KUDLA–RAPOPORT CYCLES AND DERIVATIVES OF LOCAL DENSITIES

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Abstract. We prove the local Kudla–Rapoport conjecture, which is a precise identity between the arithmetic intersection numbers of special cycles on unitary Rapoport–Zink spaces and the derivatives of local representation densities of hermitian forms. As a first application, we prove the global Kudla–Rapoport conjecture, which relates the arithmetic intersection numbers of special cycles on unitary Shimura varieties and the central derivatives of the Fourier coefficients of incoherent Eisenstein series. Combining previous results of Liu and Garcia–Sankaran, we also prove cases of the arithmetic Siegel–Weil formula in any dimension.

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1. Introduction

1.1. **Background.** The classical *Siegel–Weil formula* ([Sie51, Wei65]) relates certain Siegel Eisenstein series to the arithmetic of quadratic forms, namely it expresses special *values* of these series as theta functions — generating series of representation numbers of quadratic forms. Kudla ([Kud97b, Kud04]) initiated an influential program to establish the *arithmetic Siegel–Weil formula* relating certain Siegel Eisenstein series to objects in arithmetic geometry, which among others, aims to express the *central derivative* of these series as the arithmetic analogue of theta functions — generating series of arithmetic intersection numbers of n special divisors on Shimura varieties associated to $\mathrm{SO}(n-1,2)$ or $\mathrm{U}(n-1,1)$. These special divisors include Heegner points on modular or Shimura curves appearing in the Gross–Zagier formula ([GZ86, YZZ13]) ($n=2$), modular correspondence on the product of two modular curves in the Gross–Keating formula ([GK93]) and Hirzebruch–Zagier cycles on Hilbert modular surfaces ([HZ76]) ($n=3$).

The arithmetic Siegel–Weil formula was established by Kudla, Rapoport and Yang ([KRY99, Kud97b, KR00b, KRY06]) for $n=1,2$ (orthogonal case) in great generality. The *archimedean* component of the formula was also known, due to Liu [Liu11a] (unitary case), and Garcia–Sankaran [GS19] in full generality (cf. Bruinier–Yang [BY21] for an alternative proof in the orthogonal case). However, the full formula (in particular, the nonarchimedean part) was widely open in higher dimension.

In the works [KR11, KR14] Kudla–Rapoport made the nonarchimedean part of the conjectural formula more precise by defining arithmetic models of the special cycles (for any n in the unitary case), now known as *Kudla–Rapoport cycles*. They formulated the *global Kudla–Rapoport conjecture* for the nonsingular part of the formula, and explained how it would follow (at least at an unramified place) from the *local Kudla–Rapoport conjecture*, relating the derivatives of local representation densities of hermitian forms and arithmetic intersection numbers of Kudla–Rapoport cycles on unitary Rapoport–Zink spaces. They further proved the conjectures in the special case when the arithmetic intersection is *non-degenerate* (i.e., of the expected dimension 0). Outside the non-degenerate case, the only known result was due to Terstiege [Ter13a], who proved the Kudla–Rapoport conjectures for $n=3$. Analogous results were known in the orthogonal case, see [GK93, KR99, KR00a, BY21] (non-degenerate case) and [Ter11] ($n=3$).

The main result of this paper settles the local Kudla–Rapoport conjecture for any n in the unitary case. As a first application, we will be able to deduce the global Kudla–Rapoport conjecture, and prove the first cases of the arithmetic Siegel–Weil formula in all higher dimensions. In a companion paper [LZ21], we will also use similar methods to prove analogous results in the orthogonal case.

As explained in [Kud97b] and [Liu11a], the arithmetic Siegel–Weil formula (together with the doubling method) has important application to the *arithmetic inner product formula*, relating the central derivative of the standard L-function of cuspidal automorphic representations on orthogonal or unitary groups to the height pairing of certain cycles on Shimura varieties constructed from arithmetic theta liftings. It can be viewed as a higher dimensional generalization of the Gross–Zagier formula, and an arithmetic analogue of the Rallis inner product formula. Further applications to the arithmetic inner product formula are investigated in [LL, LL21]. We also mention that the local Kudla–Rapoport conjecture has application to the so-called *unitary arithmetic fundamental lemma* for cycles on unitary Shimura varieties arising from the embedding $\mathrm{U}(n) \times \mathrm{U}(n) \hookrightarrow \mathrm{U}(2n)$.

1.2. **The local Kudla–Rapoport conjecture.** Let p be an odd prime. Let F_0 be a finite extension of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_q$ and a uniformizer ϖ. Let F be an unramified quadratic extension of F_0. Let $\breve{F}$ be the completion of the maximal unramified extension of F. For any integer $n \geq 1$, the *unitary Rapoport–Zink space* $\mathcal{N} = \mathcal{N}_n$ (§2.1) is the formal scheme over $S = \operatorname{Spf} O_{\breve{F}}$ parameterizing hermitian formal O_F-modules of signature $(1, n-1)$ within the supersingular quasi-isogeny class. Let $\mathbb{E}$ and $\mathbb{X}$ be the framing hermitian O_F-modules of signature $(1,0)$ and $(1, n-1)$ over $\bar{k}$. The space of *quasi-homomorphisms* $\mathbb{V} = \mathbb{V}_n := \mathrm{Hom}^\circ_{O_F}(\bar{\mathbb{E}}, \mathbb{X})$ carries a natural F/F_0-hermitian form, which makes $\mathbb{V}$ the unique (up to isomorphism) non-degenerate nonsplit (see §1.7) F/F_0-hermitian space of dimension n (§2.2). For any subset $L \subseteq \mathbb{V}$, the local *Kudla–Rapoport cycle* $\mathcal{Z}(L)$ (§2.3) is a closed formal subscheme of $\mathcal{N}$, over which each quasi-homomorphism $x \in L$ deforms to homomorphisms.

Let $L \subseteq \mathbb{V}$ be an O_F-lattice (of full rank n). We now associate to L two integers: the *arithmetic intersection number* $\mathrm{Int}(L)$ and the *derivative of the local density* $\partial\mathrm{Den}(L)$.

Let $x_1, \ldots, x_n$ be an O_F-basis of L. Define the *arithmetic intersection number*

$$\mathrm{Int}(L) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}), \tag{1.2.0.1}$$

where $\mathcal{O}_{\mathcal{Z}(x_i)}$ denotes the structure sheaf of the Kudla–Rapoport divisor $\mathcal{Z}(x_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{N}$, and χ denotes the Euler–Poincaré characteristic (§2.4). By [Ter13a, Proposition 3.2] (or [How19, Corollary D]), we know that $\mathrm{Int}(L)$ is independent of the choice of the basis $x_1, \ldots, x_n$ and hence is a well-defined invariant of L itself.

For M another hermitian O_F-lattice (of arbitrary rank), define $\mathrm{Rep}_{M,L}$ to be the *scheme of integral representations of M by L*, an O_{F_0}-scheme such that for any O_{F_0}-algebra R, $\mathrm{Rep}_{M,L}(R) = \mathrm{Herm}(L \otimes_{O_{F_0}} R, M \otimes_{O_{F_0}} R)$, where Herm denotes the set of hermitian module homomorphisms. The *local density* of integral representations of M by L is defined to be

$$\mathrm{Den}(M, L) := \lim_{N \to +\infty} \frac{\#\mathrm{Rep}_{M,L}(O_{F_0}/\varpi^N)}{q^{N \cdot \dim(\mathrm{Rep}_{M,L})_{F_0}}}.$$

Let $\langle 1 \rangle^k$ be the self-dual hermitian O_F-lattice of rank k with hermitian form given by the identity matrix $\mathbf{1}_k$. Then $\mathrm{Den}(\langle 1 \rangle^k, L)$ is a polynomial in $(-q)^{-k}$ with $\mathbb{Q}$-coefficients. Define the (normalized) *local Siegel series* of L to be the polynomial $\mathrm{Den}(X, L) \in \mathbb{Z}[X]$ (Theorem 3.5.1) such that

$$\mathrm{Den}((-q)^{-k}, L) = \frac{\mathrm{Den}(\langle 1 \rangle^{n+k}, L)}{\mathrm{Den}(\langle 1 \rangle^{n+k}, \langle 1 \rangle^n)}.$$

It satisfies a functional equation relating $X \leftrightarrow \frac{1}{X}$,

$$\mathrm{Den}(X, L) = (-X)^{\mathrm{val}(L)} \cdot \mathrm{Den}\left(\frac{1}{X}, L\right). \tag{1.2.0.2}$$

Here $\mathrm{val}(L)$ is the valuation of L defined in §1.7. Since $\mathbb{V}$ is nonsplit, we know that $\mathrm{val}(L)$ is odd and so the value $\mathrm{Den}(1, L) = 0$. We thus consider the *derivative of the local density*

$$\partial\mathrm{Den}(L) := -\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=1} \mathrm{Den}(X, L).$$

Our main theorem in Part 1 is a proof of the local Kudla–Rapoport conjecture [KR11, Conjecture 1.3], which asserts an exact identity between the two integers just defined.

Theorem 1.2.1 (Theorem 3.4.1, local Kudla–Rapoport conjecture)**.** *Let $L \subseteq \mathbb{V}$ be an O_F-lattice of full rank n. Then*

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

We refer to $\mathrm{Int}(L)$ as the *geometric side* of the identity (related to the geometry of Rapoport–Zink spaces and Shimura varieties) and $\partial\mathrm{Den}(L)$ the *analytic side* (related to the derivatives of Eisenstein series and L-functions).

Our main theorem in Part 2 proves a variant of the local Kudla–Rapoport conjecture in the presence of a minimal nontrivial level structure, given by the stabilizer of an almost self-dual lattice (see §1.7) in a nonsplit F/F_0-hermitian space. The relevant Rapoport–Zink space on the geometric side is no longer formally smooth. See Theorems 10.3.1 and 10.5.1 for the precise statement.

1.3. **The arithmetic Siegel–Weil formula.** Next let us describe some global applications of our local theorems. We now switch to global notations. Let F be a CM number field, with F_0 its totally real subfield of index 2. Fix a CM type $\Phi \subseteq \mathrm{Hom}(F, \overline{\mathbb{Q}})$ of F. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and identify the CM type Φ with the set of archimedean places of F, and also with the set of archimedean places of F_0. Let V be an F/F_0-hermitian space of dimension n and $G = \mathrm{Res}_{F_0/\mathbb{Q}}\,\mathrm{U}(V)$. Assume the signatures of V are $\{(n-1,1)_{\phi_0}, (n,0)_{\phi\in\Phi-\{\phi_0\}}\}$ for some distinguished element $\phi_0 \in \Phi$. Define a torus $Z^{\mathbb{Q}} = \{z \in \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m : \mathrm{Nm}_{F/F_0}(z) \in \mathbb{G}_m\}$. Associated to $\widetilde{G} := Z^{\mathbb{Q}} \times G$ there is a natural Shimura datum $(\widetilde{G}, \{h_{\widetilde{G}}\})$ of PEL type (§11.1). Let $K = K_{Z^{\mathbb{Q}}} \times K_G \subseteq \widetilde{G}(\mathbb{A}_f)$ be a compact open subgroup. Then the associated Shimura variety $\mathrm{Sh}_K = \mathrm{Sh}_K(\widetilde{G}, \{h_{\widetilde{G}}\})$ is of dimension $n-1$ and has a canonical model over its reflex field E.

Assume that $K_{Z^{\mathbb{Q}}} \subseteq Z^{\mathbb{Q}}(\mathbb{A}_f)$ is the unique maximal open compact subgroup. Assume that $K_G = \prod_v K_{G,v}$, where v runs over the finite places of F_0 such that $K_{G,v} \subseteq \mathrm{U}(V)(F_{0,v})$ is given by

- the stabilizer of a self-dual or almost self-dual lattice $\Lambda_v \subseteq V_v$ if v is inert in F,
- the stabilizer of a self-dual lattice $\Lambda_v \subseteq V_v$ if v is ramified in F,
- a principal congruence subgroup of $\mathrm{U}(V)(F_{0,v}) \simeq \mathrm{GL}_n(F_{0,v})$ if v is split in F.

Then we construct a global regular integral model $\mathcal{M}_K$ of Sh_K over O_E following [RSZ20] (see §14.1, §14.2 for more precise technical assumptions needed). When $F_0 = \mathbb{Q}$, we have $E = F$ and the integral model $\mathcal{M}_K$ recovers that in [BHK$^+$20] when K_G is the stabilizer of a global self-dual lattice, which is closely related to that in [KR14].

Let $\mathbb{V}$ be the *incoherent* $\mathbb{A}_F/\mathbb{A}_{F_0}$-hermitian space nearby V, namely $\mathbb{V}$ is totally positive definite and $\mathbb{V}_v \cong V_v$ for all finite places v. Let $\varphi_K \in \mathscr{S}(\mathbb{V}_f^n)$ be a K-invariant (where K acts on $\mathbb{V}_f$ via the second factor K_G) factorizable Schwartz function such that $\varphi_{K,v} = \mathbf{1}_{(\Lambda_v)^n}$ at all v inert in F. Let $T \in \mathrm{Herm}_n(F)$ be a nonsingular hermitian matrix of size n. Associated to (T, φ_K)

we construct arithmetic cycles $\mathcal{Z}(T, \varphi_K)$ over $\mathcal{M}_K$ (§14.3) generalizing the *Kudla–Rapoport cycles* $\mathcal{Z}(T)$ in [KR14]. Analogous to the local situation (1.2.0.1), we may define its local arithmetic intersection numbers $\mathrm{Int}_{T,v}(\varphi_K)$ at finite places v (§13.5). Using the star product of Kudla's Green functions, we also define its local arithmetic intersection number $\mathrm{Int}_{T,v}(\mathsf{y}, \varphi_K)$ at infinite places (§15.3), which depends on a parameter $\mathsf{y} \in \mathrm{Herm}_n(F_\infty)_{>0}$ where $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^\Phi$. Combining all the local arithmetic numbers together, define the *global arithmetic intersection number*, or the *arithmetic degree* of the Kudla–Rapoport cycle $\mathcal{Z}(T, \varphi_K)$,

$$\widehat{\deg}_T(\mathsf{y}, \varphi_K) \coloneqq \sum_{v \nmid \infty} \mathrm{Int}_{T,v}(\varphi_K) + \sum_{v \mid \infty} \mathrm{Int}_{T,v}(\mathsf{y}, \varphi_K).$$

It is closely related to the usual arithmetic degree on the Gillet–Soulé arithmetic Chow group $\widehat{\mathrm{Ch}}^n_{\mathbb{C}}(\mathcal{M}_K)$ (§15.4).

On the other hand, associated to $\varphi = \varphi_K \otimes \varphi_\infty \in \mathscr{S}(\mathbb{V}^n)$, where φ_∞ is the Gaussian function, there is a classical *incoherent Eisenstein series* $E(\mathsf{z}, s, \varphi_K)$ (§12.4) on the hermitian upper half space

$$\mathbb{H}_n = \{\mathsf{z} = \mathsf{x} + i\mathsf{y} : \ \mathsf{x} \in \mathrm{Herm}_n(F_\infty), \ \mathsf{y} \in \mathrm{Herm}_n(F_\infty)_{>0}\}.$$

This is essentially the Siegel Eisenstein series associated to a standard Siegel–Weil section of the degenerate principal series (§12.1). The Eisenstein series here has a meromorphic continuation and a functional equation relating $s \leftrightarrow -s$. The central value $E(\mathsf{z}, 0, \varphi_K) = 0$ by the incoherence. We thus consider its *central derivative*

$$\partial\mathrm{Eis}(\mathsf{z}, \varphi_K) \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\mathsf{z}, s, \varphi_K).$$

Associated to an additive character $\psi : \mathbb{A}_{F_0}/F_0 \to \mathbb{C}^\times$, it has a decomposition into the central derivative of the Fourier coefficients

$$\partial\mathrm{Eis}(\mathsf{z}, \varphi_K) = \sum_{T \in \mathrm{Herm}_n(F)} \partial\mathrm{Eis}_T(\mathsf{z}, \varphi_K).$$

Now we can state our first application to the global Kudla–Rapoport conjecture [KR14, Conjecture 11.10], which asserts an identity between the arithmetic degree of Kudla–Rapoport cycles and the derivative of nonsingular Fourier coefficients of the incoherent Eisenstein series.

Theorem 1.3.1 (Theorem 14.5.1, global Kudla–Rapoport conjecture)**.** *Let* $\mathrm{Diff}(T, \mathbb{V})$ *be the set of places* v *such that* $\mathbb{V}_v$ *does not represent* T *(§12.3). Let* $T \in \mathrm{Herm}_n(F)$ *be nonsingular such that* $\mathrm{Diff}(T, \mathbb{V}) = \{v\}$ *where* v *is inert in* F *and not above 2. Then*

$$\widehat{\deg}_T(\mathsf{y}, \varphi_K) q^T = c_K \cdot \partial\mathrm{Eis}_T(\mathsf{z}, \varphi_K),$$

where $q^T \coloneqq \psi_\infty(\mathrm{tr}\, T\mathsf{z})$*,* $c_K = \frac{(-1)^n}{\mathrm{vol}(K)}$ *is a nonzero constant independent of* T *and* φ_K*, and* $\mathrm{vol}(K)$ *is the volume of* K *under a suitable Haar measure on* $\widetilde{G}(\mathbb{A}_f)$*.*

We form the generating series of arithmetic degrees

$$\widehat{\deg}(\mathsf{z}, \varphi_K) \coloneqq \sum_{\substack{T \in \mathrm{Herm}_n(F) \\ \det T \neq 0}} \widehat{\deg}_T(\mathsf{y}, \varphi_K) q^T.$$

Now we can state our second application to the arithmetic Siegel–Weil formula, which relates this generating series to the central derivative of the incoherent Eisenstein series.

Theorem 1.3.2 (Theorem 15.5.1, arithmetic Siegel–Weil formula)**.** *Assume that F/F_0 is unramified at all finite places and split at all places above 2. Further assume that φ_K is nonsingular (§12.3) at two places split in F. Then*

$$\widehat{\deg}(\mathsf{z}, \varphi_K) = c_K \cdot \partial \mathrm{Eis}(\mathsf{z}, \varphi_K).$$

In particular, $\widehat{\deg}(\mathsf{z}, \varphi_K)$ is a nonholomorphic hermitian modular form of genus n.

Remark 1.3.3. The unramifiedness assumption on F/F_0 forces $F_0 \neq \mathbb{Q}$. To treat the general case, one needs to formulate and prove an analogue of Theorem 1.2.1 when the local extension F/F_0 is ramified. We remark that at a ramified place, in addition to the *Krämer model* with level given by the stabilizer of a self-dual lattice, we may also consider the case of *exotic good reduction* with level associated to an (almost) π-modular lattice. We hope to extend our methods to cover these cases, which in particular requires an extension of the local density formula of Cho–Yamauchi [CY20] to the ramified case (see [LL21] for the case of π-modular lattices).

Remark 1.3.4. The nonsingularity assumption on φ_K allows us to kill all the singular terms on the analytic side. Such φ_K exists for a suitable choice of K since we allow arbitrary Drinfeld levels at split places.

1.4. **Strategy of the proof of the main Theorem 1.2.1.** The previously known special cases of the local Kudla–Rapoport conjecture ([KR11, Ter13a]) are proved via explicit computation of both the geometric and analytic sides. Explicit computation seems infeasible for the general case. Our proof instead proceeds via induction on n using the *uncertainty principle*.

More precisely, for a fixed O_F-lattice $L^\flat \subseteq \mathbb{V} = \mathbb{V}_n$ of rank $n-1$ (we assume $L^\flat_F$ is non-degenerate throughout the paper), consider functions on $x \in \mathbb{V} \setminus L^\flat_F$,

$$\mathrm{Int}_{L^\flat}(x) \coloneqq \mathrm{Int}(L^\flat + \langle x \rangle), \quad \partial\mathrm{Den}_{L^\flat}(x) \coloneqq \partial\mathrm{Den}(L^\flat + \langle x \rangle).$$

Then it remains to show the equality of the two functions $\mathrm{Int}_{L^\flat} = \partial\mathrm{Den}_{L^\flat}$. Both functions vanish when x is non-integral, i.e., $\mathrm{val}(x) < 0$. Here $\mathrm{val}(x)$ denotes the valuation of the norm of x. By utilizing the inductive structure of the Rapoport–Zink spaces and local densities, it is not hard to see that if $x \perp L^\flat$ with $\mathrm{val}(x) = 0$, then

$$\mathrm{Int}_{L^\flat}(x) = \mathrm{Int}(L^\flat), \quad \partial\mathrm{Den}_{L^\flat}(x) = \partial\mathrm{Den}(L^\flat)$$

for the lattice $L^\flat \subseteq \mathbb{V}_{n-1} \cong \langle x \rangle_F^\perp$ of full rank $n-1$. By induction on n, we have $\mathrm{Int}(L^\flat) = \partial\mathrm{Den}(L^\flat)$, and thus the difference function $\phi = \mathrm{Int}_{L^\flat} - \partial\mathrm{Den}_{L^\flat}$ vanishes on $\{x \in \mathbb{V} : x \perp L^\flat, \mathrm{val}(x) \leq 0\}$. We would like to deduce that ϕ indeed vanishes identically.

The uncertainty principle (Proposition 8.1.6), which is a simple consequence of the Schrödinger model of the local Weil representation of SL_2, asserts that if $\phi \in C_c^\infty(\mathbb{V})$ satisfies that both ϕ and its Fourier transform $\hat{\phi}$ vanish on $\{x \in \mathbb{V} : \mathrm{val}(x) \leq 0\}$, then $\phi = 0$. In other words, $\phi, \hat{\phi}$ cannot simultaneously have "small support" unless $\phi = 0$. We can then finish the proof by applying the uncertainty principle to $\phi = \mathrm{Int}_{L^\flat} - \partial\mathrm{Den}_{L^\flat}$, if we can show that both $\mathrm{Int}_{L^\flat}$ and $\partial\mathrm{Den}_{L^\flat}$ are invariant under the Fourier transform (up to the Weil constant $\gamma_{\mathbb{V}} = -1$). However, both functions have singularities along the hyperplane $L_F^\flat \subseteq \mathbb{V}$, which cause trouble in computing their Fourier transforms or even in showing that $\phi \in C_c^\infty(\mathbb{V})$.

To overcome this difficulty, we isolate the singularities by decomposing

$$\mathrm{Int}_{L^\flat} = \mathrm{Int}_{L^\flat,\mathscr{H}} + \mathrm{Int}_{L^\flat,\mathscr{V}}, \quad \partial\mathrm{Den}_{L^\flat} = \partial\mathrm{Den}_{L^\flat,\mathscr{H}} + \partial\mathrm{Den}_{L^\flat,\mathscr{V}}$$

into "horizontal" and "vertical" parts. Here on the geometric side $\mathrm{Int}_{L^\flat,\mathscr{H}}$ is the contribution from the horizontal part of the Kudla–Rapoport cycles, which we determine explicitly in terms of quasi-canonical lifting cycles (Theorem 4.2.1). On the analytic side we define $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ to match with $\mathrm{Int}_{L^\flat,\mathscr{H}}$. We show the horizontal parts have logarithmic singularity along $L_F^\flat$, and vertical parts are indeed in $C_c^\infty(\mathbb{V})$ (Corollary 6.2.2, Proposition 7.3.4). We can then finish the proof if we can determine the Fourier transforms as

$$\widehat{\mathrm{Int}}_{L^\flat,\mathscr{V}} = -\,\mathrm{Int}_{L^\flat,\mathscr{V}}, \quad \widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{V}} = -\partial\mathrm{Den}_{L^\flat,\mathscr{V}}. \tag{1.4.0.1}$$

On the geometric side we show (1.4.0.1) (Corollary 6.3.3) by reducing to the case of intersection with Deligne–Lusztig curves. This reduction requires the Bruhat–Tits stratification of $\mathcal{N}^{\mathrm{red}}$ into certain Deligne–Lusztig varieties (§2.7, due to Vollaard–Wedhorn [VW11]) and the Tate conjecture for these Deligne–Lusztig varieties (Theorem 5.3.2, which we reduce to a cohomological computation of Lusztig [Lus76]).

