t \rangle$, with fundamental invariants $(a_1, \cdots, a_{t-1}, a_t - 2)$. Then

$$\pi L^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_t \rangle \text{ and } \pi L'^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_{t-1}+1}\ell_{t-1}, \pi^{-a_t+2}\ell_t \rangle.$$

For a fixed $x_0 = \sum_{1 \leq i < t} s_i \pi^{-a_i+1}\ell_i$ where $s_i \in O_F$, let

$$S_{x_0}^\circ := \{x \in (\pi L^\sharp)^\circ - (\pi L'^\sharp)^\circ : x = x_0 + s_t\pi^{-a_t+1}\ell_t, s_t \in O_F\}/L,$$
$$S_{x_0}^{\circ\circ} := \{x \in (\pi L^\sharp)^{\circ\circ} - (\pi L'^\sharp)^{\circ\circ} : x = x_0 + s_t\pi^{-a_t+1}\ell_t, s_t \in O_F\}/L.$$

It suffices to show $|S_{x_0}^\circ| = q \cdot |S_{x_0}^{\circ\circ}|$. Notice that $x = x_0 + s_t\pi^{-a_t+1}\ell_t \in S_{x_0}^\circ$ if and only if

$$s_t \in O_F^\times, \quad (x, x) = u_t(-\pi_0)^{-\frac{a_t}{2}+1}(u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0, x_0) + s_t\bar{s}_t) \in O_{F_0},$$

and $x \in S_{x_0}^{\circ\circ}$ if and only if

$$s_t \in O_F^\times, \quad (x, x) = u_t(-\pi_0)^{-\frac{a_t}{2}+1}(u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0, x_0) + s_t\bar{s}_t) \in (\pi_0).$$

Consider the π-adic expansions

$$s_t = \sum_{i \geq 0}^{\infty} b_i \pi^i, \quad -u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0, x_0) = \sum_{i \geq 0}^{\infty} c_i \pi^i,$$

where $b_i, c_i \in O_{F_0}/(\pi_0)$. Then $x \in S_{x_0}^\circ$ if and only if $s_t \in O_F^\times$, and

$$\begin{aligned} c_0 &= b_0^2, \\ c_1 &= b_1 b_0 - b_0 b_1, \\ c_2 &= b_2 b_0 - b_1 b_1 + b_0 b_2, \\ &\cdots \\ c_{a_t-3} &= \sum_{i=0}^{a_t-3} (-1)^i b_{a_t-3-i} b_i. \end{aligned}$$

Similarly, $x \in S_{x_0}^{\circ\circ}$ if and only if $x \in S_{x_0}^\circ$ and

$$c_{a_t-2} - \sum_{i=1}^{a_t-3} (-1)^i b_{a_t-2-i} b_i = 2b_{a_t-2} b_0.$$

Since $s_t \in O_F^\times$, $b_0 \neq 0$ and b_{a_t-2} is uniquely determined by the above equation. Hence $|S_{x_0}^\circ| = q \cdot |S_{x_0}^{\circ\circ}|$ as a result.

(ii) If a_t is odd (since L is non-maximal, $a_t > 1$ in this case) or $a_t = 2$, then we may choose a normal basis $\{\ell_1, \cdots, \ell_t\}$ of L such that the moment matrix of $\{\ell_{t-1}, \ell_t\}$ is H_{a_t}, where

$H_{a_t} := \begin{pmatrix} 0 & \pi^{a_t} \\ (-\pi)^{a_t} & 0 \end{pmatrix}$. We may choose $L' = \langle \ell_1, \cdots, \ell_{t-2}, \pi^{-1}\ell_{t-1}, \ell_t\rangle$ with fundamental invariants $(a_1, \cdots, a_{t-2}, a_t - 1, a_t - 1)$. In this case,

$$\pi L^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_t\rangle \text{ and } \pi L'^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_{t-1}, \pi^{-a_t+2}\ell_t\rangle.$$

For a fixed $x_0 = \sum_{1\le i<t-1} s_i\pi^{-a_i+1}\ell_i$ where $s_i \in O_F$, let

$$S^\circ_{x_0} := \{x \in (\pi L^\sharp)^\circ - (\pi L'^\sharp)^\circ : x = x_0 + s_{t-1}\pi^{-a_t+1}\ell_{t-1} + s_t\pi^{-a_t+1}\ell_t, \text{ where } s_{t-1}, s_t \in O_F\}/L,$$
$$S^{\circ\circ}_{x_0} := \{x \in (\pi L^\sharp)^{\circ\circ} - (\pi L'^\sharp)^{\circ\circ} : x = x_0 + s_{t-1}\pi^{-a_t+1}\ell_{t-1} + s_t\pi^{-a_t+1}\ell_t, \text{ where } s_{t-1}, s_t \in O_F\}/L.$$

It suffices to show $|S^\circ_{x_0}| = q \cdot |S^{\circ\circ}_{x_0}|$. Notice that $x = x_0 + s_{t-1}\pi^{-a_t+1}\ell_{t-1} + s_t\pi^{-a_t+1}\ell_t \in S^\circ_{x_0}$ if and only if

$$s_t \in O_F^\times, \quad (x, x) = (x_0, x_0) + (s_{t-1}\bar{s}_t + (-1)^{a_t}\bar{s}_{t-1}s_t)(-1)^{-a_t+1}\pi^{-a_t+2} \in O_{F_0},$$

and $x \in S^{\circ\circ}_{x_0}$ if and only if

$$s_t \in O_F^\times, \quad (x, x) = (x_0, x_0) + (s_{t-1}\bar{s}_t + (-1)^{a_t}\bar{s}_{t-1}s_t)(-1)^{-a_t+1}\pi^{-a_t+2} \in (\pi_0).$$

Write

$$s_{t-1} = \sum_{i\ge0}^{\infty} b_i\pi^i, \quad s_t = \sum_{i\ge0}^{\infty} c_i\pi^i, \quad -(-1)^{a_t-1}\pi^{a_t-2}(x_0, x_0) = \sum_{i\ge0}^{\infty} d_i\pi^i,$$

where $b_i, c_i, d_i \in O_{F_0}/(\pi_0)$. Then $x \in S^{\circ\circ}_{x_0}$ if and only if $x \in S^\circ_{x_0}$ and

$$d_{a_t-2} + S = -2b_{a_t-2}c_0.$$

where S is certain expression involving $b_0, \cdots, b_{a_t-3}$ and $c_1, \cdots, c_{a_t-2}$. Since $s_t \in O_F^\times$, $c_0 \ne 0$. Hence, for any given S, the number of choices of b_{a_t-2} is determined if $x \in S^{\circ\circ}_{x_0}$. As a result, $|S^\circ_{x_0}| = q \cdot |S^{\circ\circ}_{x_0}|$. $\square$