On the analytic side we are only able to show (1.4.0.1) (Theorem 7.4.1) directly when $x \perp L^\flat$ and $\mathrm{val}(x) < 0$. The key ingredient is a local density formula (Theorem 3.5.1) due to Cho–Yamauchi [CY20] together with the functional equation (1.2.0.2). We then deduce the general case by performing another induction on $\mathrm{val}(L^\flat)$ (§8.2).

We remark the extra symmetry (1.4.0.1) under the Fourier transform can be thought of as a *local modularity*, in analogy with the global modularity of arithmetic generating series (such as in [BHK$^+$20]) encoding an extra global SL_2-symmetry. The latter global modularity plays a crucial role in the second author's recent proof [Zha21] of the *arithmetic fundamental lemma*. In contrast to [Zha21], our proof of the local Kudla–Rapoport conjecture does not involve global arguments, thanks to a more precise understanding of the horizontal part of Kudla–Rapoport cycles. In other similar (non-arithmetic) situations, induction arguments involving Fourier transforms and the uncertainty principle are not unfamiliar: here we only mention the second author's proof [Zha14] of the Jacquet–Rallis smooth transfer conjecture, and more recently Beuzart-Plessis' new proof [BP21] of the Jacquet–Rallis fundamental lemma.

1.5. **The structure of the paper.** In Part 1, we review necessary background on the local Kudla–Rapoport conjecture and prove the main Theorem 1.2.1. In Part 2, we prove a variant of the local Kudla–Rapoport conjecture in the almost self-dual case (Theorems 10.3.1, 10.5.1), by relating both the geometric and analytic sides in the almost self-dual to the self-dual case (but in one dimension higher). In Part 3, we review semi-global and global integral models of Shimura varieties and Kudla–Rapoport cycles, and incoherent Eisenstein series. We then apply the local results in Parts

1 and 2 to prove the local arithmetic Siegel–Weil formula (Theorem 13.6.1), the global Kudla–Rapoport conjecture (Theorem 14.5.1), and cases of the arithmetic Siegel–Weil formula (Theorem 15.5.1).

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1.7. **Notation on hermitian lattices.** Let p be a prime. In the local parts of the paper (Part 1 and 2), we let F_0 be a non-archimedean local field of residue characteristic p, with ring of integers O_{F_0}, residue field $k = \mathbb{F}_q$ of size q, and uniformizer ϖ. Unless otherwise specified, we let F be a quadratic extension of F_0, with ring of integers O_F and residue field k_F. Let σ be the nontrivial automorphism of F/F_0. Let $\breve F$ be the completion of the maximal unramified extension of F, and $O_{\breve F}$ its ring of integers.

Unless otherwise specified, we assume that F/F_0 is unramified (with an exception of §8.1.5 on the uncertainty principle). We further assume that F_0 has characteristic 0 and residue characteristic $p > 2$ (with exceptions of §3, §7, §8.1.5, §9, which concern only the analytic side).

Let $\mathbb{V}$ be a (non-degenerate) F/F_0-hermitian space with hermitian form $(\ ,\ )$. We write $\mathrm{val}(x) := \mathrm{val}((x,x))$ for any $x \in \mathbb{V}$, where val is the valuation on F_0. Recall that the (non-degenerate) F/F_0-hermitian spaces are classified up to isomorphism by its dimension n and its discriminant $\mathrm{disc}(\mathbb{V}) = (-1)^{\binom{n}{2}} \det(\mathbb{V}) \in F_0^\times/\mathrm{Nm}_{F/F_0}F^\times$ ([Jac62, Theorem 3.1]). We say $\mathbb{V}$ is *split* if $\mathrm{disc}(\mathbb{V}) = 1 \in F_0^\times/\mathrm{Nm}_{F/F_0}F^\times$, and *nonsplit* otherwise.

Let $L \subseteq \mathbb{V}$ be an O_F-lattice of rank n. We denote by $L^\vee$ its dual lattice under $(\ ,\ )$. We say that L is *integral* if $L \subseteq L^\vee$. If L is integral, define its *fundamental invariants* to be the unique sequence of integers $(a_1, \ldots, a_n)$ such that $0 \le a_1 \le \cdots \le a_n$, and $L^\vee/L \simeq \oplus_{i=1}^n O_F/\varpi^{a_i}$ as O_F-modules; define its *valuation* to be $\mathrm{val}(L) := \sum_{i=1}^n a_i$; and define its *type*, denoted by $t(L)$, to be the number of nonzero terms in its fundamental invariants $(a_1, \ldots, a_n)$.

We say L is *minuscule* or a *vertex lattice* if it is integral and $L^\vee \subseteq \varpi^{-1}L$. Note that L is a vertex lattice of type t if and only if it has fundamental invariants $(0^{(n-t)}, 1^{(t)})$, if and only if $L \subseteq^t L^\vee \subseteq \varpi^{-1}L$, where $\subseteq^t$ indicates that the O_F-colength is equal to t. The set of vertex lattices of type t in $\mathbb{V}$ is denoted by $\mathrm{Vert}^t = \mathrm{Vert}^t(\mathbb{V})$. We say L is *self-dual* if $L = L^\vee$, or equivalently L is a vertex lattice of type 0. We say L is *almost self-dual* if L is a vertex lattice of type 1. When F/F_0 is unramified, if $\mathbb{V}$ is split then $\mathrm{val}(L)$ is even and $\mathbb{V}$ contains a self-dual lattice; if $\mathbb{V}$ is nonsplit then $\mathrm{val}(L)$ is odd and $\mathbb{V}$ contains an almost self-dual lattice.

Unless otherwise specified, we denote by $L^\flat \subseteq \mathbb{V}$ an O_F-lattice of rank $n-1$, and we always assume that $L^\flat_F$ is non-degenerate. Here we use the subscript $(-)_F$ to stand for the base change to F, so $L^\flat_F = L^\flat \otimes_{O_F} F$.

Fix an unramified additive character $\psi : F_0 \to \mathbb{C}^\times$. Here "unramifiedness" means that the conductor of ψ (i.e., the largest fractional ideal in F_0 on which ψ is trivial) is O_{F_0}. For an integrable function f on $\mathbb{V}$, we define its Fourier transform $\widehat{f}$ to be

$$\widehat{f}(x) := \int_{\mathbb{V}} f(y)\psi(\mathrm{tr}_{F/F_0}(x,y))\mathrm{d}y, \quad x \in \mathbb{V}. \tag{1.7.0.2}$$

We normalize the Haar measure on $\mathbb{V}$ to be self-dual, so $\hat{\hat{f}}(x) = f(-x)$. For an O_F-lattice $L \subseteq \mathbb{V}$ of rank n, we have (under the assumption that F/F_0 is unramified)

$$\widehat{\mathbf{1}}_L = \mathrm{vol}(L)\mathbf{1}_{L^\vee}, \quad \text{and} \quad \mathrm{vol}(L) = [L^\vee : L]^{-1/2} = q^{-\mathrm{val}(L)}.$$

Note that $\mathrm{val}(L)$ can be defined for any lattice L (not necessarily integral) so that the above equality for $\mathrm{vol}(L)$ holds.

1.8. **Notation on formal schemes.** Let X be a formal scheme. Denote by X^{red} the underlying reduced scheme. For closed formal subschemes $\mathcal{Z}_1, \cdots, \mathcal{Z}_m$ of X, denote by $\cup_{i=1}^m \mathcal{Z}_i$ the formal scheme-theoretic union, i.e., the closed formal subscheme with ideal sheaf $\cap_{i=1}^m \mathcal{I}_{\mathcal{Z}_i}$, where $\mathcal{I}_{\mathcal{Z}_i}$ is the ideal sheaf of $\mathcal{Z}_i$. A closed formal subscheme on X is called a Cartier divisor if it is defined by an *invertible* ideal sheaf.

Let X be a formal scheme over $\mathrm{Spf}\, O_{\breve{F}}$. Then X defines a functor on the category of $\mathrm{Spf}\, O_{\breve{F}}$-schemes (i.e. $O_{\breve{F}}$-schemes on which ϖ is locally nilpotent). For a noetherian ϖ-adically complete $O_{\breve{F}}$-algebra R, write $X(R) := \mathrm{Hom}_{\mathrm{Spf}\, O_{\breve{F}}}(\mathrm{Spf}\, R, X) = \varprojlim_n X(\mathrm{Spec}\, R/\varpi^n)$.

When X is noetherian, denote by $K_0^Y(X)$ the Grothendieck group (modulo quasi-isomorphisms) of finite complexes of coherent locally free $\mathcal{O}_X$-modules, acyclic outside Y (i.e., the homology sheaves are formally supported on Y). As defined in [Zha21, (B.1), (B.2)], denote by $\mathrm{F}^i K_0^Y(X)$ be the (descending) codimension filtration on $K_0^Y(X)$, and denote by $\mathrm{Gr}^i K_0^Y(X)$ its i-th graded piece. As in [Zha21, Appendix B], the definition of $K_0^Y(X)$, $\mathrm{F}^i K_0^Y(X)$ and $\mathrm{Gr}^i K_0^Y(X)$ can be extended to locally noetherian formal schemes X by writing X as an increasing union of open noetherian formal subschemes. Similarly, we let $K_0'(X)$ denote the Grothendieck group of coherent sheaves of $\mathcal{O}_X$-modules. Now let X be regular. Then there is a natural isomorphism $K_0^Y(X) \simeq K_0'(Y)$. For closed formal subschemes $\mathcal{Z}_1, \cdots, \mathcal{Z}_m$ of X, denote by $\mathcal{Z}_1 \overset{\mathbb{L}}{\cap}_X \cdots \overset{\mathbb{L}}{\cap}_X \mathcal{Z}_m$ (or simply $\mathcal{Z}_1 \overset{\mathbb{L}}{\cap} \cdots \overset{\mathbb{L}}{\cap} \mathcal{Z}_m$) the derived tensor product $\mathcal{O}_{\mathcal{Z}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_{\mathcal{Z}_m}$, viewed as an element in $K_0^{\mathcal{Z}_1 \cap \cdots \cap \mathcal{Z}_m}(X)$.

For $\mathcal{F}$ a finite complex of coherent $\mathcal{O}_X$-modules, we define its Euler–Poincaré characteristic

$$\chi(X, \mathcal{F}) := \sum_{i,j} (-1)^{i+j} \,\mathrm{length}_{O_{\breve{F}}} H^i(X, H_j(\mathcal{F}))$$

if the lengths are all finite. Assume that X is regular with pure dimension n. If $\mathcal{F}_i \in \mathrm{F}^{r_i} K_0^{\mathcal{Z}_i}(X)$ with $\sum_i r_i \geq n$, then by [Zha21, (B.3)] we know that $\chi(X, \bigotimes_i^{\mathbb{L}} \mathcal{F}_i)$ depends only on the image of $\mathcal{F}_i$ in $\mathrm{Gr}^{r_i} K_0^{\mathcal{Z}_i}(X)$. In fact, we will only need this assertion when X is a scheme (cf. Remark 6.4.11). When X is a formal scheme, the assertion holds trivially when one of the r_i is $\dim X$; this special case will be used repeatedly.

For a morphism $\pi : X \to Y$ between two formal schemes and a closed formal subscheme $\mathcal{Z} \hookrightarrow Y$, let $\pi^{-1}(\mathcal{Z}) \hookrightarrow X$ be the preimage of $\mathcal{Z}$. Let $\pi^* : K_0^{\mathcal{Z}}(Y) \to K_0^{\pi^{-1}(\mathcal{Z})}(X)$ be the homomorphism induced by pulling back locally free sheaves. If π is proper (i.e., a morphism of finite type such that the induced morphism $X^{\mathrm{red}} \to Y^{\mathrm{red}}$ on the reduced schemes is proper), there is a direct image homomorphism $\pi_* : K_0'(X) \to K_0'(Y)$ sending (the class of) a coherent $\mathcal{O}_X$-module $\mathcal{F}$ to $\sum_{i\geq 0}(-1)^i \mathrm{R}^i\pi_*\mathcal{F}$.

1.9. **Reminder on hermitian spaces over finite fields.** Let V be a (non-degenerate) $k_F/k = \mathbb{F}_{q^2}/\mathbb{F}_q$-hermitian space of dimension m (which is unique up to isomorphism). The following (well-known) formula will be used throughout this article often without explicit reference.

Lemma 1.9.1. *Let $\mathcal{S}_b(V)$ be set of totally isotropic k_F-subspaces of dimension b in V, and $S_{m,b} := \#\mathcal{S}_b(V)$. Then*

$$S_{m,b} = \frac{\prod_{i=m-2b+1}^{m}(1-(-q)^i)}{\prod_{i=1}^{b}(1-q^{2i})}.$$

Proof. The unitary group $\mathrm{U}(V)(k) = \mathrm{U}_m(k)$ acts transitively on the set $\mathcal{S}_b(V)$, with stabilizer given by a parabolic subgroup $P_b(k) \subseteq \mathrm{U}(V)(k)$. As an affine variety, we have

$$P_b \simeq \mathrm{Res}_{k_F/k}\,\mathrm{GL}_b \times \mathrm{Res}_{k_F/k}\,\mathbb{G}_a^{b\delta} \times \mathbb{G}_a^{b^2} \times \mathrm{U}_\delta,$$

where $\delta = m-2b$. Therefore

$$S_{m,b} = \frac{\#\mathrm{U}_m(k)}{\#P_b(k)} = \frac{q^{m^2}\prod_{i=1}^{m}(1-(-q)^{-i})}{[q^{2b^2}\prod_{i=1}^{b}(1-q^{-2i})]\cdot q^{2b\delta}\cdot q^{b^2}\cdot[q^{\delta^2}\prod_{i=1}^{\delta}(1-(-q)^{-i})]},$$

which simplifies to the desired formula. □

Part 1. Local Kudla–Rapoport conjecture: the self-dual case

2. Kudla–Rapoport cycles

2.1. **Rapoport–Zink spaces $\mathcal{N}$.** Let $n \geq 1$ be an integer. A *hermitian O_F-module of signature* $(1, n-1)$ over a $\mathrm{Spf}\, O_{\breve F}$-scheme S is a triple (X, ι, λ) where

(1) X is a formal ϖ-divisible O_{F_0}-module over S of relative height $2n$ and dimension n,

(2) $\iota : O_F \to \mathrm{End}(X)$ is an action of O_F extending the O_{F_0}-action and satisfying the Kottwitz condition of signature $(1, n-1)$: for all $a \in O_F$, the characteristic polynomial of $\iota(a)$ on $\mathrm{Lie}\, X$ is equal to $(T-a)(T-\sigma(a))^{n-1} \in \mathcal{O}_S[T]$,

(3) $\lambda : X \xrightarrow{\sim} X^\vee$ is a principal polarization on X whose Rosati involution induces the automorphism σ on O_F via ι.

Up to O_F-linear quasi-isogeny compatible with polarizations, there is a unique such triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $S = \mathrm{Spec}\, \bar k$. Let $\mathcal{N} = \mathcal{N}_n = \mathcal{N}_{F/F_0,n}$ be the (relative) *unitary Rapoport–Zink space of signature* $(1, n-1)$, parameterizing hermitian O_F-modules of signature $(1, n-1)$ within the supersingular quasi-isogeny class. More precisely, $\mathcal{N}$ is the formal scheme over $\mathrm{Spf}\, O_{\breve F}$ which represents the functor sending each S to the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho)$, where the *framing* $\rho : X \times_S \bar S \to \mathbb{X} \times_{\mathrm{Spec}\, \bar k} \bar S$ is an O_F-linear quasi-isogeny of height 0 such that $\rho^*((\lambda_{\mathbb{X}})_{\bar S}) = \lambda_{\bar S}$. Here $\bar S := S_{\bar k}$ is the special fiber.

The Rapoport–Zink space $\mathcal{N} = \mathcal{N}_n$ is formally locally of finite type and formally smooth of relative dimension $n-1$ over $\mathrm{Spf}\, O_{\breve F}$ ([RZ96], [Mih, Proposition 1.3]).

2.2. **The hermitian space $\mathbb{V}$.** Let $\mathbb{E}$ be the formal O_{F_0}-module of relative height 2 and dimension 1 over $\operatorname{Spec}\bar{k}$. Then $D \coloneqq \operatorname{End}^\circ_{O_{F_0}}(\mathbb{E}) \coloneqq \operatorname{End}_{O_{F_0}}(\mathbb{E}) \otimes \mathbb{Q}$ is the quaternion division algebra over F_0. We fix an F_0-embedding $\iota_{\mathbb{E}} : F \to D$, which makes $\mathbb{E}$ into a formal O_F-module of relative height 1. We fix an O_{F_0}-linear principal polarization $\lambda_{\mathbb{E}} : \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee$. Then $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ is a hermitian O_F-module of signature $(1, 0)$. We have $\mathcal{N}_1 \simeq \operatorname{Spf} O_{\breve{F}}$ and there is a unique lifting (*the canonical lifting*) $\mathcal{E}$ of the formal O_F-module $\mathbb{E}$ over $\operatorname{Spf} O_{\breve{F}}$, equipped with its O_F-action $\iota_{\mathcal{E}}$, its framing $\rho_{\mathcal{E}} : \mathcal{E}_{\bar{k}} \xrightarrow{\sim} \mathbb{E}$, and its principal polarization $\lambda_{\mathcal{E}}$ lifting $\rho^*_{\mathcal{E}}(\lambda_{\mathbb{E}})$. Define $\bar{\mathbb{E}}$ to be the same O_{F_0}-module as $\mathbb{E}$ but with O_F-action given by $\iota_{\bar{\mathbb{E}}} \coloneqq \iota_{\mathbb{E}} \circ \sigma$, and $\lambda_{\bar{\mathbb{E}}} \coloneqq \lambda_{\mathbb{E}}$, and similarly define $\bar{\mathcal{E}}$ and $\lambda_{\bar{\mathcal{E}}}$.

Define

$$\mathbb{V} = \mathbb{V}_n \coloneqq \operatorname{Hom}^\circ_{O_F}(\bar{\mathbb{E}}, \mathbb{X}) = \operatorname{Hom}_{O_F}(\bar{\mathbb{E}}, \mathbb{X}) \otimes \mathbb{Q}$$

to be the space of *special quasi-homomorphisms* ([KR11, Definition 3.1]). Then $\mathbb{V}$ carries a F/F_0-hermitian form: for $x, y \in \mathbb{V}$, the pairing $(x, y) \in F$ is given by

$$(\bar{\mathbb{E}} \xrightarrow{x} \mathbb{X} \xrightarrow{\lambda_{\mathbb{X}}} \mathbb{X}^\vee \xrightarrow{y^\vee} \bar{\mathbb{E}}^\vee \xrightarrow{\lambda_{\bar{\mathbb{E}}}^{-1}} \bar{\mathbb{E}}) \in \operatorname{End}^\circ_{O_F}(\bar{\mathbb{E}}) = \iota_{\bar{\mathbb{E}}}(F) \simeq F.$$

The hermitian space $\mathbb{V}$ is the unique (up to isomorphism) non-degenerate non-split F/F_0-hermitian space of dimension n. The space of special homomorphisms $\operatorname{Hom}_{O_F}(\bar{\mathbb{E}}, \mathbb{X})$ is an integral hermitian O_F-lattice in $\mathbb{V}$. The unitary group $\mathrm{U}(\mathbb{V})(F_0)$ acts on the framing hermitian O_F-module $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ (via the identification in [KR11, Lemma 3.9]) and hence acts on the Rapoport–Zink space $\mathcal{N}$ via $g(X, \iota, \lambda, \rho) = (X, \iota, \lambda, g \circ \rho)$ for $g \in \mathrm{U}(\mathbb{V})(F_0)$.

2.3. **Kudla–Rapoport cycles $\mathcal{Z}(L)$.** For any subset $L \subseteq \mathbb{V}$, define the *Kudla–Rapoport cycle* (or *special cycle*) $\mathcal{Z}(L) \subseteq \mathcal{N}$ to be the closed formal subscheme which represents the functor sending each S to the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho)$ such that for any $x \in L$, the quasi-homomorphism

$$\rho^{-1} \circ x \circ \rho_{\bar{\mathcal{E}}} : \bar{\mathcal{E}}_S \times_S \bar{S} \xrightarrow{\rho_{\bar{\mathcal{E}}}} \bar{\mathbb{E}} \times_{\operatorname{Spec}\bar{k}} \bar{S} \xrightarrow{x} \mathbb{X} \times_{\operatorname{Spec}\bar{k}} \bar{S} \xrightarrow{\rho^{-1}} X \times_S \bar{S}$$

extends to a homomorphism $\bar{\mathcal{E}}_S \to X$ ([KR11, Definition 3.2]). Note that $\mathcal{Z}(L)$ only depends on the O_F-linear span of L in $\mathbb{V}$.

2.4. **Arithmetic intersection numbers $\operatorname{Int}(L)$.** Let $L \subseteq \mathbb{V}$ be an O_F-lattice of rank $r \geq 1$. Let $x_1, \ldots, x_r$ be an O_F-basis of L. Since each $\mathcal{Z}(x_i)$ is a Cartier divisor on $\mathcal{N}$ ([KR11, Proposition 3.5]), we know that $\mathcal{O}_{\mathcal{Z}(x_i)} \in \mathrm{F}^1 K_0^{\mathcal{Z}(x_i)}(\mathcal{N})$ (see §1.8), and hence by [Zha21, (B.3)] we obtain

$$\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_r)} \in \mathrm{F}^r K_0^{\mathcal{Z}(L)}(\mathcal{N}).$$

This is independent of the choice of the basis $x_1, \ldots, x_n$ by [How19, Corollary C] and hence is a well-defined invariant of L itself. We will provide a different proof of this independence (see Corollary 2.8.2) after recalling the structure of the reduced scheme of $\mathcal{Z}(L)$.

Definition 2.4.1. Define the *derived Kudla–Rapoport cycle* ${}^{\mathbb{L}}\mathcal{Z}(L)$ to be the image of $\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_r)}$ in the r-th graded piece $\operatorname{Gr}^r K_0^{\mathcal{Z}(L)}(\mathcal{N})$.

Definition 2.4.2. When $L \subseteq \mathbb{V}$ has rank $r = n$, define the *arithmetic intersection number*

$$\mathrm{Int}(L) \coloneqq \chi\big(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}(L)\big), \tag{2.4.2.1}$$

where χ denotes the Euler–Poincaré characteristic (§1.8). Notice that if L is not integral then $\mathcal{Z}(L)$ is empty and hence $\mathrm{Int}(L) = 0$.

Example 2.4.3 (The case $\operatorname{rank} L = 1$)**.** If $\operatorname{rank} L = 1$, then by the theory of canonical lifting ([Gro86]), we have

$$\mathrm{Int}(L) = \frac{\mathrm{val}(L) + 1}{2}.$$

2.5. **Generalized Deligne–Lusztig varieties Y_V.** Let V be the unique (up to isomorphism) k_F/k-hermitian space of odd dimension $2d+1$. Define Y_V to be the closed k_F-subvariety of the Grassmannian $\mathrm{Gr}_{d+1}(V)$ parameterizing subspaces $U \subseteq V$ of dimension $d+1$ such that $U^\perp \subseteq U$ ([Vol10, (2.19)]). It is a smooth projective variety of dimension d, and has a locally closed stratification

$$Y_V = \bigsqcup_{i=0}^{d} X_{P_i}(w_i),$$

where each $X_{P_i}(w_i)$ is a generalized Deligne–Lusztig variety of dimension i associated to a certain parabolic subgroup $P_i \subseteq \mathrm{U}(V)$ ([Vol10, Theorem 2.15]). The open stratum $Y_V^\circ \coloneqq X_{P_d}(w_d)$ is a classical Deligne–Lusztig variety associated to a Borel subgroup $P_d \subseteq \mathrm{U}(V)$ and a Coxeter element w_d. Each of the other strata $X_{P_i}(w_i)$ is also isomorphic to a parabolic induction of a classical Deligne–Lusztig variety of Coxeter type for a Levi subgroup of $\mathrm{U}(V)$ ([HLZ19, Proposition 2.5.1]).

2.6. **Minuscule Kudla–Rapoport cycles $\mathcal{V}(\Lambda)$.** Let $\Lambda \subseteq \mathbb{V}$ be a vertex lattice. Then $V_\Lambda \coloneqq \Lambda^\vee/\Lambda$ is a k_F-vector space of dimension $t(\Lambda)$, equipped with a (non-degenerate) k_F/k-hermitian form induced from $\mathbb{V}$. Since $\mathbb{V}$ is a non-split hermitian space, the type $t(\Lambda)$ is odd. Thus we have the associated generalized Deligne–Lusztig variety Y_{V_Λ} of dimension $(t(\Lambda)-1)/2$. The reduced subscheme of the minuscule Kudla–Rapoport cycle $\mathcal{V}(\Lambda) \coloneqq \mathcal{Z}(\Lambda)^{\mathrm{red}}$ is isomorphic to $Y_{V_\Lambda, \bar{k}}$[1]. In fact $\mathcal{Z}(\Lambda)$ itself is already reduced ([LZ17, Theorem B]), so $\mathcal{V}(\Lambda) = \mathcal{Z}(\Lambda)$.

2.7. **The Bruhat–Tits stratification on $\mathcal{N}^{\mathrm{red}}$.** The reduced subscheme of $\mathcal{N}$ satisfies $\mathcal{N}^{\mathrm{red}} = \bigcup_\Lambda \mathcal{V}(\Lambda)$, where Λ runs over all vertex lattices $\Lambda \subseteq \mathbb{V}$. For two vertex lattices Λ, Λ', we have $\mathcal{V}(\Lambda) \subseteq \mathcal{V}(\Lambda')$ if and only if $\Lambda \supseteq \Lambda'$; and $\mathcal{V}(\Lambda) \cap \mathcal{V}(\Lambda')$ is nonempty if and only if $\Lambda + \Lambda'$ is also a vertex lattice, in which case it is equal to $\mathcal{V}(\Lambda + \Lambda')$. In this way we obtain a *Bruhat–Tits stratification* of $\mathcal{N}^{\mathrm{red}}$ by locally closed subvarieties ([VW11, Theorem B]),

$$\mathcal{N}^{\mathrm{red}} = \bigsqcup_{\Lambda} \mathcal{V}(\Lambda)^\circ, \quad \mathcal{V}(\Lambda)^\circ \coloneqq \mathcal{V}(\Lambda) - \bigcup_{\Lambda \subsetneq \Lambda'} \mathcal{V}(\Lambda').$$

Each Bruhat–Tits stratum $\mathcal{V}(\Lambda)^\circ \simeq Y^\circ_{V_\Lambda, \bar{k}}$ is a classical Deligne–Lusztig variety of Coxeter type associated to $\mathrm{U}(V_\Lambda)$, which has dimension $(t(\Lambda)-1)/2$. It follows that the irreducible components of $\mathcal{N}^{\mathrm{red}}$ are exactly the projective varieties $\mathcal{V}(\Lambda)$, where Λ runs over all vertex lattices of maximal

[1] We naturally identify $\mathbb{V}$ with $\mathbf{N}_0$ in [VW11] and C in [KR11] via [KR11, Lemma 3.9]. Notice that $\mathcal{V}(\Lambda)$ in [VW11] and [KR11] is the same as our $\mathcal{V}(\Lambda^\vee)$.

type ([VW11, Corollary C]). The points in the 0-dimensional Bruhat–Tits strata, i.e., $\mathcal{V}(\Lambda)$ for type 1 vertex lattices Λ, are known as *superspecial* points.

For $L \subseteq \mathbb{V}$ be an O_F-lattice of rank $r \geq 1$. By [KR11, Proposition 4.1], the reduced subscheme $\mathcal{Z}(L)^{\mathrm{red}}$ of a Kudla–Rapoport cycle $\mathcal{Z}(L)$ is a union of Bruhat–Tits strata,

$$\mathcal{Z}(L)^{\mathrm{red}} = \bigcup_{L \subseteq \Lambda} \mathcal{V}(\Lambda). \tag{2.7.0.1}$$

When $n \geq 3$, a point $z \in \mathcal{N}(\overline{k})$ is called *super-general* if there is no special homomorphism u of valuation 0 such that $z \in \mathcal{Z}(u)(\overline{k})$. By (2.7.0.1), we know that z is super-general if and only if $z \in \mathcal{V}(\Lambda)^{\circ}$ for Λ a vertex lattice of type n. In particular, there is no super-general point on $\mathcal{N}$ when $n \geq 3$ is even.

2.8. **Independence of the choice of the basis.** We generalize the results of Terstiege [Ter13a, Lemma 3.1, Proposition 3.2].

Lemma 2.8.1. *Let $x, y \in \mathbb{V} = \mathbb{V}_n$ be linearly independent. Then the sheaves $\mathrm{Tor}_i^{\mathcal{O}_{\mathcal{N}_n}}(\mathcal{O}_{\mathcal{Z}(x)}, \mathcal{O}_{\mathcal{Z}(y)})$ vanish for all $i \geq 1$. In particular,*

$$\mathcal{O}_{\mathcal{Z}(x)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(y)} = \mathcal{O}_{\mathcal{Z}(x)} \otimes \mathcal{O}_{\mathcal{Z}(y)}.$$

Proof. The proof is similar to [Ter13a, Lemma 3.1]. Let $z \in \mathcal{N}_n(\overline{k})$ and let $R = \mathcal{O}_{\mathcal{N}_n, z}$ be the local ring at z. Let $f, g \in R$ be the local equations at z of $\mathcal{Z}(x), \mathcal{Z}(y)$ respectively. Then (cf. *loc. cit.*)

$$\mathrm{Tor}_1^{\mathcal{O}_{\mathcal{N}_n}}(\mathcal{O}_{\mathcal{Z}(x)}, \mathcal{O}_{\mathcal{Z}(y)})_z = \ker\left(R/(g) \xrightarrow{\cdot f} R/(g)\right),$$

and $\mathrm{Tor}_i^{\mathcal{O}_{\mathcal{N}_n}}(\mathcal{O}_{\mathcal{Z}(x)}, \mathcal{O}_{\mathcal{Z}(y)})_z = 0$ for $i > 1$. We *claim* that f and g have no common divisor in the regular ring R for every $z \in \mathcal{N}_n(\overline{k})$. The claim implies the desired vanishing of Tor_1.

We prove the claim by induction on n. When $n \leq 3$, this is known by the proof of [Ter13a, Lemma 3.1]. Now assume that $n \geq 4$.

Let $\mathfrak{V}$ be the set of $z \in \mathcal{N}_n(\overline{k})$ where the local equations of $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$ share a common divisor.

First suppose that $\mathfrak{V}$ contains a point z that is not super-general (§2.7). Choose u with valuation 0 such that $z \in \mathcal{Z}(u)(\overline{k})$. We may further assume that u is linearly independent from x and y. In fact, if $u \in \langle x, y\rangle_F$, we may choose a non-zero $u' \in \langle x, y\rangle_F^{\perp}$ such that $z \in \mathcal{Z}(u')(\overline{k})$, and then we replace u by $u + u'$. Denote by $x^{\flat}$ (resp. $y^{\flat}$) the orthogonal projection of x (resp. y) to $\langle u\rangle^{\perp}$. Then $x^{\flat}$ and $y^{\flat}$ remain linearly independent. The restrictions of $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$ to $\mathcal{Z}(u) \simeq \mathcal{N}_{n-1}$ (cf. (2.11.0.2)) are the special divisors $\mathcal{Z}(x^{\flat})$ and $\mathcal{Z}(y^{\flat})$. By our assumption $z \in \mathfrak{V}$, the local equations at $z \in \mathcal{N}_{n-1}(\overline{k})$ of $\mathcal{Z}(x^{\flat})$ and $\mathcal{Z}(y^{\flat})$ share a common divisor. This contradicts the induction hypothesis.

Now suppose that $\mathfrak{V}$ consists of only super-general points. In particular, n is odd. Recall that the difference divisor $\mathcal{D}(y) := \mathcal{Z}(y) - \mathcal{Z}(y/\varpi)$ (as Cartier divisors) is effective and regular by [Ter13b]. We have an equality of Cartier divisors $\mathcal{Z}(y) = \sum_{i \geq 0} \mathcal{D}(y/\varpi^i)$ (this is a locally finite sum). Let $z_0 \in \mathfrak{V}$ and let Λ be the unique vertex lattice of type n such that $z_0 \in \mathcal{V}(\Lambda)(\overline{k})$. Possibly replacing y by y/ϖ^i for some $i \geq 0$, we may assume that locally at z_0 the divisor $\mathcal{D}(y)$ is a component of $\mathcal{Z}(x)$.

By the argument of [KR11, Lemma 3.6], the set of points $z \in \mathcal{D}(y)^{\mathrm{red}}$ where the local equations at z of $\mathcal{Z}(x)$ and $\mathcal{D}(y)$ share a common divisor is open and closed in $\mathcal{D}(y)^{\mathrm{red}}$. In fact we can directly apply *loc. cit* by letting $\mathfrak{X}$ be the formal completion of $\mathcal{N}_n$ along $\mathcal{D}(y)^{\mathrm{red}}$ and noting that the local equation of $\mathcal{D}(y)$ is given by an irreducible element. It follows that there exists an irreducible component of the scheme $\mathcal{D}(y) \cap \mathcal{V}(\Lambda)$, denoted by $\mathfrak{D}_0$, passing through z_0. Then $\mathfrak{D}_0(\bar{k}) \subset \mathfrak{V}$, and $\mathfrak{D}_0$ is closed subscheme of $\mathcal{V}(\Lambda)$. By our assumption on $\mathfrak{V}$, we have $\mathfrak{D}_0 \subset \mathcal{V}(\Lambda)^\circ$. However, by [Lus76, Corollary 2.8], the open variety $\mathcal{V}(\Lambda)^\circ$ is affine with $\dim \mathcal{V}(\Lambda) = \frac{n-1}{2} \geq 2$ and hence can not have any positive dimensional projective irreducible subscheme (such as $\mathfrak{D}_0$). Contradiction! This completes the induction. $\square$

Corollary 2.8.2. *Let $L \subseteq \mathbb{V}$ be an O_F-lattice of rank $r \geq 1$. Let $x_1, \ldots, x_r$ be an O_F-basis of L. Then $\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_r)} \in K_0^{\mathcal{Z}(L)}(\mathcal{N})$ is independent of the choice of the basis.*

Proof. This is similar to [Ter13a, Proposition 3.2]. We can transform a basis into any other basis by a suitable sequence of the following operations: permutations; the multiplication on a basis vector by a unit in $O_F^\times$; for every pair $(i,j), i \neq j$, the substitution of x_i by $x_i + \alpha x_j$ for $\alpha \in O_F$. $\square$

2.9. **Horizontal and vertical parts of $\mathcal{Z}(L)$.**

Definition 2.9.1. A formal scheme Z over $\operatorname{Spf} O_{\breve F}$ is called *vertical* (resp. *horizontal*) if ϖ is locally nilpotent on Z (resp. flat over $\operatorname{Spf} O_{\breve F}$). Clearly the formal scheme-theoretic union of two vertical (resp. horizontal) formal subschemes of a formal scheme is also vertical (resp. horizontal).

We define the *horizontal part* $Z_{\mathscr{H}} \subseteq Z$ to be the closed formal subscheme defined by the ideal sheaf $\mathcal{O}_Z[\varpi^\infty] \subseteq \mathcal{O}_Z$. Then $Z_{\mathscr{H}}$ is the maximal horizontal closed formal subscheme of Z.

When Z is noetherian, there exists $N \gg 0$ such that $\varpi^N \mathcal{O}_Z[\varpi^\infty] = 0$, and we define the *vertical part* $Z_{\mathscr{V}} \subseteq Z$ to be the closed formal subscheme defined by the ideal sheaf $\varpi^N \mathcal{O}_Z$. Since $\mathcal{O}_Z[\varpi^\infty] \cap \varpi^N \mathcal{O}_Z = 0$, we have a decomposition

$$Z = Z_{\mathscr{H}} \cup Z_{\mathscr{V}},$$

as a union of horizontal and vertical formal subschemes. Notice that the horizontal part $Z_{\mathscr{H}}$ is canonically defined, while the vertical part $Z_{\mathscr{V}}$ depends on the choice of N.

Lemma 2.9.2. *Let $L \subseteq \mathbb{V}$ be a O_F-lattice of rank $r \geq n-1$ such that L_F is non-degenerate. Then $\mathcal{Z}(L)$ is noetherian.*

Proof. As a closed formal subscheme of the locally noetherian formal scheme $\mathcal{N}$, we know that $\mathcal{Z}(L)$ is locally noetherian. Since L has rank $r \geq n-1$, the number of vertex lattices $\Lambda \subseteq \mathbb{V}$ such that $L \subseteq \Lambda$ is finite. In fact, when $r = n$ we have $L \subseteq \Lambda \subseteq L^\vee$; when $r = n-1$ we have $L_1 \subseteq \Lambda \subseteq L_1^\vee$, where $L_1 = L \oplus \langle x \rangle$ and $\langle x \rangle \subseteq L_F^\perp$ is an O_F-lattice with $\mathrm{val}(x) \gg 0$ (depending only on L). By (2.7.0.1), we know that $\mathcal{Z}(L)^{\mathrm{red}}$ is a closed subset in finitely many irreducible components of $\mathcal{N}^{\mathrm{red}}$. Since each irreducible component of $\mathcal{N}^{\mathrm{red}}$ is quasi-compact, we know that $\mathcal{Z}(L)$ is quasi-compact, hence noetherian. $\square$

By Lemma 2.9.2, for $L \subseteq \mathbb{V}$ an O_F-lattice of rank $r \geq n-1$, we obtain a decomposition of the Kudla–Rapoport cycle into *horizontal* and *vertical* parts

$$\mathcal{Z}(L) = \mathcal{Z}(L)_{\mathscr{H}} \cup \mathcal{Z}(L)_{\mathscr{V}}.$$

Again notice that the vertical part $\mathcal{Z}(L)_{\mathscr{V}}$ depends on the choice of an integer $N \gg 0$. Since the choice of N is not important for our purpose we suppress it from the notation (cf. §5.2).

2.10. **Finiteness of** $\mathrm{Int}(L)$**.** The following result should be well-known to the experts.

Lemma 2.10.1. *Let $L \subseteq \mathbb{V}$ be an O_F-lattice of rank n. Then the formal scheme $\mathcal{Z}(L)$ is a proper scheme over* $\mathrm{Spf}\, O_{\breve{F}}$*. In particular,* $\mathrm{Int}(L)$ *is finite.*

Proof. The vertical part $\mathcal{Z}(L)_{\mathscr{V}}$ is a scheme by Lemma 5.1.1 below. We show that the horizontal part $\mathcal{Z}(L)_{\mathscr{H}}$ is empty. If not, there exists $z \in \mathcal{Z}(L)(O_K)$ for some finite extension K of $\breve{F}$. Let $\mathcal{X}$ be the corresponding O_F-hermitian module of signature $(1, n-1)$ over O_K. Since L has rank n, we know that $\mathcal{X}$ admits n linearly independent special homomorphisms $\tilde{x}_i : \bar{\mathcal{E}} \to \mathcal{X}$, which gives rise to an O_F-linear isogeny

$$(\tilde{x}_1, \ldots, \tilde{x}_n) : \bar{\mathcal{E}}^n \to \mathcal{X}.$$

It then follows that the O_F-action on $\mathcal{X}$ satisfies the Kottwitz signature condition $(0, n)$ rather than $(1, n-1)$ in characteristic 0, a contradiction. Thus $\mathcal{Z}(L)_{\mathscr{H}}$ is empty, and so $\mathcal{Z}(L)$ is a scheme. Since $\mathcal{Z}(L)^{\mathrm{red}}$ is contained in finitely many irreducible components of $\mathcal{N}^{\mathrm{red}}$ and each irreducible component of $\mathcal{N}^{\mathrm{red}}$ is proper over $\mathrm{Spec}\, \bar{k}$, it follows that the scheme $\mathcal{Z}(L)$ is proper over $\mathrm{Spf}\, O_{\breve{F}}$. The finiteness of $\mathrm{Int}(L)$ then follows from the discussion before [Zha21, (B.4)]. □

2.11. **A cancellation law for** $\mathrm{Int}(L)$**.** Let $M \subset \mathbb{V}_n$ be a self-dual lattice of rank r. We may choose the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of $\mathcal{N}_{n-r}$, the framing object $(\mathbb{X} \times \bar{\mathbb{E}}^r, \iota_{\mathbb{X}} \times \iota_{\bar{\mathbb{E}}^r}, \lambda_{\mathbb{X}} \times \lambda_{\bar{\mathbb{E}}^r})$ of $\mathcal{N}_n$ and $g \in \mathrm{U}(\mathbb{V}_n)(F_0)$ such that the self-dual lattice $\mathrm{Hom}_{O_F}(\bar{\mathbb{E}}, \bar{\mathbb{E}}^r) \subseteq \mathrm{Hom}^\circ_{O_F}(\bar{\mathbb{E}}, \mathbb{X} \times \bar{\mathbb{E}}^r)$ is identified with the self-dual lattice $gM \subseteq \mathbb{V}_n$. Recall that $\mathrm{U}(\mathbb{V}_n)(F_0)$ naturally acts on $\mathcal{N}_n$ (§2.2). The map $(X, \iota, \lambda, \rho) \mapsto g^{-1}(X \times \bar{\mathcal{E}}^r, \iota \times \iota_{\bar{\mathcal{E}}^r}, \lambda \times \lambda_{\bar{\mathcal{E}}^r}, \rho \times \rho_{\bar{\mathcal{E}}^r})$ gives a closed embedding

$$\delta_M\colon\ \mathcal{N}_{n-r} \longrightarrow \mathcal{N}_n\ , \tag{2.11.0.1}$$

which identifies $\mathcal{N}_{n-r}$ with the special cycle $\mathcal{Z}(M)$ ([RSZ18, Remark 4.5]). Let $\mathbb{V}_n = \mathbb{V}_{n-r} \obot M_F$ be the induced orthogonal decomposition. For $u \in \mathbb{V}_n$, denote by $u^\flat$ the projection to $\mathbb{V}_{n-r}$. If $u^\flat \neq 0$, then the special divisor $\mathcal{Z}(u)$ intersects transversely with $\mathcal{N}_{n-r}$ and its pull-back to $\mathcal{N}_{n-r}$ is the special divisor $\mathcal{Z}(u^\flat)$. By [Zha21, Lemma B.2 (i)], we obtain

$$\mathcal{N}_{n-r} \overset{\mathbb{L}}{\cap} \mathcal{Z}(u) = \mathcal{Z}(u^\flat). \tag{2.11.0.2}$$

Lemma 2.11.1. *Let $M \subset \mathbb{V}_n$ be a self-dual lattice of rank r and $L^\flat$ an integral lattice in $\mathbb{V}_{n-r}$. Then*

$$\mathrm{Int}(L^\flat \obot M) = \mathrm{Int}(L^\flat).$$

Proof. This follows from the equation (2.11.0.2) and the definition of Int by (2.4.2.1). □

3. Local densities

In this section (except §3.4) we allow F_0 to be a non-archimedean local field of characteristic not equal to 2 (but possibly with residue characteristic 2), and F an unramified quadratic extension.

3.1. **Local densities for hermitian lattices.** Let L, M be two hermitian O_F-lattices. Let $\mathrm{Rep}_{M,L}$ be the *scheme of integral representations of M by L*, an O_{F_0}-scheme such that for any O_{F_0}-algebra R,

$$\mathrm{Rep}_{M,L}(R) = \mathrm{Herm}(L \otimes_{O_{F_0}} R, M \otimes_{O_{F_0}} R), \tag{3.1.0.1}$$

where Herm denotes the set of hermitian module homomorphisms. The *local density* of integral representations of M by L is defined to be

$$\mathrm{Den}(M, L) := \lim_{N \to +\infty} \frac{\#\mathrm{Rep}_{M,L}(O_{F_0}/\varpi^N)}{q^{N \cdot \dim(\mathrm{Rep}_{M,L})_{F_0}}}.$$

Note that if L, M have rank n, m respectively and the generic fiber $(\mathrm{Rep}_{M,L})_{F_0} \neq \varnothing$, then $n \leq m$ and

$$\dim(\mathrm{Rep}_{M,L})_{F_0} = \dim \mathrm{U}_m - \dim \mathrm{U}_{m-n} = n \cdot (2m - n). \tag{3.1.0.2}$$

3.2. **Local Siegel series for hermitian lattices.** Let $k \geq 0$ be an integer. Let $\langle 1 \rangle^k$ be the self-dual hermitian O_F-lattice of rank k with hermitian form given the identity matrix $\mathbf{1}_k$. Let L be a hermitian O_F-lattice of rank n. By [Hir98, Theorem II], $\mathrm{Den}(\langle 1 \rangle^{n+k}, L)$ is a polynomial in $(-q)^{-k}$ with $\mathbb{Q}$-coefficients (zero if L is not integral). A special case (see [KR11, p.677]) is

$$\mathrm{Den}(\langle 1 \rangle^{n+k}, \langle 1 \rangle^n) = \prod_{i=1}^{n} (1 - (-q)^{-i} X) \Big|_{X=(-q)^{-k}}. \tag{3.2.0.3}$$

Define the (normalized) *local Siegel series* of L to be the polynomial $\mathrm{Den}(X, L) \in \mathbb{Z}[X]$ (Theorem 3.5.1) such that

$$\mathrm{Den}((-q)^{-k}, L) = \frac{\mathrm{Den}(\langle 1 \rangle^{n+k}, L)}{\mathrm{Den}(\langle 1 \rangle^{n+k}, \langle 1 \rangle^n)}.$$

The local Siegel series satisfies a functional equation ([Hir12, Theorem 5.3])

$$\mathrm{Den}(X, L) = (-X)^{\mathrm{val}(L)} \cdot \mathrm{Den}\left(\frac{1}{X}, L\right). \tag{3.2.0.4}$$

Definition 3.2.1. Define the central value of the local density to be

$$\mathrm{Den}(L) := \mathrm{Den}(1, L).$$

In particular, if $\mathrm{val}(L)$ is odd, then $\mathrm{Den}(L) = 0$. In this case, define the *central derivative of the local density* or *derived local density* by

$$\partial\mathrm{Den}(L) := -\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=1} \mathrm{Den}(X, L).$$

Notice that if L is not integral then $\mathrm{Den}(X, L) = 0$ and hence $\partial\mathrm{Den}(L) = 0$.

Notice that by definition $\mathrm{Den}(M,L)$ only depends on the isometry classes of M and L, and hence only depends on the fundamental invariants of M and L. In particular, $\mathrm{Den}(X,L)$ and $\partial\mathrm{Den}(L)$ only depends on the fundamental invariants of L. Moreover, there is an analog of Lemma 2.11.1: for any self-dual lattice M of $\mathrm{rank}(M)=m$ and any integral lattice $L^\flat$ of $\mathrm{rank}(L^\flat)=n$, we have

$$\mathrm{Den}(X,L^\flat \obot M)=\mathrm{Den}(X,L^\flat)$$

and therefore we obtain a cancellation law:

$$\partial\mathrm{Den}(L^\flat \obot M)=\partial\mathrm{Den}(L^\flat). \tag{3.2.1.1}$$

3.3. **Relation with local Whittaker functions.** Let $\Lambda=\langle 1\rangle^n$ be an self-dual hermitian O_F-lattice. Let L be a hermitian O_F-lattice of rank n. Let $T=((x_i,x_j))_{1\le i,j\le n}$ be the fundamental matrix of an O_F-basis $\{x_1,\ldots,x_n\}$ of L, an $n\times n$ hermitian matrix over F. Associated to the standard Siegel–Weil section of the characteristic function $\varphi_0=\mathbf{1}_{\Lambda^n}$ and the unramified additive character $\psi: F_0\to\mathbb{C}^\times$, there is a local (generalized) Whittaker function $W_T(g,s,\varphi_0)$ (see §12.2, §12.3 for the precise definition). By [KR14, Proposition 10.1], when $g=1$, it satisfies the interpolation formula for integers $s=k\ge 0$ (notice $\gamma_p(V)=1$ in the notation there),

$$W_T(1,k,\varphi_0)=\mathrm{Den}(\langle 1\rangle^{n+2k},L).$$

So its value at $s=0$ is

$$W_T(1,0,\varphi_0)=\mathrm{Den}(\langle 1\rangle^{n},L)=\mathrm{Den}(L)\cdot\mathrm{Den}(\langle 1\rangle^{n},\langle 1\rangle^{n}),$$

and its derivative at $s=0$ is[2]

$$W_T'(1,0,\varphi_0)=\partial\mathrm{Den}(L)\cdot\mathrm{Den}(\langle 1\rangle^{n},\langle 1\rangle^{n})\cdot\log q^2.$$

Plugging in (3.2.0.3), we obtain

$$W_T(1,0,\varphi_0)=\mathrm{Den}(L)\cdot\prod_{i=1}^n(1-(-q)^{-i}), \tag{3.3.0.2}$$

$$W_T'(1,0,\varphi_0)=\partial\mathrm{Den}(L)\cdot\prod_{i=1}^n(1-(-q)^{-i})\cdot\log q^2. \tag{3.3.0.3}$$

3.4. **The local Kudla–Rapoport conjecture.** Now we can state the main theorem of this article, which proves the Kudla–Rapoport conjecture on the identity between arithmetic intersection numbers of Kudla–Rapoport cycles and central derivatives of local densities. Recall that $\mathbb{V}=\mathbb{V}_n$ is the hermitian space defined in §2.2.