Proposition 8.7. *Assume that $t \ge 1$ is odd and L is a full type lattice of rank t. Then for any $\chi \in \{\pm1\}$, we have*

$$(1 - \chi q^{\frac{t+1}{2}})(1 + \chi q^{\frac{t-1}{2}})\mu^+(L) + (1 + \chi q^{\frac{t-1}{2}})\mu^0(L) + \mu^-(L) = 0.$$

Proof. We prove this for maximal L first. We can choose a basis $\{\ell_1, \cdots, \ell_t\}$ of L with moment matrix $\mathrm{Diag}(H_1^{\frac{t-1}{2}}, u_t(-\pi_0))$. Set $L_1 = \langle \ell_1, \cdots, \ell_{t-1}\rangle$ and $L_2 = \langle \ell_t\rangle$. Then we can directly compute that

$$(\pi L^\sharp)^{\circ\circ} = L, \quad (\pi L^\sharp)^\circ = L_1 \obot \pi^{-1}L_2, \quad (L^\sharp)^\circ = \pi^{-1}L_1 \obot \pi^{-1}L_2.$$

Hence

$$\mu^+(L) = |(\pi L^\sharp)^{\circ\circ}/L| = 1, \quad \mu^0(L) = |((\pi L^\sharp)^\circ - (\pi L^\sharp)^{\circ\circ})/L| = q - 1,$$

and

$$\mu^-(L) = |((L^\sharp)^\circ - (\pi L^\sharp)^\circ)/L| = q^t - q.$$

As a result,

$$\begin{aligned}&(1-\chi q^{\frac{t+1}{2}})(1+\chi q^{\frac{t-1}{2}})\mu^+(L)+(1+\chi q^{\frac{t-1}{2}})\mu^0(L)+\mu^-(L)\\&=(1-\chi q^{\frac{t+1}{2}})(1+\chi q^{\frac{t-1}{2}})+(1+\chi q^{\frac{t-1}{2}})(q-1)+q^t-q\\&=0.\end{aligned}$$

Now we assume L is not maximal and the proposition holds for L' such that $L\subsetneq L'$ by an induction on $\mathrm{val}(L)$. With this assumption, it suffices to show

$$(1-\chi q^{\frac{t+1}{2}})(1+\chi q^{\frac{t-1}{2}})\mu^+(L,L')+(1+\chi q^{\frac{t-1}{2}})\mu^0(L,L')+\mu^-(L,L')=0,$$

which follows from a combination of Lemmas 8.4 and 8.6. $\square$

Lemma 8.8. *If L is non-maximal full type lattice of rank t, then there exists a L' such that $L\subset L'\subset \pi^{-1}L$ and*

$$\mu^{0,+}(L,L')=\mu^{0,-}(L,L').$$

Proof. Let

$$S^\nu:=\{x\in((\pi L^\sharp)^\circ-(\pi L^\sharp)^{\circ\circ})-((\pi L'^\sharp)^\circ-(\pi L'^\sharp)^{\circ\circ}):\chi(\langle x\rangle)=\nu\}/L.$$

We need to show $|S^{+1}|=|S^{-1}|$. Let $\{\ell_1,\cdots,\ell_t\}$ be a normal basis of L, and let $\{a_1,\cdots,a_t\}$ be the set of fundamental invariants of L. We consider two cases seperately.
(i) If a_t is even and $a_t\geq 4$, then we may choose a normal basis $\{\ell_1,\cdots,\ell_t\}$ of L such that $\langle\ell_1,\cdots,\ell_{t-1}\rangle\perp\ell_t$. In this case, we choose $L'=\langle\ell_1,\cdots,\ell_{t-1},\pi^{-1}\ell_t\rangle$, with fundamental invariants $(a_1,\cdots,a_{t-1},a_t-2)$. Then

$$\pi L^\sharp=\langle\pi^{-a_1+1}\ell_1,\cdots,\pi^{-a_t+1}\ell_t\rangle \text{ and } \pi L'^\sharp=\langle\pi^{-a_1+1}\ell_1,\cdots,\pi^{-a_{t-1}+1}\ell_{t-1},\pi^{-a_t+2}\ell_t\rangle.$$

For a fixed $x_0=\sum_{1\leq i<t}s_i\pi^{-a_i+1}\ell_i$ where $s_i\in O_F$, we set

$$S^\nu_{x_0}:=\{x\in((\pi L^\sharp)^\circ-(\pi L^\sharp)^{\circ\circ})-((\pi L'^\sharp)^\circ-(\pi L'^\sharp)^{\circ\circ}):x=x_0+s_t\pi^{-a_t+1}\ell_t,s_t\in O_F,\chi(\langle x\rangle)=\nu\}/L.$$

We need to show $|S^{+1}_{x_0}|=|S^{-1}_{x_0}|$. Write $(\ell_t,\ell_t)=u_t(-\pi_0)^{\frac{a_t}{2}}$. Notice that $x=x_0+s_t\pi^{-a_t+1}\ell_t\in S^\nu_{x_0}$ if and only if

$$s_t\in O_F^\times,\quad (x,x)\in O_{F_0}^\times,\quad \chi((x,x))=\nu. \tag{8.7}$$

Notice that

$$(x,x)=u_t(-\pi_0)^{-\frac{a_t}{2}+1}(u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0,x_0)+s_t\bar{s}_t).$$

Write

$$s_t=\sum_{i\geq 0}^{\infty}b_i\pi^i,\quad \text{ and }\ -u_t^{-1}(-\pi_0)^{\frac{a_t}{2}-1}(x_0,x_0)=\sum_{i\geq 0}^{\infty}c_i\pi^i,$$

where $b_i, c_i \in O_{F_0}/(\pi_0)$. Then the conditions in (8.7) are equivalent to the following equations:

$$\begin{aligned}
c_0 &= b_0^2 \neq 0,\\
c_1 &= b_1 b_0 - b_0 b_1,\\
c_2 &= b_2 b_0 - b_1 b_1 + b_0 b_2,\\
&\cdots\\
c_{a_t-3} &= \sum_{i=0}^{a_t-3} (-1)^i b_{a_t-3-i} b_i,\\
\nu &= \chi\Big(u_t(-\pi_0)^{-\frac{a_t}{2}+1}\big(-c_{a_t-2} + \sum_{i=0}^{a_t-2} (-1)^i b_{a_t-2-i} b_i\big)\Big).
\end{aligned}$$

Since a_t is even by assumption, the last equation is the same with

$$\nu = \chi\Big(u_t(-\pi_0)^{-\frac{a_t}{2}+1}\big(-c_{a_t-2} + \sum_{i=1}^{a_t-3} (-1)^i b_{a_t-2-i} b_i + 2b_0 b_{a_t-2}\big)\Big).$$