Theorem 3.4.1 (local Kudla–Rapoport conjecture)**.** *Let $L\subseteq\mathbb{V}$ be an O_F-lattice of full rank n. Then*

$$\mathrm{Int}(L)=\partial\mathrm{Den}(L).$$

This will be proved in §8.2.

[2]In [KR14, Proposition 9.3], the factor $\log p$ should be $\log p^2$.

Remark 3.4.2. In the notation of §3.3, it follows immediately from Theorem 3.4.1 and (3.3.0.3) that

$$\mathrm{Int}(L) = \frac{W_T'(1,0,\varphi_0)}{\log q^2} \cdot \prod_{i=1}^{n} (1-(-q)^{-i})^{-1}.$$

3.5. **Formulas in terms of weighted lattice counting: Theorem of Cho–Yamauchi.** Define weight factors

$$\mathfrak{m}(a;X) \coloneqq \prod_{i=0}^{a-1}(1-(-q)^i X), \quad \mathfrak{m}(a) \coloneqq -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathfrak{m}(a;X) = \prod_{i=1}^{a-1}(1-(-q)^i),$$

where by convention $\mathfrak{m}(0;X)=1$ and $\mathfrak{m}(0)=0$, $\mathfrak{m}(1)=1$. Then we have the following explicit formula for the local Siegel series.

Theorem 3.5.1 (Cho–Yamauchi)**.** *The following identity holds:*

$$\mathrm{Den}(X,L) = \sum_{L\subset L'\subset L'^\vee} X^{2\ell(L'/L)} \cdot \mathfrak{m}(t(L');X),$$

where the sum runs over all integral lattices $L' \supset L$. Here

$$\ell(L'/L) \coloneqq \mathrm{length}_{O_F} L'/L.$$

Proof. This is proved the same way as in the orthogonal case [CY20, Corollary 3.11], using the following hermitian analogue of [Kit93, §5.6 Exercise 4]. Let U be an $\mathbb{F}_{q^2}/\mathbb{F}_q$-hermitian space of dimension n whose radical has dimension a. Let V be a (non-degenerate) $\mathbb{F}_{q^2}/\mathbb{F}_q$-hermitian space of dimension $m \geq n$. Then the number of isometries from U to V is equal to

$$q^{n(2m-n)} \cdot \prod_{i=0}^{n+a-1}(1-(-q)^{i-m}).$$

Writing $m = n+k$, this is equal to

$$q^{n(2m-n)} \cdot \mathrm{Den}(\langle 1\rangle^{n+k}, \langle 1\rangle^n) \cdot \mathfrak{m}(a;(-q)^{-k}),$$

which explains the correct weight factor $\mathfrak{m}(a;X)$ appearing in the theorem.

We remark that since F/F_0 is unramified, the analogue of the smoothness theorem [CY20, Theorem 3.9] is valid in the hermitian case even when the residue characteristic is $p=2$, as [GY00, Lemma 5.5.2] is still valid for $p=2$ by [GY00, §9]. □

Example 3.5.2 (The case $\mathrm{rank}\, L = 1$)**.** If $\mathrm{rank}\, L = 1$, the formula specializes to

$$\mathrm{Den}(X,L) = \sum_{i=0}^{\mathrm{val}(L)} (-X)^i.$$

In particular, if $\mathrm{val}(L^\flat)$ is odd, we obtain $\mathrm{Den}(L)=0$ and

$$\partial\mathrm{Den}(L) = \frac{\mathrm{val}(L)+1}{2}.$$

Also note that if $L' \supset L$, then $\mathrm{val}(L')$ and $\mathrm{val}(L)$ have the same parity (see §1.7). In particular, if $\mathrm{val}(L)$ is odd, then $t(L')>0$ and hence $\mathfrak{m}(t(L');1)=0$. Thus we obtain the following explicit formula for $\partial\mathrm{Den}(L)$.

Corollary 3.5.3. *If* $\mathrm{val}(L)$ *is odd, then*

$$\partial\mathrm{Den}(L) = \sum_{L \subset L' \subset L'^\vee} \mathfrak{m}(t(L')).$$

3.6. **Some special cases.** Since $\mathfrak{m}(a; (-q)^{-k}) = 0$ if $0 \le k \le (a-1)$, we also obtain

Corollary 3.6.1. *For* $k \ge 0$*,*

$$\mathrm{Den}((-q)^{-k}, L) = \sum_{\substack{L \subset L' \subset L'^\vee \\ t(L') \le k}} q^{-2\ell(L'/L)k} \cdot \mathfrak{m}(t(L'); (-q)^{-k})$$

In particular, for $k = 0$*,*

$$\mathrm{Den}(L) = \mathrm{Den}(1, L) = \sum_{\substack{L \subset L' \subset L'^\vee \\ t(L') = 0}} 1 = \#\{L' \text{ self-dual} : L \subseteq L'\}. \tag{3.6.1.1}$$

For $k = 1$*,*

$$\frac{1}{\mathrm{vol}(L)}\mathrm{Den}((-q)^{-1}, L) = \sum_{\substack{L \subset L' \subset L'^\vee \\ t(L') = 0}} 1 + \sum_{\substack{L \subset L' \subset L'^\vee \\ t(L') = 1}} (1 + q^{-1})\frac{1}{\mathrm{vol}(L')}. \tag{3.6.1.2}$$

Corollary 3.6.2. *The following identities hold:*

$$\mathrm{Den}(-q, L) = \sum_{L \subset L' \subset L'^\vee} [L' : L] \cdot \mathfrak{m}(t(L') + 1), \tag{3.6.2.1}$$

and

$$\mathrm{Den}(-q, L) = \frac{1}{\mathrm{vol}(L)}\mathrm{Den}((-q)^{-1}, L). \tag{3.6.2.2}$$

Proof. The first part follows from Theorem 3.5.1 and the fact that

$$\mathfrak{m}(t(L'); -q) = \mathfrak{m}(t(L') + 1).$$

The second part follows from the functional equation (3.2.0.4). ☐

3.7. **An induction formula.**

Proposition 3.7.1. *Let* $L^\flat$ *be a hermitian* O_F*-lattice of rank* $n-1$ *with fundamental invariants* $(a_1, \cdots, a_{n-1})$*. Let* $L = L^\flat + \langle x \rangle$ *and* $L' = L^\flat + \langle \varpi^{-1}x \rangle$ *where* $x \perp L^\flat$ *with* $\mathrm{val}(x) > a_{n-1}$*. Then*

$$\mathrm{Den}(X, L) = X^2\mathrm{Den}(X, L') + (1 - X)\mathrm{Den}(-qX, L^\flat).$$

This is [Ter13a, Theorem 5.1] in the hermitian case, and Katsurada [Kat99, Theorem 2.6 (1)] in the orthogonal case (see also [CY20]).

4. Horizontal parts of Kudla–Rapoport cycles

4.1. **Quasi-canonical lifting cycles.** Let $\langle y\rangle \subseteq \mathbb{V}_2$ be a rank one O_F-lattice. By [KR11, Proposition 8.1], we have a decomposition as Cartier divisors on $\mathcal{N}_2$,

$$\mathcal{Z}(y) = \sum_{i=0}^{\lfloor \mathrm{val}(y)/2\rfloor} \mathcal{Z}_{\mathrm{val}(y)-2i}.$$

Here $\mathcal{Z}_s$ ($s \geq 0$) is the *quasi-canonical lifting cycle* of level s on $\mathcal{N}_2$, the horizontal divisor corresponding to the *quasi-canonical lifting of level* s of the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of $\mathcal{N}_2$ (the quasi-canonical lifting of level $s = 0$ is the *canonical lifting*). We define the *primitive part* of $\mathcal{Z}(y)$ to be

$$\mathcal{Z}(y)^\circ \coloneqq \mathcal{Z}_{\mathrm{val}(y)} \subseteq \mathcal{Z}(y).$$

Let $O_{F,s} = O_{F_0} + \varpi^s O_F \subseteq O_F$. Let $\breve{F}_s$ be the finite abelian extension of $\breve{F}$ corresponding to the subgroup $O_{F,s}^\times$ under local class field theory. Let $O_{\breve{F},s}$ be the ring of integers of the ring class field $\breve{F}_s$. Then $O_{\breve{F},0} = O_{\breve{F}}$, and the degree of $O_{\breve{F},s}$ over $O_{\breve{F}}$ is equal to $q^s(1+q^{-1})$ when $s \geq 1$. We have

$$\mathcal{Z}_s \cong \operatorname{Spf} O_{\breve{F},s}.$$

4.2. **Horizontal cycles.** Let $M^\flat \subseteq \mathbb{V}_n$ be an integral hermitian O_F-lattice of rank $n-1$. When $t(M^\flat) \leq 1$, we may find a rank $n-2$ self-dual O_F-lattice M_{n-2}, and a rank one O_F-lattice $\langle y\rangle$, such that we have an orthogonal direct sum decomposition

$$M^\flat = M_{n-2} \oplus \langle y\rangle.$$

Let $M_{n-2,F}^\perp \subseteq \mathbb{V}_n$ be the orthogonal complement of $M_{n-2,F}$ in $\mathbb{V}_n$. Then we have an isomorphism $M_{n-2,F}^\perp \simeq \mathbb{V}_2$, and thus an isomorphism (see §2.11)

$$\mathcal{Z}(M_{n-2}) \simeq \mathcal{N}_2.$$

Under this isomorphism, we can identify the Cartier divisor $\mathcal{Z}(M^\flat) \subseteq \mathcal{Z}(M_{n-2})$ with the Cartier divisor $\mathcal{Z}(y) \subseteq \mathcal{N}_2$.

We define the *primitive part* $\mathcal{Z}(M^\flat)^\circ \subseteq \mathcal{Z}(M^\flat)$ to be the primitive part $\mathcal{Z}(y)^\circ \subseteq \mathcal{Z}(y)$ under the above identification. Since $\mathrm{val}(y) = \mathrm{val}(M^\flat)$, we have a decomposition as Cartier divisors on $\mathcal{Z}(M_{n-2})$,

$$\mathcal{Z}(M^\flat) \simeq \sum_{i=0}^{\lfloor \mathrm{val}(M^\flat)/2\rfloor} \mathcal{Z}_{\mathrm{val}(M^\flat)-2i}, \tag{4.2.0.1}$$

and we can characterize $\mathcal{Z}(M^\flat)^\circ$ as the unique component of $\mathcal{Z}(M^\flat)$ isomorphic to $\mathcal{Z}_{\mathrm{val}(M^\flat)}$ (the component of the maximal degree). In particular, $\mathcal{Z}(M)^\circ$ is independent of the choice of the self-dual lattice M_{n-2} and we have

$$\deg_{O_{\breve{F}}}(\mathcal{Z}(M^\flat)^\circ) = \begin{cases} 1, & t(M^\flat) = 0,\\ \mathrm{vol}(M^\flat)^{-1}(1+q^{-1}), & t(M^\flat) = 1.\end{cases} \tag{4.2.0.2}$$

Theorem 4.2.1. *Let $L^\flat \subseteq \mathbb{V}_n$ be a hermitian O_F-lattice of rank $n-1$. Then*

$$\mathcal{Z}(L^\flat)_{\mathscr{H}} = \bigcup_{\substack{L^\flat \subseteq M^\flat \subseteq (M^\flat)^\vee \\ t(M^\flat)\le 1}} \mathcal{Z}(M^\flat)^\circ. \tag{4.2.1.1}$$

Moreover, the identity

$$\mathcal{O}_{\mathcal{Z}(L^\flat)_{\mathscr{H}}} = \sum_{\substack{L^\flat \subseteq M^\flat \subseteq (M^\flat)^\vee \\ t(M^\flat)\le 1}} \mathcal{O}_{\mathcal{Z}(M^\flat)^\circ} \tag{4.2.1.2}$$

holds in $\mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)_{\mathscr{H}}}(\mathcal{N}_n)$.

Lemma 4.2.2. *The primitive cycles $\mathcal{Z}(M^\flat)^\circ$ on the right-hand-side of (4.2.1.1) are all distinct.*

Proof. If not, suppose $\mathcal{Z}(M_1^\flat)^\circ = \mathcal{Z}(M_2^\flat)^\circ$. Let $M^\flat = M_1^\flat + M_2^\flat$, which also has type $t(M^\flat) \le 1$. Then by definition we have $\mathcal{Z}(M^\flat) = \mathcal{Z}(M_1^\flat) \cap \mathcal{Z}(M_2^\flat)$. By the assumption $\mathcal{Z}(M_1^\flat)^\circ = \mathcal{Z}(M_2^\flat)^\circ$ we know that $\mathcal{Z}(M_i^\flat)^\circ \subseteq \mathcal{Z}(M^\flat)$ (for $i=1,2$). So by (4.2.0.1) the inclusion $\mathcal{Z}(M_i^\flat)^\circ \simeq \mathcal{Z}_{\mathrm{val}(M_i^\flat)} \subseteq \mathcal{Z}(M^\flat)$ implies that $\mathrm{val}(M_i^\flat) \le \mathrm{val}(M^\flat)$. But $M^\flat \supseteq M_i^\flat$, it follows that $M^\flat = M_i^\flat$, and so $M_1^\flat = M_2^\flat$, a contradiction. □

Theorem 4.2.1 will be proved in §4.5. By Lemma 4.2.2, we know that (4.2.1.1) implies (4.2.1.2). It is clear from construction that in (4.2.1.1) the right-hand-side is contained in the left-hand-side. To show the reverse inclusion, we will use the Breuil modules and Tate modules, to be explained in next two sections.

4.3. **Breuil modules.** First let us review the (absolute) Breuil modules ([Bre00], [Kis06, Appendix], [BC09, §12.2]). Let $W = W(\bar{k})$ be the ring of Witt vectors. Let O_K be a totally ramified extension of W of degree e defined by an Eisenstein polynomial $E(u) \in W[u]$ (i.e., $W[u]/(E(u)) \cong O_K$). Let S be Breuil's ring, the p-adic completion of $W[u][\frac{E(u)^i}{i!}]_{i\ge 1}$ (the divided power envelope of $W[u]$ with respect to the ideal $(E(u))$. The ring S is local and W-flat, and $S/uS \cong W$. Let $\mathrm{Fil}^1 S \subseteq S$ be the ideal generated by all $\frac{E(u)^i}{i!}$. Then $S/\mathrm{Fil}^1 S \cong O_K$. For a p-divisible group G over O_K, we have its Breuil module $\mathscr{M}(G) = \mathbb{D}(G)(S)$, where $\mathbb{D}(G)$ is the (covariant) Dieudonné crystal of G. It is a finite free S-module together with an S-submodule $\mathrm{Fil}^1\mathscr{M}(G)$, and a ϕ_S-linear homomorphism $\phi_{\mathscr{M}} : \mathrm{Fil}^1\mathscr{M}(G) \to G$ satisfying certain conditions. By Breuil's theorem, the functor $G \mapsto \mathscr{M}(G) = \mathbb{D}(G)(S)$ is an equivalence of categories between p-divisible groups over O_K and Breuil modules ([Kis06, Proposition A.6]). The classical Dieudonné module $M(G_{\bar{k}})$ of the special fiber $G_{\bar{k}}$ is given by $\mathbb{D}(G_{\bar{k}})(W) = \mathbb{D}(G)(S) \otimes_S W = \mathscr{M}(G)/u\mathscr{M}(G)$, with Hodge filtration $\mathrm{Fil}^1 M(G_{\bar{k}})$ equal to the image of $\mathrm{Fil}^1\mathscr{M}(G)$. We also have $\mathbb{D}(G)(O_K) = \mathbb{D}(G)(S) \otimes_S O_K = \mathscr{M}(G) \otimes_S O_K$.

For ϖ-divisible O_{F_0}-modules, one has an analogous theory of relative Breuil modules (see [Hen16]) by replacing $W = W(\bar{k})$ with $O_{\breve{F}} = W_{O_{F_0}}(\bar{k})$, and by defining S to be the ϖ-adic completion of the O_{F_0}-divided power envelope (in the sense of [Fal02]) of $O_{\breve{F}}[u]$ with respect to the ideal $(E(u))$.

4.4. **Tate modules.** Let K be a finite extension of $\breve{F}$. Let $z \in \mathcal{N}_n(O_K)$ and let G be the corresponding O_F-hermitian module of signature $(1, n-1)$ over O_K. Let

$$L \coloneqq \mathrm{Hom}_{O_F}(T_p\bar{\mathcal{E}}, T_pG),$$

where $T_p(-)$ denotes the integral p-adic Tate modules. Since the polarizations λ_G and $\lambda_{\bar{E}}$ are principal, we know that L is a *self-dual* O_F-hermitian lattice of rank n under the hermitian form $\{x, y\} \in O_F$ given by

$$(T_p\bar{\mathcal{E}} \xrightarrow{x} T_pG \xrightarrow{\lambda_G} T_pG^\vee \xrightarrow{y^\vee} T_p\bar{\mathcal{E}}^\vee \xrightarrow{\lambda_{\bar{\mathcal{E}}}^\vee} T_p\bar{\mathcal{E}}) \in \mathrm{End}_{O_F}(T_p\bar{\mathcal{E}}) \cong O_F.$$

There are two injective O_F-linear homomorphisms (preserving their hermitian forms)

$$\begin{tikzcd} & \mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G) \arrow[dl, "i_K"'] \arrow[dr, "i_{\overline{k}}"] & \\ L = \mathrm{Hom}_{O_F}(T_p\bar{\mathcal{E}}, T_pG) & & \mathbb{V}_n = \mathrm{Hom}^\circ_{O_F}(\bar{\mathbb{E}}, \mathbb{X}_n), \end{tikzcd}$$

where the right map $i_{\overline{k}}$ is induced by the reduction to $\mathrm{Spec}\,\bar{k}$ and the framings $\rho_{\bar{\mathcal{E}}}$ and $\rho_z : G_{\overline{k}} \to \mathbb{X}_n$ corresponding to $\overline{\mathcal{E}}$ and $z \in \mathcal{N}_n(O_K)$ respectively. These extend to F-linear homomorphisms (still denoted by the same notation)

(4.4.0.1)
$$\begin{tikzcd} & \mathrm{Hom}^\circ_{O_F}(\overline{\mathcal{E}}, G) \arrow[dl, "i_K"'] \arrow[dr, "i_{\overline{k}}"] & \\ L_F & & \mathbb{V}_n. \end{tikzcd}$$

Lemma 4.4.1. *The following identity holds:*

(4.4.1.1)
$$\mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G) = i_K^{-1}(L),$$

Proof. We may identify $\mathrm{Hom}^\circ_{O_F}(\overline{\mathcal{E}}, G)$ as subspaces of the bottom two vector spaces. So

$$i_K^{-1}(L) \cong L \cap \mathrm{Hom}^\circ_{O_F}(\overline{\mathcal{E}}, G)$$

where the intersection is taken inside the F-vector space L_F. By [Tat67, Theorem 4, Corollary 1], i_K induces an isomorphism

$$\mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G) \cong \mathrm{Hom}_{O_F[\Gamma_K]}(T_p\overline{\mathcal{E}},, T_pG),$$

where $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$, and so an isomorphism

$$\mathrm{Hom}^\circ_{O_F}(\overline{\mathcal{E}}, G) \cong \mathrm{Hom}_{O_F[\Gamma_K]}(V_p\overline{\mathcal{E}},, V_pG),$$

where $V_p(-)$ denotes the rational p-adic Tate module. Thus we obtain

$$\begin{aligned} L \cap \mathrm{Hom}^\circ_{O_F}(\overline{\mathcal{E}}, G) &\cong \mathrm{Hom}_{O_F}(T_p\overline{\mathcal{E}}, T_pG) \cap \mathrm{Hom}_{O_F[\Gamma_K]}(V_p\overline{\mathcal{E}}, V_pG) \\ &= \mathrm{Hom}_{O_F[\Gamma_K]}(T_p\overline{\mathcal{E}},, T_pG) \\ &\cong \mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G), \end{aligned}$$

which proves the result. □

Let $M \subseteq \mathbb{V}_n$ be an O_F-lattice (of arbitrary rank). By definition we have $z \in \mathcal{Z}(M)(O_K)$ if and only if $M \subseteq i_{\bar{k}}(\mathrm{Hom}_{O_F}(\overline{\mathcal{E}}, G))$. It follows from Lemma 4.4.1 that $z \in \mathcal{Z}(M)(O_K)$ if and only if

(4.4.1.2)
$$M \subseteq i_{\overline{k}}(i_K^{-1}(L)).$$

4.5. **Proof of Theorem 4.2.1.** Let $z \in \mathcal{Z}(L^\flat)(O_K)$ and let G be the corresponding O_F-hermitian module of signature $(1, n-1)$ over O_K. By (4.4.1.2), we know that

$$L^\flat \subseteq i_{\bar{k}}(i_K^{-1}(L)).$$

Define $M^\flat := L_F^\flat \cap i_{\bar{k}}(i_K^{-1}(L))$. By (4.4.1.2) again, we obtain that $z \in \mathcal{Z}(M^\flat)(O_K)$. Moreover, the diagram (4.4.0.1) induces an isomorphism

$$M^\flat \xrightarrow{\sim} L \cap i_K(i_{\bar{k}}^{-1}(L_F^\flat)).$$

Set $\mathbb{W} = i_K(i_{\bar{k}}^{-1}(L_F^\flat))$. Then it has the same dimension as $L_F^\flat$.

Lemma 4.5.1. *Assume that L is a self-dual O_F-hermitian lattice of rank n and $\mathbb{W} \subset L_F$ is a vector subspace of dimension $n-1$. Let $M^\flat := \mathbb{W} \cap L$. Then $t(M^\flat) \leq 1$.*

Proof. Since $M^\flat = \mathbb{W} \cap L$, we know that $L/M^\flat$ is a free O_F-module of rank one. Hence we may write $L = M^\flat + \langle x \rangle$ for some $x \in L$. Choose an orthogonal basis $\{e_1, \ldots, e_{n-1}\}$ of $M^\flat$ such that $(e_i, e_i) = \varpi^{a_i}$. The fundamental matrix of $\{e_1, \ldots, e_{n-1}, x\}$ has the form

$$T = \begin{pmatrix} \varpi^{a_1} & & & (e_1, x) \\ & \varpi^{a_2} & & (e_2, x) \\ & & \ddots & \vdots \\ (x, e_1) & (x, e_2) & \cdots & (x, x) \end{pmatrix}.$$

If $t(M^\flat) \geq 2$ (i.e., at least two a_i's are > 0), then the rank of $T \bmod \varpi$ is at most $n-1$, contradicting that L is self-dual. $\square$

It follows from Lemma 4.5.1 that $z \in \mathcal{Z}(M^\flat)(O_K)$ is a quasi-canonical lifting contained in the right-hand-side of (4.2.1.1). By construction, $M^\flat$ is the largest lattice in $L_F^\flat$ contained in $i_{\bar{k}}(i_K^{-1}(L))$, thus in fact we have $z \in \mathcal{Z}(M^\flat)^\circ(O_K)$ by the equation (4.4.1.2). Therefore the O_K-points of both sides of (4.2.1.1) are equal.