Notice that the possible choices of $\{b_0, \cdots, b_{a_t-3}\}$ are determined by the first $a_t - 2$ equations. And for a given choice of $\{b_0, \cdots, b_{a_t-3}\}$, the number of choices of b_{a_t-2} that satisfies the last equation is clearly independent of ν since $b_0 \neq 0$.
(ii) If a_t is odd or $a_t = 2$, then we may choose a normal basis $\{\ell_1, \cdots, \ell_t\}$ of L such that the moment matrix of $\{\ell_{t-1}, \ell_t\}$ is H_{a_t}, where $H_{a_t} := \begin{pmatrix} 0 & \pi^{a_t} \\ (-\pi)^{a_t} & 0 \end{pmatrix}$. We may choose $L' = \langle \ell_1, \cdots, \ell_{t-2}, \pi^{-1}\ell_{t-1}, \ell_t\rangle$ with fundamental invariants $(a_1, \cdots, a_{t-2}, a_{t-1}-1, a_t-1)$. In this case,

$$\pi L^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_t+1}\ell_t\rangle \text{ and } \pi L'^\sharp = \langle \pi^{-a_1+1}\ell_1, \cdots, \pi^{-a_{t-1}+1}\ell_{t-1}, \pi^{-a_t+2}\ell_t\rangle.$$

For a fixed $x_0 = \sum_{1\leq i<t-1} s_i \pi^{-a_i+1}\ell_i$, we set

$$\begin{aligned}
S_{x_0}^\nu := \{x \in ((\pi L^\sharp)^\circ - (\pi L^\sharp)^{\circ\circ}) - ((\pi L'^\sharp)^\circ - (\pi L'^\sharp)^{\circ\circ}) :& x = x_0 + s_{t-1}\pi^{-a_{t-1}+1}\ell_{t-1} + s_t \pi^{-a_t+1}\ell_t,\\
& s_{t-1}, s_t \in O_F, \chi(\langle x\rangle) = \nu\}/L.
\end{aligned}$$

It suffices to show $|S_{x_0}^{+1}| = |S_{x_0}^{-1}|$. Notice that $x = x_0 + s_{t-1}\pi^{-a_{t-1}+1}\ell_{t-1} + s_t\pi^{-a_t+1}\ell_t \in S_{x_0}^\nu$ if and only if

(8.8)
$$s_t \in O_F^\times, \quad (x,x) = (x_0,x_0) + (s_{t-1}\bar{s}_t + (-1)^{a_t}\bar{s}_{t-1}s_t)(-1)^{-a_t+1}\pi^{-a_t+2} \in O_{F_0}^\times, \quad \chi((x,x)) = \nu.$$

Write

$$s_{t-1} = \sum_{i\geq 0}^{\infty} b_i \pi^i, \quad s_t = \sum_{i\geq 0}^{\infty} c_i\pi^i, \quad -(-1)^{a_t-1}\pi^{a_t-2}(x_0,x_0) = \sum_{i\geq 0}^{\infty} d_i \pi^i,$$

where $b_i, c_i, d_i \in O_{F_0}/(\pi_0)$. Then the condition in (8.8) is equivalent to the following equations:

$$\begin{aligned}
d_0 &= b_0c_0 + (-1)^{a_t}b_0c_0,\\
d_1 &= b_1c_0 - b_0c_1 + (-1)^{a_t}(-b_1c_0 + b_0c_0),\\
&\cdots\\
\nu &= \chi\Big((-1)^{a_t-1}\pi^{-a_t+2}\big(-d_{a_t-2} + S + 2b_{a_t-2}c_0\big)\Big),
\end{aligned}$$

where S is certain expression involving $b_0, \cdots, b_{a_t-3}$ and $c_1, \cdots, c_{a_t-2}$. Since $s_t \in O_F^\times$, $c_0 \neq 0$. Hence, for any given S, the number of choices of b_{a_t-2} that satisfies the last equation is clearly independent of ν. $\square$

Proposition 8.9. *Assume that $t \geq 1$ is even and that L is a full type lattice of rank t. Then for any $\chi \in \{\pm 1\}$, we have*

$$(1-q^t)\mu^+(L) + (1-\chi q^{\frac{t}{2}})\mu^{0,+1}(L) + (1+\chi q^{\frac{t}{2}})\mu^{0,-1}(L) + \mu^-(L) = 0.$$

Proof. We prove this for maximal L first. There are two cases we need to consider.
(i) If we can choose a basis $\{\ell_1, \cdots, \ell_t\}$ of L with moment matrix $\mathrm{Diag}(H_1^{\frac{t}{2}-1}, u_{t-1}(-\pi_0), u_t(-\pi_0))$ where $\chi(-u_{t-1}u_t) = -1$, then set $L_1 = \langle \ell_1, \cdots, \ell_{t-2}\rangle$ and $L_2 = \langle \ell_{t-1}, \ell_t\rangle$. In this case, a direct computation shows that

$$(\pi L^\sharp)^{\circ\circ} = L, \quad (\pi L^\sharp)^\circ = L_1 \oplus \pi^{-1}L_2, \quad (L^\sharp)^\circ = \pi^{-1}L.$$

Hence

$$\mu^+(L) = |(\pi L^\sharp)^{\circ\circ}/L| = 1, \quad \mu^-(L) = |((L^\sharp)^\circ - (\pi L^\sharp)^\circ)/L| = q^t - q^2. \tag{8.9}$$

Moreover,

$$\mu^{0,\nu}(L) = |\{(x,y) \in \mathbb{F}_q^2 - (0,0) \mid \chi(u_{t-1}x^2 + u_ty^2) = \nu\}|.$$

It is well known that

$$|\{(x,y) \in \mathbb{F}_q^2 - (0,0) \mid u_{t-1}x^2 + u_ty^2 = 1\}| = q - \chi(-u_{t-1}u_t) = q+1.$$

Hence

$$\mu^{0,+1}(L) = \mu^{0,-1}(L) = \frac{q^2-1}{2}. \tag{8.10}$$

Combining (8.9) and (8.10), we have

$$\begin{aligned}
&(1-q^t)\mu^+(L) + (1-\chi q^{\frac{t}{2}})\mu^{0,+}(L) + (1+\chi q^{\frac{t}{2}})\mu^{0,-}(L) + \mu^-(L)\\
&= (1-q^t) + (q^2-1) + (q^t - q^2) = 0.
\end{aligned}$$

(ii) If we can choose a basis $\{\ell_1, \cdots, \ell_t\}$ of L with moment matrix $H_1^{\frac{t}{2}}$, then we can directly compute that

$$(\pi L^\sharp)^{\circ\circ} = L, \quad (\pi L^\sharp)^\circ = L, \quad (L^\sharp)^\circ = \pi^{-1}L.$$

Hence

$$\mu^+(L) = |(\pi L^\sharp)^{\circ\circ}/L| = 1, \quad \mu^0(L) = 0, \quad \mu^-(L) = |((L^\sharp)^\circ - (\pi L^\sharp)^\circ)/L| = q^t - 1.$$

As a result we have

$$(1-q^t)\mu^+(L) + (1-\chi q^{\frac{t}{2}})\mu^{0,+}(L) + (1+\chi q^{\frac{t}{2}})\mu^{0,-}(L) + \mu^-(L) = (1-q^t) + (q^t-1) = 0.$$

Now we assume L is not maximal and the proposition holds for L' such that $L \subsetneq L'$ by an induction on $\mathrm{val}(L)$. With this assumption, it suffices to show

$$(1-q^t)\mu^+(L,L') + (1-\chi q^{\frac{t}{2}})\mu^{0,+1}(L,L') + (1+\chi q^{\frac{t}{2}})\mu^{0,-1}(L,L') + \mu^-(L,L') = 0,$$

which follows from a combination of Lemmas 8.4 and 8.8. □

9. Proof of the main theorem

We prove the main theorem in this section by an induction on $\mathrm{val}(L)$ using the results we obtained about the partial Fourier transform in previous sections.