To finish the proof of Theorem 4.2.1, by the flatness of both sides of (4.2.1.1) it remains to check that the $O_K[\varepsilon]$-points of both sides are equal (where $\varepsilon^2 = 0$). By Lemma 4.2.2, each O_K-point on the right-hand-side of (4.2.1.1) lifts uniquely to an $O_K[\varepsilon]$-point. Thus it remains to show that for each $z \in \mathcal{Z}(L^\flat)(O_K)$, there is a unique lift of z in $\mathcal{Z}(L^\flat)(O_K[\varepsilon])$. Let $\mathbb{D}(G)$ be the (covariant) O_{F_0}-relative Dieudonné crystal of G. The action of O_F via $\iota : O_F \to \mathrm{End}(G)$ induces an action $O_F \otimes_{O_{F_0}} O_K \simeq O_K \oplus O_K$ on $\mathbb{D}(G)(O_K)$, and hence a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{D}(G)(O_K)$. Let $\mathscr{A} = \mathrm{gr}_0\mathbb{D}(G)(O_K)$ be the 0th graded piece of $\mathbb{D}(G)(O_K)$, a free O_K-module of rank n. By the Kottwitz signature condition, it is equipped with an O_K-hyperplane $\mathscr{H} = \mathrm{Fil}^1\mathscr{A} := \mathrm{Fil}^1\mathbb{D}(G)(O_K) \cap \mathscr{A}$ (as usual, hyperplane means a free direct summand of rank $n-1$). The O_K-hyperplane $\mathscr{H}$ contains the image of $L^\flat$ under the identification of [KR11, Lemma 3.9]. Let $\widetilde{\mathscr{A}} = \mathrm{gr}_0\mathbb{D}(G)(O_K[\varepsilon])$. Since the kernel of $O_K[\varepsilon] \to O_K$ has a nilpotent divided power structure, by Grothendieck–Messing theory, a lift $\tilde{z} \in \mathcal{Z}(L^\flat)(O_K[\varepsilon])$ of z corresponds to an $O_K[\varepsilon]$-hyperplane $\widetilde{\mathscr{H}}$ of $\widetilde{\mathscr{A}}$ lifting the O_K-hyperplane $\mathscr{H}$ of $\mathscr{A}$ and contains the image of $L^\flat$ in $\widetilde{\mathscr{A}}$ (cf. [LZ17, Theorem 3.1.3], [KR11, Proof of Proposition 3.5]). Since $L^\flat \subseteq \mathrm{Hom}_{O_F}(\bar{\mathcal{E}}, G)$ has rank $n-1$, by Breuil's theorem (§4.3) we know that the image of $L^\flat$ in $\mathrm{gr}_0\mathbb{D}(G)(S)$ has rank $n-1$ over S and thus its image in the base change $\mathscr{A}$ has rank

$n-1$ over O_K. Hence the $O_K[\varepsilon]$-hyperplane $\widetilde{\mathscr{H}}$ is the unique $O_K[\varepsilon]$-hyperplane containing the $O_K[\varepsilon]$-submodule of rank $n-1$ spanned by the image of $L^\flat$ in $\widetilde{\mathscr{A}}$. Hence the lift $\tilde{z}$ is unique as desired.

4.6. **Relation with the local density.** Notice that $\deg_{O_{\breve{F}}}(\mathcal{Z}(L^\flat)_{\mathscr{H}})$ is equal to the degree of the 0-cycle $\mathcal{Z}(L^\flat)_{\breve{F}}$ in the generic fiber $\mathcal{N}_{\breve{F}}$ of the Rapoport–Zink space, which may be interpreted as a *geometric* intersection number on the generic fiber. We have the following identity between this geometric intersection number and a local density.

Corollary 4.6.1. $\deg_{O_{\breve{F}}}(\mathcal{Z}(L^\flat)_{\mathscr{H}}) = \mathrm{vol}(L^\flat)^{-1}\mathrm{Den}((-q)^{-1}, L^\flat) = \mathrm{Den}(-q, L^\flat)$.

Proof. The first equality follows immediately from Theorem 4.2.1, Equation (4.2.0.2), and Equation (3.6.1.2). The second equality follows from the functional equation (3.6.2.2). □

Remark 4.6.2. Using the p-adic uniformization theorem (§13.1) and the flatness of the horizontal part of the global Kudla–Rapoport cycles, one may deduce from Corollary 4.6.1 an identity between the geometric intersection number (i.e. the degree) of a special 0-cycle on a compact Shimura variety associated to $\mathrm{U}(n,1)$ and the value of a Fourier coefficient of a *coherent* Siegel Eisenstein series on $\mathrm{U}(n,n)$ at the near central point $s=1/2$. This should give a different proof (of a unitary analogue) of a theorem of Kudla [Kud97a, Theorem 10.6] for compact orthogonal Shimura varieties.

5. Vertical parts of Kudla–Rapoport cycles

5.1. **The support of the vertical part $\mathcal{Z}(L^\flat)_{\mathscr{V}}$.** Let $L^\flat$ be an O_F-lattice of rank $n-1$ in $\mathbb{V}_n$. Recall that $\mathcal{Z}(L^\flat)_{\mathscr{V}}$ is the vertical part of the Kudla–Rapoport cycle $\mathcal{Z}(L^\flat) \subseteq \mathcal{N}_n$ (§2.9).

Lemma 5.1.1. *$\mathcal{Z}(L^\flat)_{\mathscr{V}}$ is supported on $\mathcal{N}_n^{\mathrm{red}}$, i.e., $\mathcal{O}_{\mathcal{Z}(L^\flat)_{\mathscr{V}}}$ is annihilated by a power of the ideal sheaf of $\mathcal{N}_n^{\mathrm{red}} \subseteq \mathcal{N}_n$.*

Proof. If not, we may find an affine formal curve $C = \operatorname{Spf} R$ (i.e., an affine formal scheme of dimension 1) as a closed formal subscheme of $\mathcal{Z}(L^\flat)_{\mathscr{V}}$ such that C^{red} consists of a single point $z \in \mathcal{N}_n^{\mathrm{red}}$. The universal p-divisible O_{F_0}-module X^{univ} over $\mathcal{N}_n$ pulls back to a p-divisible O_{F_0}-module $\mathcal{X}_\eta$ over a geometric generic point η of $\operatorname{Spec} R$. Since $C^{\mathrm{red}} = \{z\}$, we know that the p-divisible O_{F_0}-module $\mathcal{X}_\eta$ is not supersingular. Otherwise, if $\mathcal{X}_\eta$ is supersingular, then $\mathcal{X}_\eta$ corresponds to a geometric point of $\mathcal{N}_n^{\mathrm{red}} - \{z\}$ (we may identify the complete local ring $O_{\mathcal{N}_n,z}$ of $\mathcal{N}_n$ at z with the complete local ring at a closed point of the integral model of a unitary Shimura variety and identify $O_{\mathcal{N}_n^{\mathrm{red}},z}$ with the complete local ring of the supersingular locus), and hence a geometric point of $C^{\mathrm{red}} - \{z\}$, which is impossible. On the other hand, if $L^\flat = \langle x_1, \ldots, x_{n-1}\rangle$, then $\mathcal{X}_\eta$ admits $n-1$ linearly independent special homomorphisms $\tilde{x}_i : \bar{\mathcal{E}}_\eta \to \mathcal{X}$, which gives rise to a homomorphism

$$(\tilde{x}_1, \ldots, \tilde{x}_{n-1}) : \bar{\mathcal{E}}_\eta^{n-1} \to \mathcal{X}_\eta.$$

Since η has characteristic p, by the Dieudonné–Manin classification we know that $\mathcal{X}_\eta$ is isogenous to $\bar{\mathcal{E}}_\eta^{n-1} \times \mathcal{X}'_\eta$ for $\mathcal{X}'_\eta$ a p-divisible O_{F_0}-module of relative height 2 and dimension 1 with an O_F-action. It follows that $\mathcal{X}'_\eta$ is supersingular, and so $\mathcal{X}_\eta$ itself is also supersingular, a contradiction. □

5.2. **Horizontal and vertical parts of ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)$.** Since $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ is one dimensional, the intersection $\mathcal{Z}(L^\flat)_{\mathscr{H}} \cap \mathcal{Z}(L^\flat)_{\mathscr{V}}$ must be zero dimensional (if non-empty). It follows that there is a decomposition of the $(n-1)$-th graded piece

$$\mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}_n) = \mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)_{\mathscr{H}}}(\mathcal{N}_n) \oplus \mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N}_n). \tag{5.2.0.1}$$

Definition 5.2.1. The decomposition (5.2.0.1) induces a decomposition of the derived Kudla–Rapoport cycle into *horizontal* and *vertical* parts

$$ {}^{\mathbb{L}}\mathcal{Z}(L^\flat) = {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}} + {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}_n).$$

From this decomposition, we see that even though the vertical part $\mathcal{Z}(L^\flat)_{\mathscr{V}}$ depends on the choice of an integer $N \gg 0$ (§2.9), the element ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}_n)$ is canonical and independent of the choice of N.

Since $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ has the expected dimension, the first summand ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{H}}$ is represented by the structure sheaf of $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ by [Zha21, Lemma B.2 (ii)]. Abusing notation we shall write the sum as

$$ {}^{\mathbb{L}}\mathcal{Z}(L^\flat) = \mathcal{Z}(L^\flat)_{\mathscr{H}} + {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}}. \tag{5.2.1.1}$$

By Lemma 5.1.1, we have a change-of-support homomorphism

$$\mathrm{Gr}^{n-1} K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N}_n) \longrightarrow \mathrm{Gr}^{n-1} K_0^{\mathcal{N}_n^{\mathrm{red}}}(\mathcal{N}_n).$$

Abusing notation we will also denote the image of ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}}$ in the target by the same symbol.

Corollary 5.2.2. *There exist finitely many curves $C_i \subseteq \mathcal{N}_n^{\mathrm{red}}$ and $\mathrm{mult}_{C_i} \in \mathbb{Q}$ such that*

$$ {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} = \sum_i \mathrm{mult}_{C_i}[\mathcal{O}_{C_i}] \in \mathrm{Gr}^{n-1} K_0^{\mathcal{N}_n^{\mathrm{red}}}(\mathcal{N}_n).$$

Proof. It follows immediately from Lemma 5.1.1, where the finiteness of such curves C_i is due to Lemma 2.9.2. ☐

5.3. **The Tate conjecture for certain Deligne–Lusztig varieties.** Consider the generalized Deligne–Lusztig variety $Y_d := Y_V$ and the classical Deligne–Lusztig variety $Y_d^\circ := Y_V^\circ$ as defined in §2.5, where V is the unique k_F/k-hermitian space of dimension $2d+1$. Recall that we have a stratification

$$Y_d = \bigsqcup_{i=0}^{d} X_{P_i}(w_i).$$

Let

$$X_i^\circ := X_{P_i}(w_i), \quad X_i := \overline{X_i^\circ} = \bigsqcup_{m=0}^{i} X_m^\circ.$$

Then X_i° is a disjoint union of isomorphic copies of the classical Deligne–Lusztig variety Y_i°, and each irreducible component of X_i is isomorphic to Y_i.

For any k_F-variety S, we write $H^j(S)(i) := H^j(S_{\bar{k}_F}, \overline{\mathbb{Q}_\ell}(i))$ ($\ell \neq p$ is a prime). Let $\mathbf{F} = \mathrm{Fr}_{k_F}$ be the q^2-Frobenius acting on $H^j(S)(i)$.

Lemma 5.3.1. *For any $d, i \geq 0$ and $s \geq 1$, the action of $\mathbf{F}^s$ on the following cohomology groups are semisimple, and the space of $\mathbf{F}^s$-invariants is zero when $j \geq 1$.*
(i) $H^{2j}(Y_d^\circ)(j)$.
(ii) $H^{2j}(X_i^\circ)(j)$.
(iii) $H^{2j}(Y_d - X_i)(j)$.

Proof. (i) By [Lus76, 7.3 Case ${}^2A_{2n}$] (notice the adjoint group assumption is harmless due to [Lus76, 1.18]), we know that there are exactly $2d+1$ eigenvalues of $\mathbf{F}$ on $H_c^*(Y_d^\circ)$, given by $(-q)^m$ where $m = 0, 1, \dots, 2d$, and the eigenvalue $(-q)^m$ appear exactly in $H_c^j(Y_d^\circ)$ for $j = \lfloor m/2 \rfloor + d$. By the Poincare duality, we have a perfect pairing

$$H_c^{2d-j}(Y_d^\circ) \times H^j(Y_d^\circ)(d) \to H_c^{2d}(Y_d^\circ)(d) \simeq \overline{\mathbb{Q}_\ell}.$$

Thus the eigenvalues of $\mathbf{F}$ on $H^{2j}(Y_d^\circ)(j)$ are given by $q^{2(d-j)}$ times the inverse of the eigenvalues in $H_c^{2(d-j)}(Y_d^\circ)$, which is equal to $\{(-q)^{2j}, (-q)^{2j-1}\}$ when $d \geq 2j > 0$, and $\{(-q)^{2j} = 1\}$ when $j = 0$. Hence the eigenvalue of $\mathbf{F}^s$ is never equal to 1 when $j \geq 1$. The semisimplicity of the action of $\mathbf{F}^s$ follows from [Lus76, 6.1].

(ii) It follows from (i) since X_i° is a disjoint union of Y_i°.

(iii) It follows from (ii) since $Y_d - X_i = \bigsqcup_{m=i+1}^d X_m^\circ$. $\square$

Theorem 5.3.2. *For any $0 \leq i \leq d$ and any $s \geq 1$, we have*
(i) The space of Tate classes $H^{2i}(Y_d)(i)^{\mathbf{F}^s=1}$ is spanned by the cycle classes of the irreducible components of X_{d-i}. In particular, the Tate conjecture ([Tat65, Conjecture 1]*, or* [Tat94, Conjecture T^i]*) holds for Y_d.*
(ii) Let $H^{2i}(Y_d)(i)_1 \subseteq H^{2i}(Y_d)(i)$ be the the generalized eigenspace of $\mathbf{F}^s$ for the eigenvalue 1. Then $H^{2i}(Y_d)(i)_1 = H^{2i}(Y_d)(i)^{\mathbf{F}^s=1}$.

Proof. The assertion is clear when $i = 0$. Assume $i > 0$. Associated to the closed embedding $X_{d-i} \hookrightarrow Y_d$ we have a long exact sequence

$$\cdots \to H^j_{X_{d-i}}(Y_d) \to H^j(Y_d) \to H^j(Y_d - X_{d-i}) \to H^{j+1}_{X_{d-i}}(Y_d) \to \cdots \tag{5.3.2.1}$$

Take $j = 2i$. We have a Gysin isomorphism

$$\bigoplus_{Z \in \mathrm{Irr}(X_{d-i})} H^0(Z) \xrightarrow{\sim} H^{2i}_{X_{d-i}}(Y_d)(i), \tag{5.3.2.2}$$

where the sum runs over all the irreducible components of X_{d-i}. By (5.3.2.2) and Lemma 5.3.1, the actions of $\mathbf{F}^s$ on $H^{2i}_{X_{d-i}}(Y_d)$ and $H^{2i}(Y_d - X_{d-i})$ are semisimple, and thus

$$H^{2i}_{X_{d-i}}(Y_d)(i)_1 = H^{2i}_{X_{d-i}}(Y_d)(i)^{\mathbf{F}^s=1}, \quad H^{2i}(Y_d - X_{d-i})(i)_1 = H^{2i}(Y_d - X_{d-i})(i)^{\mathbf{F}^s=1}.$$

Taking the i-th Tate twist and taking the generalized eigenspace of $\mathbf{F}^s$ for the eigenvalue 1 of (5.3.2.1), we obtain a 3-term exact sequence

$$H^{2i}_{X_{d-i}}(Y_d)(i)^{\mathbf{F}^s=1} \to H^{2i}(Y_d)(i)_1 \to H^{2i}(Y_d - X_{d-i})(i)^{\mathbf{F}^s=1}.$$

The last term is 0 by Lemma 5.3.1 (iii) as $i > 0$. Thus $H^{2i}(Y_d)(i)_1 = H^{2i}(Y_d)(i)^{\mathbf{F}^s=1}$, and we have a surjection onto Tate classes

$$\bigoplus_{Z \in \mathrm{Irr}(X_{d-i})} H^0(Z) \simeq H^{2i}_{X_{d-i}}(Y_d)(i)^{\mathbf{F}^s=1} \twoheadrightarrow H^{2i}(Y_d)(i)^{\mathbf{F}^s=1}.$$

So $H^{2i}(Y_d)(i)^{\mathbf{F}^s=1}$ is spanned by the cycle classes of the irreducible components of X_{d-i}. $\square$

Let us come back to the situation of §5.1.

Corollary 5.3.3. *There exists finitely many Deligne–Lusztig curves $C_i \subseteq \mathcal{N}_n^{\mathrm{red}}$ (i.e., $C_i = \mathcal{V}(\Lambda)$ for a vertex lattice Λ of type 3) and $\mathrm{mult}_{C_i} \in \mathbb{Q}$, such that for any $x \in \mathbb{V}_n \setminus L^\flat_F$,*

$$\chi(\mathcal{N}_n, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)) = \sum_i \mathrm{mult}_{C_i} \cdot \chi(\mathcal{N}_n, C_i \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)).$$

Proof. By the Bruhat–Tits stratification of $\mathcal{N}_n^{\mathrm{red}}$ (§2.7), any curve C in $\mathcal{N}_n^{\mathrm{red}}$ lies in some Deligne–Lusztig variety $\mathcal{V}(\Lambda) \cong Y_d$ for a vertex lattice Λ. By Theorem 5.3.2 (for $i = d-1$), the cycle class of such a curve C can be written as a $\mathbb{Q}$-linear combination of the cycle classes of Deligne–Lusztig curves on $\mathcal{V}(\Lambda)$. Notice that $\chi(\mathcal{N}, C \overset{\mathbb{L}}{\cap} \mathcal{Z}(x))$ only depends on the cycle class of C. In fact, since $\mathcal{Z}(x)$ is a Cartier divisor on $\mathcal{N}_n$, $\mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)$ is explicitly represented by the two-term complex of locally free sheaves

$$[\mathcal{O}_{\mathcal{N}_n}(-\mathcal{Z}(x))|_{\mathcal{V}(\Lambda)} \to \mathcal{O}_{\mathcal{N}_n}|_{\mathcal{V}(\Lambda)}] \in \mathrm{F}^1 K_0(\mathcal{V}(\Lambda)).$$

Hence by [Zha21, (B.3)], $\chi(\mathcal{N}, C \overset{\mathbb{L}}{\cap} \mathcal{Z}(x))$ only depends on the image of $\mathcal{O}_C$ in $\mathrm{Gr}^{d-1} K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}} \cong \mathrm{Ch}^{d-1}(\mathcal{V}(\Lambda))_{\mathbb{Q}}$, and the image of $\mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)$ in $\mathrm{Gr}^1 K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}} \cong \mathrm{Ch}^1(\mathcal{V}(\Lambda))_{\mathbb{Q}}$. As the cycle class map intertwines the intersection product and the cup product, cf. (6.4.0.1), we know that $\chi(\mathcal{N}, C \overset{\mathbb{L}}{\cap} \mathcal{Z}(x))$ only depends on the cycle class of C. The result then follows from Corollary 5.2.2. $\square$

5.4. **The vertical cycle in the case $n = 3$, and Theorem 3.4.1 in the case $n = 2$.** Now let $n = 3$, and let $L^\flat \subset \mathbb{V}_3$ be a rank two integral lattice. Denote by $\mathrm{Vert}^t(L^\flat)$ the set of vertex lattices Λ of type t containing $L^\flat$. For any integral lattice Λ, we denote $L^\flat_\Lambda := L^\flat_F \cap \Lambda$, which is an integral lattice in $L^\flat_F$.

Theorem 5.4.1. *(i) Let $L^\flat \subset \mathbb{V}_3$ be a rank two lattice. Then the following identity*

$$\mathcal{Z}(L^\flat)_{\mathscr{V}} = \sum_{\Lambda \in \mathrm{Vert}^3(L^\flat)} \mathrm{mult}_{L^\flat}(\Lambda) \cdot \mathcal{V}(\Lambda),$$

holds in $\mathrm{Gr}^2 K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N}_3)$, where

$$\mathrm{mult}_{L^\flat}(\Lambda) = \#\{L'^\flat \mid L^\flat \subset L'^\flat \subset L^\flat_\Lambda\}.$$

Similarly, the following identity

$${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} = \sum_{\Lambda \in \mathrm{Vert}^3(L^\flat)} \mathrm{mult}_{L^\flat}(\Lambda) \cdot \mathcal{V}(\Lambda)$$

holds in $\mathrm{Gr}^2 K_0^{\mathcal{Z}(L^\flat)_{\mathscr{V}}}(\mathcal{N}_3)$.

(ii) Theorem 3.4.1 holds when $n = 2$, i.e., $\mathrm{Int}(L^\flat) = \partial\mathrm{Den}(L^\flat)$ *for all* $L^\flat \subset \mathbb{V}_2$.

Remark 5.4.2. (i) Part (ii) is known by [KR11, Theorem 1.1]. However, our proof is logically independent from *loc. cit.*.

(ii) Later we will only need (in the proof of Lemma 6.2.1) a very special case of part (i) of Theorem 5.4.1, i.e., the minuscule case in the proof below.

We first establish two lemmas. The first one is trivial and we state it because it will be used repeatedly.

Lemma 5.4.3. *Let $e \in \mathbb{V}_3$ be a unit-normed vector. Then there is a unique vertex lattice Λ_e of type 1 in $\mathbb{V}_3$ containing e. Moreover, if $L \subset \mathbb{V}_3$ is any integral lattice (not necessarily of full rank) and $e \in L$, then $L \subset \Lambda_e$.*

Proof. The hermitian space $\langle e\rangle^\perp \subset \mathbb{V}_3$ is two dimensional and non-split, hence has a unique maximal integral lattice $\Lambda^\flat$ (consisting of all vectors with integral norms). Then we see that $\Lambda_e = \langle e\rangle \oplus \Lambda^\flat$ has the desired property. ☐

Lemma 5.4.4. *Fix $\Lambda_0 \in \mathrm{Vert}^3(L^\flat)$. Then there exists a vector $e \in \mathbb{V}_3$ with unit norm such that, when denoting $M = \langle e\rangle$,*

(i) The lattice $\Lambda_0 + M$ is equal to the vertex lattice Λ_e of type 1 in Lemma 5.4.3, and $\Lambda_e = L^\flat_{\Lambda_e} \oplus M$;

(ii) $\Lambda_0 = L^\flat_{\Lambda_0} + \varpi M$ and $L^\flat_{\Lambda_0} = L^\flat_{\Lambda_e}$;

(iii) For any other $\Lambda \neq \Lambda_0$ in $\mathrm{Vert}^3(L^\flat + \varpi M)$, the lattice $L^\flat_\Lambda$ is equal to $L^\flat_{\varpi\Lambda_e^\vee}$ and is a sub-lattice of $L^\flat_{\Lambda_0} = L^\flat_{\Lambda_e}$ of colength one;

(iv) For any lattice $L'^\flat$ such that $L^\flat \subset L'^\flat \subset L^\flat_{\Lambda_e}$, we have

$$t(L'^\flat \oplus M) = \begin{cases} 2, & \text{if } L'^\flat \subset L^\flat_{\varpi\Lambda_e^\vee}, \\ 1, & \text{otherwise.} \end{cases}$$

Remark 5.4.5. Before presenting the proof, we indicate the geometric picture of the lemma. The reduced scheme $\mathcal{Z}(L^\flat)^{\mathrm{red}}$ of $\mathcal{Z}(L^\flat)$ is a (connected, a fact we do not need) union of the curves $\mathcal{V}(\Lambda)$ for $\Lambda \in \mathrm{Vert}^3(L^\flat)$. The lemma implies that, on any given irreducible component $\mathcal{V}(\Lambda_0)$, there exists a (superspecial) point $\mathcal{V}(\Lambda_e)$, such that among all the curves $\mathcal{V}(\Lambda) \subset \mathcal{Z}(L^\flat)^{\mathrm{red}}$ passing through $\mathcal{V}(\Lambda_e)$ (noting that such Λ necessarily belongs to $\mathrm{Vert}^3(L^\flat + \varpi M)$ due to the implication $\mathcal{V}(\Lambda_e) \subset \mathcal{V}(\Lambda) \implies e \in \Lambda^\vee \implies \varpi e \in \Lambda$), the given one $\mathcal{V}(\Lambda_0)$ has the (strictly) largest associated lattice $L^\flat_{\Lambda_0}$. This suggests the possibility to determine the multiplicity $\mathrm{mult}_{L^\flat}(\Lambda)$ by induction on $[L^\flat_{\Lambda_0} : L^\flat]$.