9.1. **Comparison of horizontal intersection numbers.**

Lemma 9.1. *Let $L \subset \mathbb{V}$ be a lattice. If $L = L_1 \oplus L_2$ where L_1 is unimodular, then*

$$\mathrm{Int}(L) - \partial\mathrm{Den}(L) = \mathrm{Int}(L_2) - \partial\mathrm{Den}(L_2).$$

Proof. The lemma follows from comparing (7.3) with (2.12). □

Definition 9.2. Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$, and $x \in \mathbb{V} \setminus L^\flat_F$. Define

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x) = \sum_{\substack{L^\flat \subset L' \subset L'^\sharp \\ L'^\flat \in \mathrm{Hor}(L^\flat)}} \partial\mathrm{Pden}(L') 1_{L'}(x). \tag{9.1}$$

Lemma 9.3. *If $L^\flat \subset \mathbb{V}$ is horizontal, then*

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) = \partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x),$$

where $\mathrm{Int}_{L^\flat,\mathscr{H}}$ is defined in Definition 4.7.

Proof. Let $L = L^\flat \oplus \langle x\rangle$. By Lemma 4.8, we know

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) = \mathrm{Int}_{L^\flat}(x) = \mathrm{Int}(L). \tag{9.2}$$

On the other hand, since $L^\flat$ is horizontal, by Lemma 4.2 any integral lattices of rank $n-1$ containing $L^\flat$ is horizontal, hence we have

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x) = \partial\mathrm{Den}_{L^\flat}(x) = \partial\mathrm{Den}(L). \tag{9.3}$$

So it suffices to prove $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$.

When $n = 2$, by (9.2) and (9.3), the lemma is a consequence of [Shi20, Theorem 1.1] and [HSY20, Theorem 1.1]. When $n > 2$, $L^\flat$ has a unimodular direct summand L_1 of rank $n-2$ such that $L = L_1 \oplus L_2$ and $L_2^\flat := L_{2,F} \cap L^\flat$ is a horizontal lattice in $L_{2,F}$. The lemma follows from the the case $n = 2$ and Lemma 9.1. □

Lemma 9.4. *If $M^\flat \subset \mathbb{V}$ is horizontal, then*

$$\chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \cdot \mathcal{Z}(x)) = \sum_{\substack{M^\flat \subset L' \subset L'^\sharp \\ L'^\flat = M^\flat}} \partial\mathrm{Pden}(L') 1_{L'}(x),$$

where ${}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ$ is as in Definition 4.5.

Proof. By Definition 4.7, we have

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \cdot \mathcal{Z}(x)) + \sum_{\substack{L'^\flat \in \mathrm{Hor}(M^\flat) \\ L'^\flat \neq M^\flat}} \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L'^\flat)^\circ \cdot \mathcal{Z}(x)). \tag{9.4}$$

We now prove the lemma by induction on $\mathrm{val}(M^\flat)$. When $M^\flat$ is unimodular, the lemma is the same as Lemma 9.3. In general notice that any integral lattices of rank $n-1$ containing $M^\flat$ is horizontal by Lemma 4.2. Applying the induction hypothesis to the right hand side of the above formula and applying Lemma 9.3 to the left hand had, we obtain

$$\sum_{\substack{M^\flat \subset L' \subset L'^\sharp \\ L'^\flat \in \mathrm{Hor}(M^\flat)}} \partial\mathrm{Pden}(L') 1_{L'}(x) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \cdot \mathcal{Z}(x)) + \sum_{\substack{M^\flat \subset L' \subset L'^\vee \\ L'^\flat \neq M^\flat}} \partial\mathrm{Pden}(L') 1_{L'}(x). \tag{9.5}$$

Subtract the left hand side by the second term of the right hand side of the equation, the lemma is proved. $\square$

Theorem 9.5. *For a non-degenerate lattice $L^\flat \subset \mathbb{V}$ of rank $n-1$, and $x \in \mathbb{V} \setminus L^\flat_F$, we have*

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) = \partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x).$$

Proof. By the definition of $\mathrm{Int}_{L^\flat,\mathscr{H}}(x)$, we have

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) = \sum_{M^\flat \in \mathrm{Hor}(L^\flat)} \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(M^\flat)^\circ \cdot \mathcal{Z}(x)).$$

The theorem now follows from (9.1) and Lemma 9.4. $\square$

9.2. **Proof of the main theorem.** The following is an analogue of Lemma 9.3.1 of [LZ22b].

Lemma 9.6. *Let $L^\flat \subset \mathbb{V}$ be a lattice of rank $n-1$ and $\mathbb{W} = (L^\flat_F)^\perp$. For $x \notin L^\flat \oplus \mathbb{W}$, there exists an O_F-lattice $L'^\flat$ of rank $n-1$ and $x' \in \mathbb{V}$ such that*

$$\mathrm{val}(L'^\flat) < \mathrm{val}(L^\flat) \text{ and } L'^\flat + \langle x' \rangle = L^\flat + \langle x \rangle.$$

Proof. Assume that $L^\flat \subset \mathbb{V}$ has fundamental invariants $(a_1, \cdots, a_{n-1})$. Let $\{\ell_1, \cdots, \ell_{n-1}\}$ be a basis of $L^\flat$ whose moment matrix is

$$\mathrm{Diag}(H_{b_1}, H_{b_3}, \cdots, H_{b_{2s-1}}, u_{2s+1}\pi^{b_{2s+1}}, \cdots, u_{n-1}\pi^{b_{n-1}}),$$

where $b_1, \cdots, b_{2s-1}$ are odd and $H_j = \begin{pmatrix} 0 & \pi^j \\ (-\pi)^j & 0 \end{pmatrix}$. Notice that $\{b_1, \cdots, b_{n-1}\} = \{a_1, \cdots, a_{n-1}\}$. The moment matrix of $\{\ell_1, \cdots, \ell_{n-1}, x\}$ is

$$T = \begin{pmatrix} H_{b_1} & & & (\ell_1, x) \\ & \ddots & & \vdots \\ & & u_{n-1}\pi^{b_{n-1}} & (\ell_{n-1}, x) \\ (x, \ell_1) & \cdots & (x, \ell_{n-1}) & (x, x) \end{pmatrix}.$$

Assume $(a_1', \cdots, a_n')$ is the fundamental invariants of $L^\flat + \langle x \rangle$. According to Lemma 2.23 of [LL22], $a_1' + \cdots + a_{n-1}'$ equals the minimal valuation of the $(n-1) \times (n-1)$ minors of T.