Proof. There exist vectors with valuation one in the lattice $L^\flat_{\Lambda_0}$. Otherwise, choose an orthogonal basis $\{e_1, e_2\}$ of $L^\flat_{\Lambda_0}$ and choose $e_3 \in \Lambda_0$ such that $\Lambda_0 = L^\flat_{\Lambda_0} \oplus \langle e_3\rangle$. Consider the fundamental matrix of the basis $\{e_1, e_2, e_3\}$ of Λ_0

$$T = \begin{pmatrix} (e_1, e_1) & 0 & (e_1, e_3) \\ 0 & (e_2, e_2) & (e_2, e_3) \\ (e_3, e_1) & (e_3, e_2) & (e_3, e_3) \end{pmatrix}.$$

Since $t(\Lambda_0) = 3$, every entry of T is divisible by ϖ. Moreover, we have $\mathrm{val}(e_1) \geq 2$ and $\mathrm{val}(e_2) \geq 2$. It follows that $\mathrm{val}(\det(T)) \geq 4$, contradicting $\mathrm{val}(\det(T)) = 3$.

Now we pick a vector x of valuation one in $L^\flat_{\Lambda_0}$ and denote by E the rank one lattice $\langle x\rangle$. Then $L^\flat_{\Lambda_0}$ splits as an orthogonal direct sum

$$L^\flat_{\Lambda_0} = E \obot M'$$

where M' is a rank one lattice.

We *claim* that there exists a vector $e \perp E$ such that

(a)The norm of e is a unit;

(b)Denoting $M = \langle e\rangle$, then the rank two lattice $M' \oplus M$ is self-dual;

(c)$\Lambda_0 = E \obot (M' \oplus \varpi M)$.

To show the claim, we consider the two dimensional subspace $\langle x\rangle^\perp_F$. From $\mathrm{val}(x) = 1$, it follows that $\langle x\rangle^\perp_F$ is a split hermitian space, and Λ_0 is an orthogonal direct sum $E \obot E^\perp$ for a vertex lattice $E^\perp$ of type 2 in $\langle x\rangle^\perp_F$. Consider the two dimensional k_F-vector space $V := \varpi^{-1}E^\perp/E^\perp$ with the induced hermitian form. The $q+1$ isotropic lines in V are bijective to self-dual lattices containing $E^\perp$. Since the lattice M' is saturated in $E^\perp$, the image of $\varpi^{-1}M'$ in V is non-zero. Hence there exists an isotropic line not containing the image of $\varpi^{-1}M'$ in V. Or equivalently, there exists a self-dual lattice $\Xi \subset \langle x\rangle^\perp_F$ containing $E^\perp$ but not $\varpi^{-1}M'$ (i.e., the rank one lattice M' remains saturated in Ξ). Finally, we choose a unit-normed e lifting a generator of the free O_F-module Ξ/M' of rank one. It is easy to verify that such a vector e satisfies all the conditions $(a), (b)$ and (c) and this proves the claim.

We fix such a vector e and we now verify that it satisfies the desired conditions. Parts (i) and (ii) are clear by the claim above. Now let Λ be a lattice in $\mathrm{Vert}^3(L^\flat + \langle \varpi e\rangle)$. Then $\Lambda + \langle e\rangle$ is an integral lattice containing a unit-normed vector, hence is a vertex lattice of type 1 (recall from §2.6 that the type of a vertex lattice is always odd). Therefore, by Lemma 5.4.3, we obtain $\Lambda + \langle e\rangle = \Lambda_e$ for all $\Lambda \in \mathrm{Vert}^3(L^\flat + \langle\varpi e\rangle)$. Now assume that $\Lambda \neq \Lambda_0$. Then we obtain the following diagram

$$\begin{tikzcd}
& \Lambda_e = E \obot (M' \oplus M) & \\
\Lambda_0 = E \obot (M' \oplus \varpi M) \arrow[ur, hook] & & \Lambda \longleftarrow L^\flat_\Lambda \oplus \varpi M \arrow[ul, hook'] \\
& \varpi\Lambda_e^\vee = E \obot (\varpi M' \oplus \varpi M) \arrow[ul, hook'] \arrow[ur, hook] & \\
& \varpi\Lambda_e = \varpi E \obot (\varpi M' \oplus \varpi M). \arrow[u, hook, "1"] &
\end{tikzcd}$$

It is easy to see that

$$E \obot \varpi M' \subset L^\flat_\Lambda \subset E \obot M',$$

and hence either $L^\flat_\Lambda = E \oplus M'$ or $L^\flat_\Lambda = E \oplus \varpi M'$. In the former case, we must have $\Lambda \supset E \oplus (M' \oplus \varpi M) = \Lambda_0$, contradicting $\Lambda \neq \Lambda_0$. This shows that $L^\flat_\Lambda = E \oplus \varpi M' = L^\flat_{\varpi \Lambda_e^\vee}$, and hence completes the proof of (iii).

Let $L'^\flat \subset L^\flat_{\Lambda_e} = E \oplus M'$. Then the type of $L'^\flat \oplus M$ is either 1 or 2. To show part (iv), we first assume that $L'^\flat \subset L^\flat_{\varpi\Lambda_e^\vee} = E \oplus \varpi M'$. Then we have

$$t(L'^\flat \oplus M) \geq t(E \oplus (\varpi M' \oplus M)) = t(E) + t(\varpi M' \oplus M)$$

and $t(E) = 1$. Now noting that $M' \oplus M$ is self-dual, its proper sub-lattice $\varpi M' \oplus M$ can not be self-dual, and hence $t(\varpi M' \oplus M) \geq 1$. Since M is self-dual, it follows that $t(L'^\flat \oplus M) < 3$, and hence $t(L'^\flat \oplus M) = 2$.

Finally we assume that $L'^\flat \subset E \oplus M'$ but $L'^\flat \not\subset E \oplus \varpi M'$, then there must be a vector $u \in L'^\flat$ whose projection to M' is a generator of M'. It follows that $\langle u \rangle \oplus M$ is a rank-two self-dual sub-lattice of $L'^\flat \oplus M$, forcing the type $t(L'^\flat \oplus M) \leq 1$. Since $\mathbb{V}_3$ is a non-split hermitian space, it does not contain any self-dual lattice of full rank. Therefore $t(L'^\flat \oplus M) = 1$ and this completes the proof of (iv). $\square$

Proof of Theorem 5.4.1. The proof is rather involved and will be divided into four steps:

(1) both parts hold when $t(L^\flat) \leq 1$.

(2) part (i) holds for minuscule $L^\flat$ and we deduce (5.4.5.5) that will be used repeatedly later.

(3) part (i) for $L^\flat$ with odd $\mathrm{val}(L^\flat)$ implies part (ii) for the same $L^\flat$.

(4) part (i) for all $L^\flat$ holds by induction on $\mathrm{val}(L^\flat)$.

We start with a remark. By Lemma 2.8.1, ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} \in \mathrm{Gr}^2 K_0^{\mathcal{N}_3^{\mathrm{red}}}(\mathcal{N}_3)$ is represented by the class of $\mathcal{O}_{[\mathcal{Z}(L^\flat)_{\mathscr{V}}]}$. Therefore

$$\text{(5.4.5.1)} \qquad {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} = \mathcal{Z}(L^\flat)_{\mathscr{V}} = \sum_{\Lambda \in \mathrm{Vert}^3(L^\flat)} \mathrm{mult}_{L^\flat}(\Lambda) \cdot \mathcal{V}(\Lambda)$$

where the multiplicity $\mathrm{mult}_{L^\flat}(\Lambda)$ is a positive integer to be determined.

We first prove that both parts hold in the special case $t(L^\flat) \leq 1$. In fact, if $t(L^\flat) \leq 1$, we may write $L^\flat = \langle u \rangle \oplus \langle e \rangle$ for a unit-normed vector e. Then $\mathcal{Z}(u)$ and $\mathcal{Z}(e) \simeq \mathcal{N}_2$ intersect transversely (cf. the discussion before (2.11.0.2)). Therefore $\mathcal{Z}(L^\flat)$ is flat over $\mathrm{Spf}\, O_{\breve{F}}$ and $\mathcal{Z}(L^\flat)_{\mathscr{V}} = {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}} = 0$. Noting that $\mathrm{mult}_{L^\flat}(\Lambda) = 0$ for any vertex lattice $\Lambda \subset \mathbb{V}_3$ of type 3, we have proved part (i). Similarly, part (ii) is reduced to the case $n = 1$ by the cancellation law Lemma 2.11.1 and (3.2.1.1). By Example 2.4.3 and 3.5.2 we have

$$\text{(5.4.5.2)} \qquad \mathrm{Int}(L^\flat) = \frac{\mathrm{val}(L^\flat) + 1}{2} = \partial\mathrm{Den}(L^\flat).$$

Next we consider the minuscule case of part (i), i.e., when the fundamental invariants of $L^\flat$ are $(1, 1)$. Then $\mathrm{Vert}^3(L^\flat)$ consists of a single type 3 lattice $\Lambda = L^\flat \oplus \langle u \rangle$ for a vector u of valuation one. By Theorem 4.2.1 the horizontal part is the sum of quasi-canonical lifting cycles $\mathcal{Z}(L'^\flat) \simeq \mathcal{N}_1$ corresponding to the $q+1$ self-dual lattices $L'^\flat$ containing $L^\flat$. Therefore by (2.7.0.1) we have an equality in $\mathrm{Gr}^2 K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}_3)$,

$$\mathcal{Z}(L^\flat) = m \cdot \mathcal{V}(\Lambda) + \sum_{L^\flat \subset L'^\flat = (L'^\flat)^\vee} \mathcal{Z}(L'^\flat), \tag{5.4.5.3}$$

where the multiplicity m of $\mathcal{V}(\Lambda)$ is a positive integer to be determined. Now let x_1, x_2 be an orthogonal basis of $L^\flat$, so that $\mathrm{val}(x_1) = \mathrm{val}(x_2) = 1$. Now choose vector $e \perp x_1$ such that e has unit norm and $\langle x_2 \rangle \oplus \langle e \rangle$ is a self-dual lattice. It follows that $L^\flat \oplus \langle e \rangle$ is a vertex lattice of type 1, and that, for any $L'^\flat = (L'^\flat)^\vee$, the strictly larger lattice $L'^\flat \oplus \langle e \rangle$ can not be integral. Therefore, $\mathcal{Z}(e)$ does not intersect with any of the quasi-canonical lifting cycles $\mathcal{Z}(L'^\flat)$ appearing in (5.4.5.3). Now consider

$$\mathrm{Int}(L^\flat \oplus \langle e \rangle) = \chi(\mathcal{N}_3, \mathcal{Z}(L^\flat) \overset{\mathbb{L}}{\cap} \mathcal{Z}(e)).$$

On one hand, by Lemma 2.11.1 applied to the self-dual lattice $\langle x_2 \rangle \oplus \langle e \rangle$, we obtain $\mathrm{Int}(L^\flat \oplus \langle e \rangle) = \mathrm{Int}(\langle x_1 \rangle) = 1$ by Example 2.4.3. On the other hand, using the decomposition (5.4.5.3), we have

$$\mathrm{Int}(L^\flat \oplus \langle e \rangle) = m \cdot \chi(\mathcal{N}_3, \mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(e)).$$

We deduce that the multiplicity $m = 1$ in (5.4.5.3), and

$$\chi(\mathcal{N}_3, \mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(e)) = 1. \tag{5.4.5.4}$$

We note that, choosing $L^\flat$ appropriately, the argument above shows that (5.4.5.4) holds for any $\Lambda \in \mathrm{Vert}^3$ and a unit-normed e such that $\Lambda + \langle e \rangle$ is an integral lattice (necessarily a vertex lattice of type 1). Moreover, it is obvious that $\chi(\mathcal{N}_3, \mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(e)) = 0$ if $\Lambda + \langle e \rangle$ is not integral (equivalently $e \notin \Lambda^\vee$). Therefore we obtain that, for any $\Lambda \in \mathrm{Vert}^3$ and any e with unit-norm,

$$\chi(\mathcal{N}_3, \mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(e)) = \begin{cases} 1, & e \in \Lambda^\vee \\ 0, & e \notin \Lambda^\vee. \end{cases} \tag{5.4.5.5}$$

This equation will be repeatedly used later.

Next we show that part (ii) for $L^\flat \subset \mathbb{V}_2$ (necessarily with odd $\mathrm{val}(L^\flat)$, as $\mathbb{V}_2$ is non-split) follows from part (i) with the same $L^\flat \subset \mathbb{V}_3$. Here we have implicitly fixed an isomorphism $\mathbb{V}_3 \simeq \mathbb{V}_2 \obot M_F$ and an embedding $\mathcal{N}_2 \to \mathcal{N}_3$ of the form (2.11.0.1) induced by a self-dual lattice $M = \langle e \rangle$ of rank one. Let $L'^\flat$ be a type one lattice containing $L^\flat$, then by Lemma 2.11.1 and (5.4.5.2),

$$\mathrm{Int}(L'^\flat \obot M) = \mathrm{Int}(L'^\flat) = \frac{\mathrm{val}(L'^\flat) + 1}{2}.$$

Let $L''^\flat$ be the unique lattice such that $L'^\flat \subset L''^\flat \subset (L'^\flat)^\vee$ and $L''^\flat / L'^\flat$ has length one. Then by part (i) for $L'^\flat$ and $L''^\flat$ we have

$$\mathcal{Z}(L'^\flat) = \mathcal{Z}(L'^\flat)_{\mathscr{H}}, \quad \mathcal{Z}(L''^\flat) = \mathcal{Z}(L''^\flat)_{\mathscr{H}}.$$

By Theorem 4.2.1 we obtain an equality in $\mathrm{Gr}^2 K_0^{\mathcal{Z}(L'^\flat)}(\mathcal{N}_3)$,

$$\mathcal{Z}(L'^\flat) = \mathcal{Z}(L''^\flat) + \mathcal{Z}(L'^\flat)^\circ,$$

where $\mathcal{Z}(L'^\flat)^\circ$ is the associated quasi-canonical lifting cycle (cf. §4.1). It follows that

$$\chi(\mathcal{N}_3, \mathcal{Z}(M) \stackrel{\mathbb{L}}{\cap} \mathcal{Z}(L'^\flat)^\circ) = \mathrm{Int}(L'^\flat) - \mathrm{Int}(L''^\flat) = 1. \tag{5.4.5.6}$$

Therefore by Theorem 4.2.1 we obtain

$$\chi(\mathcal{N}_3, \mathcal{Z}(M) \stackrel{\mathbb{L}}{\cap} \mathcal{Z}(L^\flat)_{\mathscr{H}}) = \#\{L'^\flat \mid L^\flat \subset L'^\flat \subset (L'^\flat)^\vee, t(L'^\flat) = 1\}. \tag{5.4.5.7}$$

By part (i) for $L^\flat$ (note that we are assuming this part to hold), and by (5.4.5.5), we obtain

$$\begin{aligned}\chi(\mathcal{N}_3, \mathcal{Z}(M) \stackrel{\mathbb{L}}{\cap} \mathcal{Z}(L^\flat)_{\mathscr{V}}) &= \sum_{\substack{\Lambda\in \mathrm{Vert}^3(L^\flat)\\ M\subset \Lambda^\vee}} \mathrm{mult}_{L^\flat}(\Lambda)\\ &= \sum_{\substack{L^\flat\subset L'^\flat\subset (L'^\flat)^\vee\\ t(L'^\flat)=2}} \#\{\Lambda\in \mathrm{Vert}^3(L^\flat) \mid L'^\flat \subset \Lambda, M\subset \Lambda^\vee\}.\end{aligned}$$

Here, for $\Lambda \in \mathrm{Vert}^3$, the condition $M = \langle e\rangle \subset \Lambda^\vee$ (i.e., $M + \Lambda$ being integral) is equivalent to $\Lambda \subset \Lambda_e$, where Λ_e is the lattice in Lemma 5.4.3. Note that $\Lambda_e = M \oplus \Lambda^\flat$ where $\Lambda^\flat$ is the unique maximal integral lattice in $M_F^\perp$. If $L'^\flat$ is of type 2, then $\langle L'^\flat, \Lambda^\flat\rangle \subset \varpi O_F$ (we leave the proof to the reader), or equivalently $L'^\flat \subset \varpi(\Lambda^\flat)^\vee$. Therefore any $L'^\flat$ of type 2 is automatically contained in $\varpi\Lambda_e^\vee$, hence contained in any type 3 vertex lattice $\Lambda \subset \Lambda_e$. It follows that the condition $L'^\flat \subset \Lambda$ in the sum above is redundant. Since there are $q+1$ of type 3 lattices $\Lambda \subset \Lambda_e$, we obtain

$$\chi(\mathcal{N}_3, \mathcal{Z}(M) \stackrel{\mathbb{L}}{\cap} \mathcal{Z}(L^\flat)_{\mathscr{V}}) = (q+1)\,\#\{\text{integral } L'^\flat \mid L^\flat \subset L'^\flat, t(L'^\flat) = 2\}. \tag{5.4.5.8}$$

Then the desired assertion for part (ii) for $L^\flat \subset \mathbb{V}_2$ follows from (5.4.5.7), (5.4.5.8) and the formula in Corollary 3.5.3:

$$\partial\mathrm{Den}(L^\flat) = \sum_{L^\flat\subset L'^\flat\subset (L'^\flat)^\vee} \mathfrak{m}(t(L'^\flat)),\ \text{ where } \mathfrak{m}(t(L'^\flat)) = \begin{cases}1, & t(L'^\flat)=1,\\ q+1, & t(L'^\flat) = 2.\end{cases}$$

Finally, we prove part (i) for $L^\flat \subset \mathbb{V}_3$ by induction on $\mathrm{val}(L^\flat)$. We have proved it when $t(L^\flat) = 1$. Now we fix $L^\flat \subset \mathbb{V}_3$ of type 2. By induction, we may assume that part (i) holds for all $L'^\flat \subset \mathbb{V}_3$ with $\mathrm{val}(L'^\flat) < \mathrm{val}(L^\flat)$. Note that, by what we have proved, the induction hypothesis also implies that part (ii) holds for all $L'^\flat \subset \mathbb{V}_3$ with *odd* $\mathrm{val}(L'^\flat) < \mathrm{val}(L^\flat)$ (here $L'^\flat$ need not to be a lattice in $L_F^\flat$).

To determine the multiplicity, we fix $\Lambda_0 \in \mathrm{Vert}^3(L^\flat)$. Choose e as in Lemma 5.4.4 and follow the same notation. We *claim* that $\mathcal{Z}(M)$ does not intersect the horizontal part $\mathcal{Z}(L^\flat)_{\mathscr{H}}$. In fact, if $\mathcal{Z}(M)$ were to intersect non-emptily with $\mathcal{Z}(L^\flat)_{\mathscr{H}}$, then by Theorem 4.2.1, $\mathcal{Z}(M)$ would intersect non-emptily with $\mathcal{Z}(L'^\flat)^\circ$ for a type 1 lattice $L'^\flat$ containing $L^\flat$. This would imply that $M \oplus L'^\flat$ is an integral lattice, and hence $M \oplus L'^\flat \subset \Lambda_e$ by Lemma 5.4.3. Therefore $L'^\flat \subset L^\flat_{\Lambda_e}$ and hence the type of $L^\flat_{\Lambda_e}$ is at most one. By Lemma 5.4.4 (ii), $L^\flat_{\Lambda_e} = L^\flat_{\Lambda_0}$ is a sub-lattice of $\Lambda_0 \in \mathrm{Vert}^3(L^\flat)$ and hence it has type 2. Contradiction!

It follows from the claim and (5.4.5.1) that

$$\mathrm{Int}(L^\flat \oplus M) = \sum_{\Lambda\in \mathrm{Vert}^3(L^\flat)} \mathrm{mult}_{L^\flat}(\Lambda)\cdot \chi(\mathcal{N}_3, \mathcal{V}(\Lambda) \stackrel{\mathbb{L}}{\cap} \mathcal{Z}(e)).$$

By (5.4.5.5) and noting that $e \in \Lambda^\vee \iff \varpi e \in \Lambda$, we obtain

$$\mathrm{Int}(L^\flat \oplus M) = \mathrm{mult}_{L^\flat}(\Lambda_0) + \sum_{\substack{\Lambda \in \mathrm{Vert}^3(L^\flat + \varpi M) \\ \Lambda \neq \Lambda_0}} \mathrm{mult}_{L^\flat}(\Lambda). \tag{5.4.5.9}$$

Note that the second summand is understood as zero if Λ_0 is the only element in $\mathrm{Vert}^3(L^\flat + \varpi M)$.

By Lemma 5.4.4 (i), we obtain $[\Lambda_e : L^\flat \oplus M] = [L^\flat_{\Lambda_e} : L^\flat]$. From $\mathrm{val}(\Lambda_e) = 1$ and $\mathrm{val}(L^\flat_{\Lambda_e}) \geq 2$, it follows that $\mathrm{val}(L^\flat \oplus M) < \mathrm{val}(L^\flat)$. Since M is self-dual, we can decompose $L^\flat \oplus M = M^\perp \obot M$ orthogonally for a rank two lattice $M^\perp$ (note that here $M^\perp \subset \mathbb{V}_3$ is not necessarily a lattice in $L^\flat_F$). Then $\mathrm{val}(M^\perp)$ is odd and $\mathrm{val}(M^\perp) < \mathrm{val}(L^\flat)$. By induction hypothesis, part (ii) holds for $M^\perp$. It follows that, by the cancellation law Lemma 2.11.1 and (3.2.1.1),

$$\mathrm{Int}(L^\flat \oplus M) = \mathrm{Int}(M^\perp) = \partial\mathrm{Den}(M^\perp) = \partial\mathrm{Den}(L^\flat \oplus M).$$

By Corollary 3.5.3, $\partial\mathrm{Den}(L^\flat \oplus M)$ is the sum

$$\partial\mathrm{Den}(L^\flat \oplus M) = \sum_{L^\flat \oplus M \subset L' \subset L'^\vee} \mathfrak{m}(t(L')).$$

Note that, by Lemma 5.4.3, any integral lattice L' containing M is necessarily contained in Λ_e. Since $\Lambda_e = L^\flat_{\Lambda_e} \oplus M$ by Lemma 5.4.4 (i), every L' in the sum must be a direct sum $L'^\flat \oplus M$ for a unique integral lattice $L'^\flat$ lying between $L^\flat$ and $L^\flat_{\Lambda_e}$. In other words, $\partial\mathrm{Den}(L^\flat \oplus M)$ is the sum

$$\#\{L'^\flat \mid L^\flat \subset L'^\flat \subset L^\flat_{\Lambda_e}\} + q \cdot \#\{L'^\flat \mid L^\flat \subset L'^\flat \subset L^\flat_{\Lambda_e}, t(L'^\flat \oplus M) = 2\}.$$

By (ii), (iii), and (iv) of Lemma 5.4.4, the above sum is equal to

$$\#\{L'^\flat \mid L^\flat \subset L'^\flat \subset L^\flat_{\Lambda_0}\} + \sum_{\substack{\Lambda \in \mathrm{Vert}^3(L^\flat + \varpi M) \\ \Lambda \neq \Lambda_0}} \#\{L'^\flat \mid L^\flat \subset L'^\flat \subset L^\flat_{\Lambda}\}. \tag{5.4.5.10}$$

If $L^\flat_{\Lambda_0} = L^\flat$, then both (5.4.5.9) and (5.4.5.10) have only one term and we obtain

$$\mathrm{mult}_{L^\flat}(\Lambda_0) = \#\{L'^\flat \mid L^\flat \subset L'^\flat \subset L^\flat_{\Lambda_0}\} = 1.$$

If $L^\flat_{\Lambda_0} \neq L^\flat$, by Lemma 5.4.4 (iii), the index $[L^\flat_\Lambda : L^\flat]$ is strictly smaller than $[L^\flat_{\Lambda_0} : L^\flat]$ for $\Lambda \neq \Lambda_0$ in the sum of (5.4.5.10). Therefore, by induction on $[L^\flat_{\Lambda_0} : L^\flat]$, comparing (5.4.5.9) and (5.4.5.10) we finish the proof of the multiplicity formula for Λ_0, i.e., $\mathrm{mult}_{L^\flat}(\Lambda_0) = \#\{L'^\flat \mid L^\flat \subset L'^\flat \subset L^\flat_{\Lambda_0}\}$. This completes the proof. $\square$

Corollary 5.4.6. *Let $L^\flat \subset \mathbb{V}_n$ be an integral lattice of rank $n-1$ and type $t(L^\flat) \leq 1$. Then for any $x \in \mathbb{V}_n \setminus L^\flat_F$,*

$$\chi(\mathcal{N}_n, \mathcal{Z}(x) \overset{\mathbb{L}}{\cap} \mathcal{Z}(L^\flat)^\circ) = \sum_{\substack{L^\flat + \langle x \rangle \subset L' \subset L'^\vee, \\ L' \cap L^\flat_F = L^\flat}} \mathfrak{m}(t(L')).$$

Proof. By the assumption that $t(L^\flat) \leq 1$, there exists a self-dual lattice M of rank $n-2$ such that $L^\flat = M \obot \langle u \rangle$. We then reduce the question to the case $n = 2$, in which case $L^\flat = \langle u \rangle$. By Theorem 4.2.1, we have an equality in $\mathrm{Gr}^1 K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}_2)$ (or as Cartier divisors on $\mathcal{N}_2$ in this case),

$$\mathcal{Z}(L^\flat) = \mathcal{Z}(\varpi^{-1} L^\flat) + \mathcal{Z}(L^\flat)^\circ.$$

By Theorem 5.4.1 part (ii),

$$\mathrm{Int}(L^\flat \oplus \langle x\rangle) = \partial\mathrm{Den}(L^\flat \oplus \langle x\rangle),$$

and

$$\mathrm{Int}(\varpi^{-1}L^\flat \oplus \langle x\rangle) = \partial\mathrm{Den}(\varpi^{-1}L^\flat \oplus \langle x\rangle).$$

Therefore

$$\chi(\mathcal{N}_2, \mathcal{Z}(x) \overset{\mathbb{L}}{\cap} \mathcal{Z}(L^\flat)^\circ) = \partial\mathrm{Den}(L^\flat \oplus \langle x\rangle) - \partial\mathrm{Den}(\varpi^{-1}L^\flat \oplus \langle x\rangle)$$

and the assertion follows from the formula for local density in Corollary 3.5.3. □

6. Fourier transform: the geometric side

Let $L^\flat \subset \mathbb{V} = \mathbb{V}_n$ be an O_F-lattice of rank $n-1$. Let $L^\flat_F = L^\flat \otimes_{O_F} F \subset \mathbb{V}_n$ be the F-vector subspace of dimension $n-1$. Assume that $L^\flat_F$ is non-degenerate throughout the paper.