Write $x = x^\flat + x^\perp$ where $x^\flat \in L_F^\flat$ and $x^\perp \in \mathbb{W}$. If $x^\flat \notin L^\flat$, then we can write $x^\flat = \sum_{j=1}^{n-1} \alpha_j \ell_j$ where $\alpha_i \notin O_F$ for some i. First, we assume $\alpha_i \notin O_F$ for some $i \leq 2s$. The valuation of the (n,i)-th minor of T (removing n-th row and i-th column) equals to

$$\begin{cases} \mathrm{val}_\pi((\ell_{i+1}, x)) - b_i + (b_1 + \cdots + b_{n-1}) & \text{if } i \text{ is odd,} \\ \mathrm{val}_\pi((\ell_{i-1}, x)) - b_i + (b_1 + \cdots + b_{n-1}) & \text{if } i \text{ is even.} \end{cases}$$

Since $\alpha_i \notin O_F$, we have $\mathrm{val}_\pi((\ell_{i+1}, x)) < b_i$ if i is odd and $\mathrm{val}_\pi((\ell_{i-1}, x)) < b_i$ if i is even. In particular, $\sum_{j=1}^{n-1} a_j' < \sum_{j=1}^{n-1} a_j$. Now if we choose a normal basis $\{\ell_1', \cdots, \ell_n'\}$ of $L^\flat + \langle x \rangle$, then $L'^\flat = \langle \ell_1', \cdots, \ell_{n-1}' \}$ and $x' = \ell_n'$ satisfy the property we want.

Now we assume $\alpha_i \notin O_F$ for some $2s < i \leq n-1$. The valuation of the (n,i)-th minor of T equals to

$$\mathrm{val}_\pi(\ell_i, x) - b_i + (b_1 + \cdots + b_{n-1}).$$

Since $\alpha_i \notin O_F$, $\mathrm{val}_\pi(\ell_i, x) < b_i$, hence $\sum_{j=1}^{n-1} a_j' < \sum_{j=1}^{n-1} a_j$. Now if we choose a normal basis $\{\ell_1', \cdots, \ell_n'\}$ of $L^\flat + \langle x \rangle$, then $L'^\flat = \langle \ell_1', \cdots, \ell_{n-1}' \}$ and $x' = \ell_n'$ satisfy the property we want. □

For any L, we can write it as $L^\flat + \langle x \rangle$ where $L^\flat$ is a non-degenerate hermitian O_F-lattice of rank $n-1$, and $x \in \mathbb{V} \setminus L^\flat$. Therefore, in order to show $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$, it suffices to show the following theorem.

Theorem 9.7. *Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$. For any $x \in \mathbb{V} \setminus L_F^\flat$, we have*

$$\mathrm{Int}_{L^\flat}(x) = \partial\mathrm{Den}_{L^\flat}(x).$$

Proof. For $x \in \mathbb{V} \setminus L_F^\flat$, let $\Phi_{L^\flat}(x) = \mathrm{Int}_{L^\flat}(x) - \partial\mathrm{Den}_{L^\flat}(x)$. We need to show $\Phi_{L^\flat}(x) = 0$. We prove the theorem by an induction on $\mathrm{val}(L^\flat)$. If $L^\flat$ is not integral, then $\mathrm{Int}_{L^\flat}(x) = 0$ as $\mathcal{Z}(L)$ is empty by Proposition 3.20. Moreover $\partial\mathrm{Den}_{L^\flat}(x) = 0$ by Corollary 7.2. Hence the theorem is true in this case.

Now we assume $L^\flat$ is integral. By induction hypothesis and Lemma 9.6, we may assume $\mathrm{supp}(\Phi_{L^\flat}) \subset L^\flat \ominus \mathbb{W}$ where $\mathbb{W} = (L_F^\flat)^\perp$. Since $\Phi_{L^\flat}(x)$ is invariant under the translation of $L^\flat$, we may write

$$\Phi_{L^\flat}(x) = 1_{L^\flat} \otimes \phi_{\mathbb{W}}(x), \tag{9.6}$$

where $\phi_{\mathbb{W}}(x)$ is a function on $\mathbb{W} \setminus \{0\}$. Then we have by definition

$$\Phi^{\perp}_{L^\flat} = \mathrm{vol}(L^\flat)\phi_{\mathbb{W}}.$$

Theorem 9.5 implies that

$$\Phi_{L^\flat}(x) = \Phi_{L^\flat,\mathscr{V}}(x) := \mathrm{Int}_{L^\flat,\mathscr{V}}(x) - \partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x).$$

Hence

$$\mathrm{vol}(L^\flat)\phi_{\mathbb{W}} = \Phi^{\perp}_{L^\flat,\mathscr{V}}. \tag{9.7}$$

By induction on the rank of L and Lemma 9.1, we may assume $\Phi_{L^\flat}(x) = 0$, hence $\phi_{\mathbb{W}}(x) = 0$ for $x \in \mathbb{W}^{=0}$. Combining this with the non-integral case, we know $\phi_{\mathbb{W}}(x) = 0$ for $x \in \mathbb{W}^{\leq 0}$. As a result, we have $\Phi^{\perp}_{L^\flat,\mathscr{V}}(x) = 0$ for $x \in \mathbb{W}^{\leq 0}$ by (9.7). By Theorem 8.2 and Theorem 4.16, we have $\Phi^{\perp}_{L^\flat,\mathscr{V}}(x) = 0$ for $x \in \mathbb{W}^{\geq 0} \setminus \{0\}$. Hence $\Phi^{\perp}_{L^\flat,\mathscr{V}}(x) = 0$ for all $x \in \mathbb{W} \setminus \{0\}$. Consequently, $\phi_{\mathbb{W}}(x) = 0$ by (9.7). $\square$

Combining this theorem with Theorem 4.15, we have the following corollary.

Corollary 9.8. *Let $L^\flat \subset \mathbb{V}$ be a non-degenerate lattice of rank $n-1$. Then $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x) \in \mathscr{S}(\mathbb{V})^{\geq -1}$ is a Schwartz function.*

10. Global applications

10.1. **Shimura varieties.** We will closely follow [RSZ20] and [RSZ21]. In this part we switch to global notations. Let F be a CM number field, with F_0 its totally real subfield of index 2. We fix a CM type $\Phi \subseteq \mathrm{Hom}(F, \overline{\mathbb{Q}})$ of F and a distinguished element $\phi_0 \in \Phi$. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and identify the CM type Φ with the set of archimedean places of F, and also with the set of archimedean places of F_0. Let V be an F/F_0-hermitian space of dimension $n \geq 2$. Let $V_\phi = V \otimes_{F,\phi} \mathbb{C}$ be the associated $\mathbb{C}/\mathbb{R}$-hermitian space for $\phi \in \Phi$. Assume the signature of V_ϕ is given by

$$(r_\phi, r_{\bar\phi}) = \begin{cases} (n-1,1), & \phi = \phi_0, \\ (n,0), & \phi \in \Phi \setminus \{\phi_0\}. \end{cases}$$

Define a variant $G^{\mathbb{Q}}$ of the unitary similitude group $\mathrm{GU}(V)$ by

$$G^{\mathbb{Q}} := \{g \in \mathrm{Res}_{F_0/\mathbb{Q}} \mathrm{GU}(V) : c(g) \in \mathbb{G}_m\},$$

where c denotes the similitude character. Define a cocharacter

$$h_{G^{\mathbb{Q}}} : \mathbb{C}^\times \to G^{\mathbb{Q}}(\mathbb{R}) \subseteq \prod_{\phi\in\Phi} \mathrm{GU}(V_\phi)(\mathbb{R}) \simeq \prod_{\phi\in\Phi} \mathrm{GU}(r_\phi, r_{\bar\phi})(\mathbb{R}),$$

where its ϕ-component is given by

$$h_{G^{\mathbb{Q}},\phi}(z) = \mathrm{Diag}\{z \cdot 1_{r_\phi}, \bar z \cdot 1_{r_{\bar\phi}}\}$$

under the decomposition of V_ϕ into positive definite and negative definite parts. Then its $G^{\mathbb{Q}}(\mathbb{R})$-conjugacy class defines a Shimura datum $(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$. Let $E_r = E(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$ be the reflex field, i.e., the subfield of $\overline{\mathbb{Q}}$ fixed by $\{\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma^*(r) = r\}$, where $r : \mathrm{Hom}(F, \overline{\mathbb{Q}}) \to \mathbb{Z}$ is the function defined by $r(\phi) = r_\phi$.

We similarly define the group $Z^{\mathbb{Q}}$ (a torus) associated to a totally positive definite F/F_0-hermitian space of dimension 1 (i.e., of signature $\{(1,0)_{\phi\in\Phi}\}$) and a cocharacter $h_{Z^{\mathbb{Q}}}$ of $Z^{\mathbb{Q}}$. The reflex field $E_\Phi = E(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ is equal to the reflex field of the CM type Φ, i.e., the subfield of $\overline{\mathbb{Q}}$ fixed by $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma \circ \Phi = \Phi\}$.

Now define a Shimura datum $(\widetilde{G}, \{h_{\widetilde{G}}\})$ by

$$\widetilde{G} := Z^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}} = \{(z,g) \in Z^{\mathbb{Q}} \times G^{\mathbb{Q}} \mid \mathrm{Nm}_{F/F_0}(z) = c(g)\}, \quad h_{\widetilde{G}} = (h_{Z^{\mathbb{Q}}}, h_{G^{\mathbb{Q}}}).$$

Then $\widetilde{G} \cong Z^{\mathbb{Q}} \times G$ where $G = \mathrm{Res}_{F_0/\mathbb{Q}} \mathrm{U}(V)$ by [LZ22a, (11.2.0.1)]. Its reflex field E is equal to the composite $E_r E_\Phi$, and the CM field F becomes a subfield of E via the embedding ϕ_0. We remark that $E = F$ when $F/\mathbb{Q}$ is Galois, or when $F = F_0 K$ for some imaginary quadratic $K/\mathbb{Q}$ and the CM type Φ is induced from a CM type of $K/\mathbb{Q}$ (e.g., when $F_0 = \mathbb{Q}$). Let $K \subseteq \widetilde{G}(\mathbb{A}_f)$ be a compact open subgroup. Then the associated Shimura variety $\mathrm{Sh}_K = \mathrm{Sh}_K(\widetilde{G}, \{h_{\widetilde{G}}\})$ is of dimension $n-1$ and has a canonical model over $\mathrm{Spec}\, E$. It has the following moduli interpretation, see [RSZ21, Definition 3.3] or [LZ22a, §11.2]. For a locally noetherian E-scheme S, $\mathrm{Sh}_K(S)$ is the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar\eta)$ satisfying

(1) $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_0^{\xi}(S)$, see (H1) below;
(2) A is an abelian scheme over S;
(3) $\iota : F \to \mathrm{End}^0(A)$ is an action of F on A up to isogeny satisfying the Kottwitz signature condition $\{(r_\phi, r_{\bar\phi})_{\phi\in\Phi}\}$:

$$\mathrm{charpol}(\iota(a) \mid \mathrm{Lie}\, A) = \prod_{\phi\in\Phi} (T - \phi(a))^{r_\phi} \cdot (T - \bar\phi(a))^{r_{\bar\phi}} \tag{10.1}$$

for any $a \in F$;
(4) $\lambda : A \to A^\vee$ is a quasi-polarization whose Rosati involution induces the automorphism given by the nontrivial Galois automorphism of F/F_0 via ι;
(5) $\bar\eta$ is a K-orbit of isometries of $\mathbb{A}_{F,f}/\mathbb{A}_{F_0,f}$-Hermitian modules

$$\bar\eta : \hat{V}(A_0, A) \to V \otimes_F \mathbb{A}_{F,f}.$$

Here

$$\hat{V}(A_0, A) = \mathrm{Hom}_{\mathbb{A}_{F,f}}(\hat{V}(A_0), \hat{V}(A)),$$

where $\hat{V}(A)$ (resp. $\hat{V}(A_0)$) is the rational Tate-module of A (resp. A_0) and $\hat{V}(A_0, A)$ is endowed with a Hermitian form:

$$h(x,y) = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}_{\mathbb{A}_{F,f}}(\hat{V}(A_0)) = \mathbb{A}_{F,f}.$$

10.2. **Integral model.** Assume that $K_{Z^{\mathbb{Q}}} \subseteq Z^{\mathbb{Q}}(\mathbb{A}_f)$ is the unique maximal open compact subgroup. Assume that $K_G = \prod_v K_{G,v} \subset G(\mathbb{A}_f)$ where v runs over finite places of F_0. Let $(m_v)_v$ be a collection of integers $m_v \geq 0$ indexed by finite places of F_0 such that $m_v = 0$ for all but finitely many places and $m_v = 0$ for all nonsplit places v. Let Λ be an O_F-lattice of V. Assume that for any finite place v of F_0 (write p its residue characteristic), the following conditions are satisfied.

(G0) If $p = 2$, then v is unramified in F.

(G1) If v is inert in F and V_v is split, then $\Lambda_v \subset V_v$ is self-dual and $K_{G,v}$ is the stabilizer of Λ_v. If v is further ramified over p, then the subset $\{\phi \in \Phi : \phi \text{ induces } w\} \subseteq \mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)$ is the pullback of a CM type $\Phi^{\mathrm{ur}} \subseteq \mathrm{Hom}(F_w^{\mathrm{ur}}, \overline{\mathbb{Q}}_p)$ of F_w^{ur}. Here w is the place of F above v and F_w^{ur} is the maximal subfield of F_w unramified over $\mathbb{Q}_p$.