6.1. Horizontal versus Vertical cycles.

Recall from (5.2.1.1) that there is a decomposition of the derived special cycle ${}^{\mathbb{L}}\mathcal{Z}(L^\flat)$ into a sum of vertical and horizontal parts

$${}^{\mathbb{L}}\mathcal{Z}(L^\flat) = \mathcal{Z}(L^\flat)_{\mathscr{H}} + {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}},$$

and by Theorem 4.2.1, the horizontal part is a sum of quasi-canonical lifting cycles

$$\mathcal{Z}(L^\flat)_{\mathscr{H}} = \sum_{L'^\flat} \mathcal{Z}(L'^\flat)^\circ,$$

where the sum runs over all lattices $L'^\flat$ such that

$$L^\flat \subset L'^\flat \subset (L'^\flat)^\vee \subset L^\flat_F, \quad t(L'^\flat) \le 1.$$

Definition 6.1.1. Define the *horizontal part of the arithmetic intersection number*

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) := \chi(\mathcal{N}_n, \mathcal{Z}(x) \overset{\mathbb{L}}{\cap} \mathcal{Z}(L^\flat)_{\mathscr{H}}), \quad x \in \mathbb{V} \setminus L^\flat_F. \tag{6.1.1.1}$$

Definition 6.1.2. Analogously, define the *horizontal part of the derived local density*

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x) := \sum_{\substack{L^\flat \subset L' \subset L'^\vee \\ t(L'^\flat) \le 1}} \mathfrak{m}(t(L'))\mathbf{1}_{L'}(x), \quad x \in \mathbb{V} \setminus L^\flat_F, \tag{6.1.2.1}$$

where $L' \subseteq \mathbb{V}$ are O_F-lattices of rank n, and we denote

$$L'^\flat := L' \cap L^\flat_F \subset L^\flat_F. \tag{6.1.2.2}$$

Theorem 6.1.3. *As functions on $\mathbb{V} \setminus L^\flat_F$,*

$$\mathrm{Int}_{L^\flat,\mathscr{H}} = \partial\mathrm{Den}_{L^\flat,\mathscr{H}}.$$

Proof. By Corollary 5.4.6, for a fixed integral lattice $L'^\flat \subset L^\flat_F$ of type $t \le 1$, we have

$$\chi(\mathcal{N}_n, \mathcal{Z}(x) \overset{\mathbb{L}}{\cap} \mathcal{Z}(L'^\flat)^\circ) = \sum_{\substack{L'^\flat + \langle x\rangle \subset L' \subset L'^\vee, \\ L' \cap L^\flat_F = L'^\flat}} \mathfrak{m}(t(L')).$$

The assertion follows from Theorem 4.2.1 and the corresponding formula (6.1.2.1) for the horizontal part of the local density $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$. □

Definition 6.1.4. Define the *vertical part of the arithmetic intersection number*

$$\mathrm{Int}_{L^\flat,\mathscr{V}}(x) := \chi(\mathcal{N}_n, \mathcal{Z}(x) \overset{\mathbb{L}}{\cap} {}^{\mathbb{L}}\mathcal{Z}(L^\flat)_{\mathscr{V}}), \quad x \in \mathbb{V} \setminus L^\flat_F. \tag{6.1.4.1}$$

Then by [Zha21, (B.3)] there is a decomposition

$$\mathrm{Int}_{L^\flat}(x) = \mathrm{Int}_{L^\flat,\mathscr{H}}(x) + \mathrm{Int}_{L^\flat,\mathscr{V}}(x), \quad x \in \mathbb{V} \setminus L^\flat_F. \tag{6.1.4.2}$$

We will defer the vertical part of the derived local density to the next section (Definition 7.3.2).

6.2. **Computation of** $\mathrm{Int}_{\mathcal{V}(\Lambda)}$**.** Let $\Lambda \subseteq \mathbb{V}$ be a vertex lattice. Let $\mathcal{V}(\Lambda)$ be the Deligne–Lusztig variety in the Bruhat–Tits stratification of $\mathcal{N}_n^{\mathrm{red}}$ (§2.7). Define

$$\mathrm{Int}_{\mathcal{V}(\Lambda)}(x) := \chi\big(\mathcal{N}_n, \mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)\big), \quad x \in \mathbb{V} \setminus \{0\}. \tag{6.2.0.3}$$

Next we explicitly compute $\mathrm{Int}_{\mathcal{V}(\Lambda)}$ for $\Lambda \in \mathrm{Vert}^3$, i.e., for $\mathcal{V}(\Lambda)$ a Deligne–Lusztig curve.

Lemma 6.2.1. *Let* $\Lambda \in \mathrm{Vert}^3$*. Then*

$$\mathrm{Int}_{\mathcal{V}(\Lambda)} = -q^2(1+q)\mathbf{1}_\Lambda + \sum_{\Lambda \subset \Lambda',\, t(\Lambda')=1} \mathbf{1}_{\Lambda'}.$$

Proof. We note that

$$-q^2(1+q)\mathbf{1}_\Lambda(x) + \sum_{\Lambda \subset \Lambda',\, t(\Lambda')=1} \mathbf{1}_{\Lambda'}(x) = \begin{cases} (1-q^2), & x \in \Lambda, \\ 1, & x \in \Lambda^\vee \setminus \Lambda, \text{and } \mathrm{val}(x) \geq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{6.2.1.1}$$

We start with the simple observation that $\mathcal{Z}(x) \cap \mathcal{V}(\Lambda)$ is empty unless $x \in \Lambda^\vee$ and $\mathrm{val}(x) \geq 0$. In fact, if $\mathcal{Z}(x) \cap \mathcal{V}(\Lambda)$ is non-empty, the intersection must contain a point $\mathcal{V}(\Lambda')$ for a type 1 vertex lattice Λ'. Then $\Lambda \subset \Lambda'$ and by (2.7.0.1) we have $x \in \Lambda'$. It follows that $x \in \Lambda' \subset \Lambda'^\vee \subset V^\vee$.

To show (6.2.1.1), we first consider the special case $n = 3$. If $u \notin \Lambda$, then $\mathcal{Z}(u) \cap \mathcal{V}(\Lambda)$ is non-empty only when u lies in one of the type 1 lattices nested between Λ and $\Lambda^\vee$. Then the intersection number is equal to one by (5.4.5.5), and the desired equality follows.

Now assume $u \in \Lambda$ and $u \neq 0$. Choose an orthogonal basis $\{e_1, e_2, e_3\}$ of Λ (so the norms of them all have valuation one). Let L be the rank two lattice generated by e_1, e_2. Now we note that, by Theorem 4.2.1 and Theorem 5.4.1 part (i),

$$\mathcal{Z}(L) = \mathcal{V}(\Lambda) + \sum_{L \subset M = M^\vee \subset L^\vee} \mathcal{Z}(M),$$

where each of $\mathcal{Z}(M) \simeq \mathcal{N}_1$ since M is self-dual. There are exactly $q+1$ such M.

Let $u \in \Lambda \setminus \{0\}$, and write it in terms of the chosen basis

$$u = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3, \quad \lambda_i \in O_F.$$

Assume that $\lambda_3 \neq 0$, and let $a_3 = 2\,\mathrm{val}(\lambda_3) + 1$ (an odd integer). By [Ter13a], we may calculate all of the intersection numbers

$$\chi(\mathcal{N}_3, \mathcal{Z}(L) \overset{\mathbb{L}}{\cap} \mathcal{Z}(u)) = \frac{a_3+1}{2}(q+1) + (1-q^2),$$
$$\chi(\mathcal{N}_3, \mathcal{Z}(M) \overset{\mathbb{L}}{\cap} \mathcal{Z}(u)) = \frac{a_3+1}{2}.$$

It follows that

$$\chi(\mathcal{N}_3, \mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(u)) = (1 - q^2).$$

If $\lambda_3 = 0$, then we choose L to be the span of some other pairs of basis vectors, and we run the same computation. This proves the desired equality if $u \in \Lambda \setminus \{0\}$ and completes the proof when $n = 3$.

Now assume that $n > 3$. Since Λ is a vertex lattice of type 3, it admits an orthogonal direct sum decomposition

$$\Lambda = \Lambda^\flat \oplus M \tag{6.2.1.2}$$

where $\Lambda^\flat$ is a rank 3 vertex lattice of type 3, and M is a type 0 (i.e., self-dual) lattice of rank $n-3$. Then

$$\Lambda^\vee = \Lambda^{\flat,\vee} \oplus M$$

and any element $u \in \Lambda^\vee$ has a unique decomposition

$$u = u^\flat + u_M, \quad u^\flat \in \Lambda^{\flat,\vee},\, u_M \in M.$$

First assume that $u^\flat \neq 0$, i.e., $u \notin M$. Since M is self-dual, we have a natural embedding (2.11.0.1)

$$\delta_M\colon\ \mathcal{N}_3 \longrightarrow \mathcal{N}_n$$

which identifies $\mathcal{N}_3$ with the special cycle $\mathcal{Z}(M)$. Moreover, the Deligne–Lusztig curve $\mathcal{V}(\Lambda^\flat)$ on $\mathcal{N}_3$ is sent to $\mathcal{V}(\Lambda)$, and the special divisor $\mathcal{Z}(u)$ intersects properly with $\mathcal{N}_3$ and its pull-back to $\mathcal{N}_3$ is the special divisor $\mathcal{Z}(u^\flat)$, cf. (2.11.0.2).

We obtain (by the projection formula for the morphism δ_M)

$$\chi(\mathcal{N}_n, \mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(u)) = \chi(\mathcal{N}_3, \mathcal{V}(\Lambda^\flat) \overset{\mathbb{L}}{\cap} \mathcal{Z}(u^\flat)).$$

This reduces the case $u^\flat \neq 0$ to the case $n = 3$. In particular, when $u^\flat \in \Lambda^\flat \setminus \{0\}$,

$$\chi(\mathcal{N}_n, \mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(u)) = 1 - q^2. \tag{6.2.1.3}$$

Finally it remains to show that the intersection number is the constant $(1 - q^2)$ when $u \in \big(\Lambda^\flat \oplus M\big) \setminus \{0\}$. It suffices to show this when $u \in M \setminus \{0\}$. Choose an orthogonal basis $\{e_1, e_2, e_3\}$ for $\Lambda^\flat$, and $\{f_1, \cdots, f_{n-3}\}$ for M. Write

$$u = \mu_1 f_1 + \cdots + \mu_{n-3} f_{n-3}, \quad \mu_j \in O_F.$$

One of the μ_i is non-zero, and without loss of generality we assume $\mu_1 \neq 0$. Now define $\widetilde{M}$ to be the new lattice generated by $e_1 + f_1, f_2, \cdots, f_{n-3}$. It is self-dual, and its orthogonal complement $\widetilde{\Lambda}^\flat$ in Λ is again a type 3-lattice. Now replace the decomposition $\Lambda = \Lambda^\flat \oplus M$ by the new one $\Lambda = \widetilde{\Lambda}^\flat \oplus \widetilde{M}$. Then $u \notin \widetilde{M}$, and hence we can apply (6.2.1.3). This completes the proof. □

Corollary 6.2.2. *The function* $\mathrm{Int}_{L^\flat,\mathscr{V}}$ *extends to a (necessarily unique) function in* $C_c^\infty(\mathbb{V})$.

Proof. This follows from Corollary 5.3.3 and Lemma 6.2.1. □

6.3. **Fourier transform: the geometric side; "Local modularity".** We compute the Fourier transform of $\mathrm{Int}_{L^\flat,\mathscr{V}}$ as a function on $\mathbb{V}$.

Lemma 6.3.1. *Let $\Lambda \in \mathrm{Vert}^3$. Then the function $\mathrm{Int}_{\mathcal{V}(\Lambda)} \in C_c^\infty(\mathbb{V})$ satisfies*

$$\widehat{\mathrm{Int}_{\mathcal{V}(\Lambda)}} = \gamma_{\mathbb{V}} \, \mathrm{Int}_{\mathcal{V}(\Lambda)} \, .$$

Here $\gamma_{\mathbb{V}} = -1$ is the Weil constant.

Proof. By Lemma 6.2.1, we obtain

$$\begin{aligned}\widehat{\mathrm{Int}_{\mathcal{V}(\Lambda)}} &= -\mathrm{vol}(\Lambda)\cdot q^2(1+q)\cdot 1_{\Lambda^\vee} + \sum_{\Lambda\subset\Lambda',\, t(\Lambda')=1} \mathrm{vol}(\Lambda')\cdot 1_{\Lambda'^\vee} \\ &= -(1+q^{-1})\cdot 1_{\Lambda^\vee} + \sum_{\Lambda\subset\Lambda',\, t(\Lambda')=1} q^{-1}\cdot 1_{\Lambda'^\vee}.\end{aligned}$$

Now we compute its value at $u \in \mathbb{V}$ according to four cases.

(i) If $u \in \Lambda$, then there are exactly q^3+1 type 1 lattices Λ' containing Λ, and the value is

$$q^{-1}(q^3+1) - (1+q^{-1}) = q^2 - 1.$$

(ii) Assume that $u \in \Lambda_1 \setminus \Lambda$ for some $\Lambda_1 \in \mathrm{Vert}^1$, i.e., the image of $\bar{u}$ of u in $\Lambda^\vee/\Lambda$ is an isotropic vector. Notice that $u \in \Lambda'^\vee$ if and only if $\overline{u}$ is orthogonal to the line given by the image of $(\Lambda')^\vee$ in $\Lambda^\vee/\Lambda$. So there is exactly one such $\Lambda' \in \mathrm{Vert}^1$, i.e., $\Lambda' = \Lambda_1$, and we obtain the value

$$q^{-1} - (1+q^{-1}) = -1.$$

(iii) Assume that $u \in \Lambda^\vee \setminus \Lambda$ but $u \notin \Lambda_1 \setminus \Lambda$ for any $\Lambda_1 \in \mathrm{Vert}^1$. Then $\overline{u}$ is anisotropic in $\Lambda^\vee/\Lambda$. Notice that $\overline{u}^\perp$ is a non-degenerate hermitian space of dimension two, and Λ' corresponds to an isotropic line in $\overline{u}^\perp$. So there are exactly $q+1$ of such $\Lambda' \in \mathrm{Vert}^1$, and we obtain the value

$$q^{-1}(q+1) - (1+q^{-1}) = 0.$$

(iv) If $u \notin \Lambda^\vee$, then the value at u is

$$q^{-1}\cdot 0 - (1+q^{-1})\cdot 0 = 0.$$

This completes the proof by comparing with (6.2.1.1). □

Remark 6.3.2. It follows from Lemma 6.3.1 that $\mathrm{Int}_{\mathcal{V}(\Lambda)}$ is $\mathrm{SL}_2(O_{F_0})$-invariant under the Weil representation. This invariance may be viewed as a "local modularity", an analog of the global modularity of arithmetic generating series of special divisors (such as in [BHK+20]).

Corollary 6.3.3. *The function $\mathrm{Int}_{L^\flat,\mathscr{V}} \in C_c^\infty(\mathbb{V})$ satisfies*

$$\widehat{\mathrm{Int}_{L^\flat,\mathscr{V}}} = \gamma_{\mathbb{V}} \, \mathrm{Int}_{L^\flat,\mathscr{V}} \, .$$

Proof. This follows from Corollary 5.3.3 and Lemma 6.3.1. □

6.4. **Fourier transform: the geometric side; "Higher local modularity".** In this subsection we generalize Lemmas 6.2.1 and 6.3.1 on the function $\mathrm{Int}_{\mathcal{V}(\Lambda)}$ for vertex lattices Λ of type 3 to vertex lattices Λ of arbitrary type $t(\Lambda) = 2d+1 \geq 3$.

Let

$$\mathrm{ch} : K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}} \to \bigoplus_{i=0}^{d} \mathrm{Ch}^i(\mathcal{V}(\Lambda))_{\mathbb{Q}}$$

be the Chern character from the Grothendieck ring to the Chow ring of $\mathcal{V}(\Lambda)$, which is an isomorphism of graded rings. In particular, it induces an isomorphism

$$\mathrm{ch}_i : \mathrm{Gr}^i K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Ch}^i(\mathcal{V}(\Lambda))_{\mathbb{Q}},$$

for $0 \leq i \leq d$. Let

$$\mathrm{cl}_i : \mathrm{Ch}^i(\mathcal{V}(\Lambda))_{\mathbb{Q}} \to H^{2i}(\mathcal{V}(\Lambda), \mathbb{Q}_\ell)(i)$$

be the ℓ-adic cycle class map and let

$$\mathrm{cl} = \bigoplus_{i=0}^{d} \mathrm{cl}_i : \bigoplus_{i=0}^{d} \mathrm{Ch}^i(\mathcal{V}(\Lambda))_{\mathbb{Q}} \to \bigoplus_{i=0}^{d} H^{2i}(\mathcal{V}(\Lambda), \mathbb{Q}_\ell)(i).$$

Then cl intertwines the intersection product on the Chow ring and the cup product on the cohomology ring, namely the following diagram commutes,

$$\begin{tikzcd}
\mathrm{Gr}^i K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}} \arrow[d, "\mathrm{ch}_i", "\wr"'] & \times & \mathrm{Gr}^j K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}} \arrow[r, "\cdot"] \arrow[d, "\mathrm{ch}_j", "\wr"'] & \mathrm{Gr}^{i+j} K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}} \arrow[d, "\mathrm{ch}_{i+j}", "\wr"'] \\
\mathrm{Ch}^i(\mathcal{V}(\Lambda))_{\mathbb{Q}} \arrow[d, "\mathrm{cl}_i"] & \times & \mathrm{Ch}^j(\mathcal{V}(\Lambda))_{\mathbb{Q}} \arrow[r, "\cdot"] \arrow[d, "\mathrm{cl}_j"] & \mathrm{Ch}^{i+j}(\mathcal{V}(\Lambda))_{\mathbb{Q}} \arrow[d, "\mathrm{cl}_{i+j}"] \\
H^{2i}(\mathcal{V}(\Lambda)), \mathbb{Q}_\ell)(i) & \times & H^{2j}(\mathcal{V}(\Lambda), \mathbb{Q}_\ell)(j) \arrow[r, "\cup"] & H^{2(i+j)}(\mathcal{V}(\Lambda), \mathbb{Q}_\ell)(i+j).
\end{tikzcd} \tag{6.4.0.1}$$

Denote by $\mathrm{Tate}_\ell^{2i}(\mathcal{V}(\Lambda)) \subseteq H^{2i}(\mathcal{V}(\Lambda), \mathbb{Q}_\ell)(i)$ the subspace of Tate classes, i.e., the elements fixed by $\mathbf{F}^s$ for some power $s \geq 1$. Then by Theorem 5.3.2, we have the identity

$$\mathrm{im}(\mathrm{cl}_i)_{\mathbb{Q}_\ell} = \mathrm{Tate}_\ell^{2i}(\mathcal{V}(\Lambda)),$$

and moreover $\mathrm{Tate}_\ell^{2i}(\mathcal{V}(\Lambda))$ is spanned by the cycle classes of $\mathcal{V}(\Lambda') \subseteq \mathcal{V}(\Lambda)$, where $\Lambda' \supseteq \Lambda$ runs over vertex lattices of type $2(d-i)+1$. Denote by

$$\overline{K}_0(\mathcal{V}(\Lambda)) := K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}}/\ker(\mathrm{cl} \circ \mathrm{ch}), \quad \overline{\mathrm{Ch}}^i(\mathcal{V}(\Lambda)) := \mathrm{Ch}^i(\mathcal{V}(\Lambda))_{\mathbb{Q}}/\ker \mathrm{cl}_i. \tag{6.4.0.2}$$

Then ch and cl induce isomorphisms

$$\mathrm{ch} : \overline{K}_0(\mathcal{V}(\Lambda)) \xrightarrow{\sim} \bigoplus_{i=0}^{d} \overline{\mathrm{Ch}}^i(\mathcal{V}(\Lambda)), \quad \mathrm{cl} : \bigoplus_{i=0}^{d} \overline{\mathrm{Ch}}^i(\mathcal{V}(\Lambda))_{\mathbb{Q}_\ell} \xrightarrow{\sim} \bigoplus_{i=0}^{d} \mathrm{Tate}_\ell^{2i}(\mathcal{V}(\Lambda)). \tag{6.4.0.3}$$

By Theorem 5.3.2 (ii) and that the cup product is $\mathbf{F}$-equivariant, the Poincaré duality induces a perfect pairing

$$\cup : \mathrm{Tate}_\ell^{2i}(\mathcal{V}(\Lambda)) \times \mathrm{Tate}_\ell^{2d-2i}(\mathcal{V}(\Lambda)) \to \mathbb{Q}_\ell. \tag{6.4.0.4}$$

Definition 6.4.1. For $x \in \mathbb{V} \setminus \{0\}$, define $\mathrm{K}_{\mathcal{V}(\Lambda)}(x) \in \overline{K}_0(\mathcal{V}(\Lambda))$ to be the image of

$$\mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x) \in K_0^{\mathcal{V}(\Lambda)}(\mathcal{N}_n) \xrightarrow{\sim} K_0(\mathcal{V}(\Lambda))$$

under (6.4.0.2).

Remark 6.4.2. Our main result in this subsection (Theorem 6.4.9) shows that the function $\mathrm{K}_{\mathcal{V}(\Lambda)}$ satisfies the local modularity analogous to Lemma 6.3.1.