(G2) If v is inert in F and V_v is nonsplit, then v is unramified over p and $\Lambda_v \subset V_v$ is almost self-dual, i.e., $\Lambda_v^\sharp/\Lambda_v$ is a 1 dimensional space over the residue field of F_w. Moreover $K_{G,v}$ is the stabilizer of Λ_v.

(G3) If v is ramified in F, then v is unramified over p and $\Lambda_v \subseteq V_v$ is unimodular.

(G4) If v is split in F, then $\mathrm{U}(V)(F_{0,v}) \cong \mathrm{GL}_n(F_{0,v})$ and we assume $\Lambda_v \subset V_v$ is self-dual. If $m_v = 0$, let $K_{G,v}$ be the stabilizer of Λ_v. If $m_v > 0$, further assume that all places of E above v satisfies condition D in [LZ22a, §11.5] and let $K_{G,v}$ be the principal congruence subgroup modulo $\pi_v^{m_v}$ inside $\mathrm{GL}_n(F_{0,v})$ where π_v is a uniformizer of $F_{0,v}$.

Assume that $K = K_{Z^{\mathbb{Q}}} \times K_G \subset \widetilde{G}$. Define the moduli functor $\mathcal{M}_K$ as follows. For a locally noetherian O_E-scheme S, $\mathcal{M}_K(S)$ is the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}, \eta)$ such that

(1) A (resp. A_0) is an abelian scheme over S;
(2) ι (resp. ι_0) is an action of O_F on A (resp. A_0) satisfying the Kottwitz condition (10.1) for signature $\{(r_\phi, r_{\bar{\phi}})_{\phi\in\Phi}\}$ (resp. $\{(1,0)_{\phi\in\Phi}\}$);
(3) λ (resp. λ_0) is a polarization of A (resp. A_0) whose Rosati involution induces the automorphism given by the nontrivial Galois automorphism of F/F_0 via ι (resp. ι_0);
(4) $\mathcal{F}$ is locally a direct summand $\mathcal{O}_S$-submodule of $\mathrm{Lie}\, A$ which is stable under the O_F-action. Moreover O_F acts on $\mathcal{F}$ by the structural morphism and on $\mathrm{Lie}\, A/\mathcal{F}$ by the Galois conjugate of the structural morphism.
(5) $\eta = \prod_w \eta_w$ where w runs over all places of F above a split place v of F_0 such that $m_v > 0$. For each such w, η_w is a finite $\mathcal{O}_{F,w}$-linear homomorphisms of finite flat group schemes

$$\eta_w : \pi_w^{-m_v}\Lambda_w/\Lambda_w \to \mathrm{Hom}_{\mathcal{O}_{F,w}}(A_0[w^{m_v}], A[w^{m_v}]),$$

where π_w is a uniformizer of F_w. See [RSZ20, §4.3] for more details.

We further require the following conditions to be satisfied.

(H1) $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_0^\xi$ where M_0^ξ is an integral model of $\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}, h_{Z^{\mathbb{Q}}})$ depending on the choice of a similarity class ξ of 1 dimensional F/F_0-Hermitian spaces, see [HSY21, §12.2].

(H2) For each finite place v of F_0, λ induces a polarization λ_v on the p-divisible group $A[v^\infty]$. We require $\ker \lambda_v \subseteq A[\iota(\varpi_v)]$ and is of rank equal to the size of $\Lambda_v^\sharp/\Lambda_v$, where ϖ_v is a uniformizer of $F_{0,v}$.

(H3) For each place v of F_0, we require the *sign condition* and *Eisenstein condition* as explained in [RSZ20, §4.1]. We remark that the sign condition holds automatically when v is split in F, and the Eisenstein condition holds automatically when the places of F above v are unramified over p.

(H4) We impose the *Krämer condition* on $\mathcal{F}$ as explained in [HSY21, Condition (iii) §12.2].

A morphism $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}, \eta) \to (A_0', \iota_0', \lambda_0', A', \iota', \lambda', \mathcal{F}', \eta')$ is a pair $(f_0 : A_0 \to A_0', f : A \to A')$ of O_F-linear isomorphism of abelian schemes over S such that $f^*(\lambda') = \lambda$, $f_0^*(\lambda_0') = \lambda_0$,

$f_*(\mathcal{F}) = \mathcal{F}'$ and $f_*(\eta) = \eta'$. Let $\mathcal{V}_{\mathrm{ram}}$ (resp. $\mathcal{V}_{\mathrm{asd}}$) be the set of finite places v of F_0 such that v is ramified in F (resp. v is inert in F and Λ_v is almost self-dual). The moduli problem $\mathcal{M}_K$ is representable by a Deligne-Mumford stack over O_E which is regular and is semistable at all places of E above $\mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$. The generic fiber of $\mathcal{M}_K$ is the Shimura variety Sh_K.

10.3. **Global main theorems.** For a locally noetherian O_E-scheme S and a point

$$(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}, \eta) \in \mathcal{M}_K(S),$$

define the *space of special homomorphisms* to be

$$\mathbb{V}(A_0, A) := \mathrm{Hom}_{O_F}(A_0, A),$$

a locally free O_F-module of finite rank. Then $\mathbb{V}(A_0, A)$ carries a O_F-valued hermitian form: for $x, y \in \mathbb{V}(A_0, A)$, the pairing $(x, y) \in O_F$ is given by

$$(A_0 \xrightarrow{x} A \xrightarrow{\lambda} A^\vee \xrightarrow{y^\vee} A_0^\vee \xrightarrow{\lambda_0^{-1}} A_0) \in \mathrm{End}_{O_F}(A_0) = \iota_0(O_F) \simeq O_F.$$

Let $m \geq 1$. Given an m-tuple $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{V}(A_0, A)^m$, define its *fundamental matrix* to be

$$T(\mathbf{x}) := ((x_i, x_j))_{1 \leq i,j \leq m} \in \mathrm{Herm}_m(O_F),$$

an $m \times m$ hermitian matrix over O_F.

Let $\mathbb{V}$ be the *incoherent* $\mathbb{A}_F/\mathbb{A}_{F_0}$-hermitian space associated to V, namely $\mathbb{V}$ is totally positive definite and $\mathbb{V}_v \cong V_v$ for all finite places v. Let $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$ be a K-invariant (where K acts on $\mathbb{V}_f$ via the second factor K_G) Schwartz function. We say φ_K is admissible if $\varphi_{K,v} = \mathbf{1}_{(\Lambda_v)^m}$ at all v nonsplit in F. Let $T \in \mathrm{Herm}_m(F_0)$ be a nonsingular F/F_0-hermitian matrix of size m. Associated to (T, φ_K) we can define the *arithmetic special cycles* $\mathcal{Z}(T, \varphi_K)$ over $\mathcal{M}_K$ ([LZ22a, §14.3]) generalizing the $\mathcal{Z}(T)$ in [KR14] as follows. First we consider a special Schwartz function of the form

$$\varphi_K = (\varphi_i) \in \mathscr{S}(\mathbb{V}_f^m), \quad \varphi_i = \mathbf{1}_{\Omega_i}, \quad i = 1, \ldots, m, \tag{10.2}$$

where $\Omega_i \subseteq \mathbb{V}_f$ is a K-invariant open compact subset such that $\Omega_{i,v} = \Lambda_v$ for all finite v such that v is nonsplit in F. Given such a special Schwartz function φ_K and $T \in \mathrm{Herm}_m(O_F)$, define $Z(T, \varphi_K)$ over Sh_K as follows. For a locally noetherian E-scheme S, define $Z(T, \varphi_K)(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}, \mathbf{x})$ where

(1) $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}) \in \mathrm{Sh}_K(S)$,
(2) $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{V}(A_0, A)^m$ with fundamental matrix $T(\mathbf{x}) = T$.
(3) $\eta(\mathbf{x}_*) \in (\Omega_i) \subseteq (\mathbb{V}_f)^m$. Here $\mathbf{x}_* \in \mathrm{Hom}_F(\hat{V}(A_0), \hat{V}(A))^m$ is the element induced by $\mathbf{x}$.

Let $\mathcal{Z}(T, \varphi_K)$ be the Zariski closure of $Z(T, \varphi_K)$ in $\mathcal{M}_K$. The functor $S \mapsto \mathcal{Z}(T, \varphi_K)(S)$ is represented by a (possibly empty) Deligne–Mumford stack which is finite and unramified over $\mathcal{M}$ ([KR14, Proposition 2.9]), and thus defines a cycle $\mathcal{Z}(T, \varphi_K) \in \mathrm{Z}^*(\mathcal{M}_K)$. For a general admissible Schwartz function $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$, by extending $\mathbb{C}$-linearly we obtain a cycle $\mathcal{Z}(T, \varphi_K) \in \mathrm{Z}^*(\mathcal{M})_{\mathbb{C}}$.

Now we assume $m = n$. Analogous to the local situation (1.1), we can define its local arithmetic intersection numbers $\mathrm{Int}_{T,\nu}(\varphi_K)$ for any place ν of E. Let v be the place of F_0 under ν. By [HSY21, Lemma 15.3] (see also [KR14, Proposition 2.22]), it suffices to consider the case when v is nonsplit in F. When φ_K is of the form (10.2) and v is finite, define

$$\mathrm{Int}_{T,\nu}(\varphi_K) := \chi(\mathcal{Z}(T,\varphi_K), \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_n,\varphi_n)}) \cdot \log q_\nu, \tag{10.3}$$

where q_ν denotes the size of the residue field k_ν of E_ν, $\mathcal{O}_{\mathcal{Z}(t_i,\varphi_i)}$ denotes the structure sheaf of the Kudla–Rapoport divisor $\mathcal{Z}(t_i,\varphi_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{M}_K$, χ denotes the Euler–Poincaré characteristic. For a general admissable function φ_K, we can extend the definition $\mathbb{C}$-linearly. Using the star product of Kudla's Green functions, we can also define its local arithmetic intersection number $\mathrm{Int}_{T,v}(\mathsf{y},\varphi_K)$ at infinite places ([LZ22a, §15.3]), which depends on a parameter $\mathsf{y} \in \mathrm{Herm}_n(F_{0,\infty})_{>0}$ where $F_{0,\infty} = F_0 \otimes_{\mathbb{Q}} \mathbb{R}$. Combining all the local arithmetic numbers together, define the *global arithmetic intersection number*, or the *arithmetic degree* of the special cycle $\mathcal{Z}(T,\varphi_K)$ in the arithmetic Chow group of $\mathcal{M}_K$,

$$\widehat{\deg}_T(\mathsf{y},\varphi_K) := \sum_{\nu\nmid\infty} \mathrm{Int}_{T,v}(\varphi_K) + \sum_{\nu\mid\infty} \mathrm{Int}_{T,v}(\mathsf{y},\varphi_K).$$

Theorem 10.1. *Let* $\mathrm{Diff}(T,\mathbb{V})$ *be the set of places* v *such that* $\mathbb{V}_v$ *does not represent* T. *Let* $T \in \mathrm{Herm}_n(F_0)$ *be nonsingular such that* $\mathrm{Diff}(T,\mathbb{V}) = \{v\}$ *where* v *is nonsplit in* F *and not above 2. Then*

$$\widehat{\deg}_T(\mathsf{y},\varphi_K) q^T = c_K \cdot \partial\mathrm{Eis}_T(\mathsf{z},\varphi_K),$$

where $q^T := \psi_\infty(\mathrm{Tr}\, T\mathsf{z})$, $c_K = \frac{(-1)^n}{\mathrm{vol}(K)}$ *is a nonzero constant independent of* T *and* $\mathrm{vol}(K)$ *is the volume of* K *under a suitable Haar measure. Finally,* $\partial\mathrm{Eis}_T(\mathsf{z},\varphi_K)$ *is the* T*-the coefficient of the modified central derivative of Eisenstein series in (1.11)*

Proof. When v is finite and $v \notin \mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$, the theorem is proved in [LZ22a, Theorem 13.6]. For $v \in \mathcal{V}_{\mathrm{asd}}$, our definition of $\mathrm{Int}_{T,\nu}(\varphi_K)$ differs from that of [LZ22a, (13.5.0.14)]. Correspondingly on the analytic side, our definition of $\partial\mathrm{Eis}_T(\mathsf{z},\varphi_K)$ is also modified, see (1.9) and (1.10). However using [LZ22a, Theorem 10.5.1] instead of [LZ22a, Theorem 10.3.1], the proof of [LZ22a, Theorem 13.6] works the same way in this case. When v is infinite, the theorem is proved in [Liu11, Theorem 4.17,4.20] and independently in [GS19, Theorem 1.1.2]. When v is finite and $v \in \mathcal{V}_{\mathrm{ram}}$, the theorem is a corollary of Theorem 9.7 and can be proved in the same way as [HSY21, Theorem 15.3]. ☐

We say $\varphi_v \in \mathscr{S}(\mathbb{V}_v^n)$ is *nonsingular* if its support lies in $\{\mathbf{x} \in \mathbb{V}_v^n : \det T(\mathbf{x}) \neq 0\}$, see [LZ22a, §12.3] or [Liu11, Proposition 2.1].

Theorem 10.2. *Assume that* F/F_0 *is split at all places above 2. Further assume that* φ_K *is nonsingular at two places split in* F. *Then*

$$\widehat{\deg}(\mathsf{z},\varphi_K) = c_K \cdot \partial\mathrm{Eis}(\mathsf{z},\varphi_K),$$

where $\widehat{\deg}(\mathsf{z},\varphi_K)$ *is defined in* (1.12). *In particular,* $\widehat{\deg}(\mathsf{z},\varphi_K)$ *is a nonholomorphic hermitian modular form of genus* n.

Proof. The theorem can be derived from Theorem 10.1 by the same way as [LZ22a, Theorem 15.5.1]. ☐

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