Since $\mathcal{Z}(x)$ is a Cartier divisor on $\mathcal{N}_n$, we know that $\mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)$ is explicitly represented by the two-term complex of line bundles on $\mathcal{V}(\Lambda)$,

$$[\mathcal{O}_{\mathcal{N}_n}(-\mathcal{Z}(x))|_{\mathcal{V}(\Lambda)} \to \mathcal{O}_{\mathcal{N}_n}|_{\mathcal{V}(\Lambda)}] \in \mathrm{F}^1 K_0(\mathcal{V}(\Lambda)).$$

Thus we have the Chern character

$$\begin{aligned}
\mathrm{ch}(\mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)) &= \mathrm{ch}(\mathcal{O}_{\mathcal{N}_n}(-\mathcal{Z}(x))|_{\mathcal{V}(\Lambda)}) - \mathrm{ch}(\mathcal{O}_{\mathcal{V}(\Lambda)}) \\
&= \exp(c_1(\mathcal{O}_{\mathcal{N}_n}(-\mathcal{Z}(x))|_{\mathcal{V}(\Lambda)})) - \exp(c_1(\mathcal{O}_{\mathcal{V}(\Lambda)})) \\
&= \sum_{i=1}^{d} \frac{c_1(\mathcal{O}_{\mathcal{N}_n}(-\mathcal{Z}(x))|_{\mathcal{V}(\Lambda)})^i}{i!}.
\end{aligned}$$

Definition 6.4.3. For $x \in \mathbb{V} \setminus \{0\}$, define

$$c_{1,\mathcal{V}(\Lambda)}(x) := c_1(\mathcal{O}_{\mathcal{N}_n}(-\mathcal{Z}(x))|_{\mathcal{V}(\Lambda)}) \in \mathrm{Ch}^1(\mathcal{V}(\Lambda))_{\mathbb{Q}},$$

and

$$c_{\mathcal{V}(\Lambda)}(x) := c_1(\mathcal{O}_{\mathcal{N}_n}(-\mathcal{Z}(x))|_{\mathcal{V}(\Lambda)})^d \in \mathrm{Ch}^d(\mathcal{V}(\Lambda))_{\mathbb{Q}} \xrightarrow[\sim]{\deg} \mathbb{Q}.$$

Lemma 6.4.4. *The function $c_{1,\mathcal{V}(\Lambda)}$ (resp. $c_{\mathcal{V}(\Lambda)}$) is Λ-invariant, under the translation by Λ. In particular, the function $c_{1,\mathcal{V}(\Lambda)}$ (resp. $c_{\mathcal{V}(\Lambda)}$) extends uniquely to an Λ-invariant function on $\mathbb{V}$, or equivalently, a function on $\mathbb{V}/\Lambda$ (still denoted by the same symbol).*

Proof. It suffices to show that $\mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x) \in K_0^{\mathcal{V}(\Lambda)}(\mathcal{N}_n)$ is Λ-invariant, i.e.,

$$\mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x)} = \mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+y)} \tag{6.4.4.1}$$

for any $x \neq 0$ and any $y \in \Lambda$ such that $x + y \neq 0$.

First assume that x, y are linearly independent. Then by Corollary 2.8.2, we have

$$\mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x)} = \mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+y)}.$$

Thus the left-hand-side of (6.4.4.1)

$$\mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x)} = \mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{Z}(y)}} \mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x)}$$

equals the right-hand-side of (6.4.4.1)

$$\mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+y)} = \mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{Z}(y)}} \mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+y)}$$

as desired.

It remains to consider the case that x, y are linearly dependent. Choose $x_1 \in \Lambda$ linearly independent of x. Then

$$\mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x)} = \mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+x_1)}.$$

Since $x + x_1$ is linearly independent of y, we obtain

$$\mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+x_1)} = \mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+x_1+y)}.$$

Since $x + y \neq 0$, $x + y$ is linearly independent of x_1 and hence

$$\mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+x_1+y)} = \mathcal{O}_{\mathcal{Z}(\Lambda)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x+y)}.$$

This completes the proof. □

Lemma 6.4.5. *Let $\Lambda \in \mathrm{Vert}^{2d+1}$. Then for any $x \in \Lambda$, we have*

$$c_{1,\mathcal{V}(\Lambda)}(x) = -\frac{1}{1+q^{2d+1}} \sum_{\substack{y \in V_\Lambda \setminus \{0\} \\ (y,y)=0}} c_{1,\mathcal{V}(\Lambda)}(y) \in \mathrm{Ch}^1(\mathcal{V}(\Lambda))_{\mathbb{Q}},$$

where $V_\Lambda = \Lambda^\vee/\Lambda$ is a k_F/k-hermitian space of dimension $2d+1$ (see §2.6).

Proof. Since the cycle class map for divisors $\mathrm{cl}_1 : \mathrm{Ch}^1(\mathcal{V}(\Lambda))_{\mathbb{Q}} \to H^2(\mathcal{V}(\Lambda), \mathbb{Q}_\ell)(1)$ is injective, we know from (6.4.0.3) that

$$\mathrm{Ch}^1(\mathcal{V}(\Lambda))_{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathrm{Tate}^2_\ell(\mathcal{V}(\Lambda)).$$

It follows from the perfect pairing (6.4.0.4) that to show the desired identity it suffices to show that for any Deligne–Lusztig curve $\mathcal{V}(\Lambda') \subseteq \mathcal{V}(\Lambda)$ ($t(\Lambda') = 3$), the following identity

$$c_{1,\mathcal{V}(\Lambda)}(x) \cdot [\mathcal{V}(\Lambda')] = -\frac{1}{1+q^{2d+1}} \sum_{\substack{y \in V_\Lambda \setminus \{0\} \\ (y,y)=0}} c_{1,\mathcal{V}(\Lambda)}(y) \cdot [\mathcal{V}(\Lambda')] \tag{6.4.5.1}$$

holds in $\mathrm{Ch}^d(\mathcal{V}(\Lambda))_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}$. By the projection formula,

$$c_{1,\mathcal{V}(\Lambda)}(x) \cdot [\mathcal{V}(\Lambda')] = c_{1,\mathcal{V}(\Lambda')}(x), \quad c_{1,\mathcal{V}(\Lambda)}(y) \cdot [\mathcal{V}(\Lambda')] = c_{1,\mathcal{V}(\Lambda')}(y).$$

Since $t(\Lambda') = 3$, we know from Lemma 6.2.1 that

$$c_{1,\mathcal{V}(\Lambda')}(x) = \mathrm{Int}_{\mathcal{V}(\Lambda')}(x) = (1-q^2), \quad c_{1,\mathcal{V}(\Lambda')}(y) = \mathrm{Int}_{\mathcal{V}(\Lambda')}(y) = \begin{cases} (1-q^2), & y \in \Lambda'/\Lambda, \\ 1, & y \in \Lambda'^\vee/\Lambda, y \notin \Lambda'/\Lambda, \\ 0, & y \notin \Lambda'^\vee/\Lambda. \end{cases}$$

Since $\Lambda'/\Lambda \subseteq V_\Lambda$ is totally isotropic, the number of nonzero isotropic vectors $y \in \Lambda'/\Lambda$ equals $\#(\Lambda'/\Lambda) - 1 = q^{2(d-1)} - 1$. The number of isotropic vectors $y \in \Lambda'^\vee/\Lambda$, $y \notin \Lambda'/\Lambda$ equals $\#(\Lambda'/\Lambda)$ times the number of nonzero isotropic vectors in $\Lambda'^\vee/\Lambda'$, which evaluates to $q^{2(d-1)} \cdot (1+q^3)(q^2-1)$. It follows that

$$\sum_{\substack{y \in V_\Lambda \setminus \{0\} \\ (y,y)=0}} c_{1,\mathcal{V}(\Lambda)}(y) \cdot [\mathcal{V}(\Lambda')] = (q^{2(d-1)} - 1) \cdot (1-q^2) + q^{2(d-1)}(1+q^3)(q^2-1) = -(1-q^2)(1+q^{2d+1}).$$

Hence

$$-\frac{1}{1+q^{2d+1}} \sum_{\substack{y \in V_\Lambda \setminus \{0\} \\ (y,y)=0}} c_{1,\mathcal{V}(\Lambda)}(y) \cdot [\mathcal{V}(\Lambda')] = (1-q^2) = c_{1,\mathcal{V}(\Lambda)}(x) \cdot [\mathcal{V}(\Lambda')],$$

and the desired identity (6.4.5.1) holds. ☐

Lemma 6.4.6. *Let $\Lambda \in \mathrm{Vert}^{2d+1}$. Then*

$$c_{\mathcal{V}(\Lambda)}(x) = c(d) \cdot \begin{cases} (1-q^{2d}), & x \in \Lambda, \\ 1, & x \in \Lambda^\vee \setminus \Lambda, \mathrm{val}(x) \geq 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Here $c(1) = 1$ and $c(d) = \prod_{i=1}^{d-1}(1-q^{2i})$.

Proof. We induct on d. The base case $d = 1$ follows from Lemma 6.2.1. By the same proof as in Lemma 6.2.1, we know that $c_{\mathcal{V}(\Lambda)}(x) = 0$ unless $x \in \Lambda^\vee$ and $\mathrm{val}(x) \geq 0$. By the Λ-invariance of $c_{\mathcal{V}(\Lambda)}$ in Lemma 6.4.4, to show the result it remains to show that

$$c_{\mathcal{V}(\Lambda)}(0) = c(d)(1-q^{2d}), \quad c_{\mathcal{V}(\Lambda)}(x) = c(d) \tag{6.4.6.1}$$

for any $x \in V_\Lambda \setminus \{0\}$ with $(x,x) = 0$.

By Lemma 6.4.5, we have

$$c_{\mathcal{V}(\Lambda)}(0) = c_{1,\mathcal{V}(\Lambda)}(0)^{d-1} c_{1,\mathcal{V}(\Lambda)}(0) = c_{1,\mathcal{V}(\Lambda)}(0)^{d-1} \left(-\frac{1}{1+q^{2d+1}} \sum_{\substack{y \in V_\Lambda \setminus \{0\} \\ (y,y)=0}} c_{1,\mathcal{V}(\Lambda)}(y) \right).$$

By the projection formula, we have

$$c_{1,\mathcal{V}(\Lambda)}(0)^{d-1} c_{1,\mathcal{V}(\Lambda)}(y) = c_{1,\mathcal{V}(\Lambda+\langle y \rangle)}(0)^{d-1},$$

which by induction equals

$$c_{\mathcal{V}(\Lambda+\langle y \rangle)}(0) = c(d-1)(1-q^{2(d-1)})$$

since $t(\Lambda + \langle y \rangle) = 2d-1$. The number of nonzero isotropic vectors $y \in V_\Lambda$ equals $(1+q^{2d+1})(q^{2d}-1)$. Hence

$$c_{\mathcal{V}(\Lambda)}(0) = (q^{2d+1}+1)(q^{2d}-1) \cdot \left(-\frac{1}{1+q^{2d+1}} c(d-1)(1-q^{2(d-1)}) \right) = (1-q^{2d})(1-q^{2(d-1)}) \cdot c(d-1).$$

On the other hand, for any $x \in V_\Lambda \setminus \{0\}$ with $(x,x) = 0$. by the projection formula, we have

$$c_{\mathcal{V}(\Lambda)}(x) = c_{1,\mathcal{V}(\Lambda)}(x)^{d-1} c_{1,\mathcal{V}(\Lambda)}(x) = c_{1,\mathcal{V}(\Lambda+\langle x \rangle)}(x)^{d-1},$$

which by induction equals

$$c_{\mathcal{V}(\Lambda+\langle x \rangle)}(x) = c(d-1)(1-q^{2(d-1)})$$

since $t(\Lambda + \langle x \rangle) = 2d-1$. The desired identity (6.4.6.1) then follows as $c(d-1)(1-q^{2(d-1)}) = c(d)$. ☐

Lemma 6.4.7. *Let $\Lambda \in \mathrm{Vert}^{2d+1}$. Then*

$$c_{\mathcal{V}(\Lambda)} = \frac{c(d)}{c'(d)} \sum_{\substack{\Lambda' \in \mathrm{Vert}^3 \\ \Lambda' \supseteq \Lambda}} \mathrm{Int}_{\mathcal{V}(\Lambda')}.$$

Here $c'(1) = 1$ and $c'(d) = \prod_{i=2}^{d}(1+q^{2i+1})$ when $d \geq 2$.

Proof. We distinguish three cases.

(i) For $x \in \Lambda$, we have $\mathrm{Int}_{\mathcal{V}(\Lambda')}(x) = 1 - q^2$ for any Λ' in the sum by Lemma 6.2.1. The number of such Λ' is the number of $(d-1)$-dimensional totally isotropic subspaces in V_Λ, which equals $S_{2d+1,d-1}$ (in the notation of Lemma 1.9.1). Hence the right-hand-side evaluates to

$$\frac{c(d)}{c'(d)} S_{2d+1,d-1}(1-q^2) = \frac{c(d)}{\prod_{i=2}^{d}(1+q^{2i+1})} \frac{\prod_{i=4}^{2d+1}(1-(-q)^i)}{\prod_{i=1}^{d-1}(1-q^{2i})}(1-q^2) = c(d)(1-q^{2d}),$$

which equals $c_{\mathcal{V}(\Lambda)}(x)$ by Lemma 6.4.6.

(ii) For $x \in \Lambda^\vee \setminus \Lambda$ with $\mathrm{val}(x) \geq 0$, we have

$$\mathrm{Int}_{\mathcal{V}(\Lambda')}(x) = \begin{cases} (1-q^2), & x \in \Lambda', \\ 1, & x \in \Lambda'^\vee \setminus \Lambda'. \end{cases}$$

The number of Λ' such that $x \in \Lambda'$ is the number of $(d-2)$-dimensional totally isotropic subspaces in $V_{\Lambda + \langle x \rangle}$, which equals $S_{2d-1,d-2}$. The number of Λ' such that $x \in \Lambda'^\vee \setminus \Lambda'$ is the number of $(d-1)$-dimensional totally isotropic subspaces $W \subseteq V_\Lambda$ such that $x \notin W$ but $x \in W^\perp$. In this case the map $W \mapsto W + \langle x \rangle / \langle x \rangle$ gives a surjection onto the set of $(d-1)$-dimensional totally isotropic subspaces in $\langle x \rangle^\perp / \langle x \rangle$, whose fiber has size equal to the number of $(d-1)$-dimensional subspaces of $W + \langle x \rangle$ not containing $\langle x \rangle$. Hence the number of such W is equal to $S_{2d-1,d-1} \cdot q^{2d-2}$. So the right-hand-side evaluates to

$$\begin{aligned} &\frac{c(d)}{c'(d)}(S_{2d-1,d-2}(1-q^2) + S_{2d-1,d-1} \cdot q^{2d-2}) \\ &= \frac{c(d)}{\prod_{i=2}^{d}(1+q^{2i+1})} \left(\frac{\prod_{i=4}^{2d-1}(1-(-q)^i)}{\prod_{i=1}^{d-2}(1-q^{2i})}(1-q^2) + \frac{\prod_{i=2}^{2d-1}(1-(-q)^i)}{\prod_{i=1}^{d-1}(1-q^{2i})} q^{2d-2} \right) \\ &= c(d), \end{aligned}$$

which equals $c_{\mathcal{V}(\Lambda)}(x)$ by Lemma 6.4.6.

(iii) If $x \notin \Lambda^\vee$ or $\mathrm{val}(x) < 0$, then both sides are zero. ☐

Corollary 6.4.8. *Let $\Lambda \in \mathrm{Vert}^{2d+1}$. Then $c_{\mathcal{V}(\Lambda)} \in C_c^\infty(\mathbb{V})$ satisfies*

$$\widehat{c_{\mathcal{V}(\Lambda)}} = \gamma_{\mathbb{V}}\, c_{\mathcal{V}(\Lambda)}.$$

Proof. It follows immediately from Lemmas 6.4.7 and 6.3.1. ☐

Theorem 6.4.9 (K-theoretic local modularity)**.** *Let $\Lambda \in \mathrm{Vert}^{2d+1}$. For any linear map $l : \overline{K}_0(\mathcal{V}(\Lambda)) \to \mathbb{Q}$, the function $l \circ \mathrm{K}_{\mathcal{V}(\Lambda)}$ extends to a (necessarily unique) function in $C_c^\infty(\mathbb{V})$ and satisfies*

$$\widehat{l \circ \mathrm{K}_{\mathcal{V}(\Lambda)}} = \gamma_{\mathbb{V}}\, l \circ \mathrm{K}_{\mathcal{V}(\Lambda)}.$$

Here, we refer to Definition 6.4.1 for $\mathrm{K}_{\mathcal{V}(\Lambda)}$.

Proof. Since the Tate classes are spanned by the cycle classes of Deligne–Lusztig subvarieties $\mathcal{V}(\Lambda') \subseteq \mathcal{V}(\Lambda)$, it follows that from the perfect pairing (6.4.0.4) that under the identification (6.4.0.3) any linear map $\overline{K}_0(\mathcal{V}(\Lambda)) \to \mathbb{Q}$ is a linear combination of linear maps

$$l : \overline{\mathrm{Ch}}^i(\mathcal{V}(\Lambda)) \to \mathbb{Q}$$

given by the intersection product with $[\mathcal{V}(\Lambda')]$ for varying $\mathcal{V}(\Lambda') \subseteq \mathcal{V}(\Lambda)$. So we may assume l has the form

$$l = \cdot[\mathcal{V}(\Lambda')] : \overline{\mathrm{Ch}}^{d-d'}(\mathcal{V}(\Lambda)) \to \mathbb{Q},$$

where $d' = \dim \mathcal{V}(\Lambda') \le d$. Then

$$l \circ \mathrm{K}_{\mathcal{V}(\Lambda)}(x) = \mathrm{ch}_{d'}(\mathcal{V}(\Lambda) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)) \cdot [\mathcal{V}(\Lambda')]$$

which by the projection formula equals to

$$l \circ \mathrm{K}_{\mathcal{V}(\Lambda)}(x) = \mathrm{ch}_{d'}(\mathcal{V}(\Lambda') \overset{\mathbb{L}}{\cap} \mathcal{Z}(x)) = \frac{c_{\mathcal{V}(\Lambda')}(x)}{d'!}.$$

By induction on d, we may assume $d = d'$ and we are reduced to show that $c_{\mathcal{V}(\Lambda)} \in C_c^\infty(\mathbb{V})$ satisfies

$$\widehat{c_{\mathcal{V}(\Lambda)}} = \gamma_{\mathbb{V}}\, c_{\mathcal{V}(\Lambda)}.$$

This is exactly Corollary 6.4.8. $\square$

Now we return to the function $\mathrm{Int}_{\mathcal{V}(\Lambda)}$ defined by (6.2.0.3).

Corollary 6.4.10 (Higher local modularity)**.** *Let* $\Lambda \in \mathrm{Vert}^{2d+1}$. *Then* $\mathrm{Int}_{\mathcal{V}(\Lambda)}$ *extends to a (necessarily unique) function in* $C_c^\infty(\mathbb{V})$ *and satisfies*

$$\widehat{\mathrm{Int}_{\mathcal{V}(\Lambda)}} = \gamma_{\mathbb{V}} \, \mathrm{Int}_{\mathcal{V}(\Lambda)}\, .$$

Proof. Consider the linear map given by the Euler–Poincaré characteristic

$$\chi : K_0(\mathcal{V}(\Lambda))_{\mathbb{Q}} \to \mathbb{Q}, \quad [\mathcal{F}] \longmapsto \chi(\mathcal{V}(\Lambda), \mathcal{F}).$$

By the Grothendieck–Riemann–Roch theorem, we have $\chi(\mathcal{V}(\Lambda), \mathcal{F}) = \deg(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{Td}(\mathcal{V}(\Lambda)))$, where $\mathrm{Td}(\mathcal{V}(\Lambda))$ is the Todd class of the tangent bundle of $\mathcal{V}(\Lambda)$. It follows from (6.4.0.1) that χ factors through $\overline{K}_0(\mathcal{V}(\Lambda))$ and thus defines a linear map $\chi : \overline{K}_0(\mathcal{V}(\Lambda)) \to \mathbb{Q}$. By definition we have $\chi \circ \mathrm{K}_{\mathcal{V}(\Lambda)} = \mathrm{Int}_{\mathcal{V}(\Lambda)}$. The result then follows from Theorem 6.4.9. $\square$

Remark 6.4.11. Corollary 6.4.10 allows us to give an alternative proof of Corollary 6.3.3 without a priori knowing that only the $(n-1)$-th graded piece of the derived Kudla–Rapoport cycle contributes to $\mathrm{Int}_{L^\flat, \mathscr{V}}(x)$ in the decomposition (6.1.4.2), in particular, without using [Zha21, (B.3)] for a formal scheme.

7. Fourier transform: the analytic side

In this section we allow F_0 to be a non-archimedean local field of characteristic not equal to 2 (but possibly with residual characteristic 2), and F an unramified quadratic extension.

7.1. **Lattice-theoretic notations.** Recall that $\mathbb{V} = \mathbb{V}_n$ is the hermitian space defined in §2.2 (in particular it is non-split). We let $L^\flat \subset \mathbb{V} = \mathbb{V}_n$ be an integral O_F-lattice of rank $n-1$, such that $L^\flat_F$ is non-degenerate. Define

$$(L^\flat)^{\vee,\circ} = \{x \in (L^\flat)^\vee \mid (x,x) \in O_F\}. \tag{7.1.0.1}$$

The fundamental invariants of $L^\flat$ are denoted by

$$(a_1, \cdots, a_{n-1}) \in \mathbb{Z}^{n-1},$$

where $0 \le a_1 \le \cdots \le a_{n-1}$. Denote the largest invariant by

$$e_{\max}(L^\flat) = a_{n-1}. \tag{7.1.0.2}$$

Let

$$M = M(L^\flat) = L^\flat \obot \langle u \rangle \tag{7.1.0.3}$$

be the lattice characterized by the following condition: $u \in \mathbb{V}$ is a vector satisfying $u \perp L^\flat$ and with valuation a_{n-1} or $a_{n-1}+1$ (only one of these two is possible due to the parity of $\mathrm{val}(\det(\mathbb{V}))$). In other words, the rank one lattice $\langle u \rangle$ is the set of all $x \perp L^\flat$ with $\mathrm{val}(x) \ge a_{n-1}$. Then the fundamental invariants of $M(L^\flat)$ are

$$(a_1, \cdots, a_{n-1}, a_{n-1}), \quad \text{or} \quad (a_1, \cdots, a_{n-1}, a_{n-1}+1).$$

Finally we note that, if $L^\flat \subset L'^\flat$ are two integral lattices of rank $n-1$, then

$$e_{\max}(L'^\flat) \le e_{\max}(L^\flat) \tag{7.1.0.4}$$

and $M(L^\flat) \subset M(L'^\flat)$. The above inequality follows from the characterization of $-e_{\max}(L^\flat)$ as the minimal valuation of vectors in the lattice $(L^\flat)^\vee$.

7.2. **Lemmas on lattices.** In this subsection, we do not require the lattice $L^\flat$ to be integral.

Lemma 7.2.1. *Let $L'^\flat \subset L^\flat_F$ be an O_F-lattice (of rank $n-1$). Denote*

$$\mathrm{Lat}(L'^\flat) := \{O_F\text{-lattices } L' \subset \mathbb{V} \mid \mathrm{rank}\, L' = n, \quad L'^\flat = L' \cap L^\flat_F\}.$$

Then there is a bijection

$$\begin{aligned} [(\mathbb{V}/L'^\flat) \setminus (L'^\flat_F/L'^\flat)]/O_F^\times &\xrightarrow{\sim} \mathrm{Lat}(L'^\flat) \\ u &\longmapsto L'^\flat + \langle u \rangle. \end{aligned} \tag{7.2.1.1}$$

Proof. The indicated map is well-defined and clearly injective. To show the surjectivity, we note that $L'/L'^\flat$ is free for any $L' \in \mathrm{Lat}(L'^\flat)$. Choose any element $u \in L'$ whose image in $L'/L'^\flat$ is a generator. Then it is clear that $L' = L'^\flat + \langle u \rangle$. $\square$

Let $\langle x \rangle_F = F\,x$ be the F-line generated by $x \in \mathbb{V} \setminus L^\flat_F$. Corresponding to the (not necessarily orthogonal) decomposition $\mathbb{V} = L^\flat_F \oplus \langle x \rangle_F$, there are two projection maps

$$\pi_\flat \colon\ \mathbb{V} \longrightarrow L^\flat_F\ , \quad \pi_x \colon\ \mathbb{V} \longrightarrow \langle x \rangle_F.$$

Lemma 7.2.2. *Let $L' \subset \mathbb{V}$ be an O_F-lattice (of rank n). Denote*

$$L'^{\flat} = L' \cap L^{\flat}_F, \quad L'_x = L' \cap \langle x \rangle_F.$$

The natural projection maps induce isomorphisms of O_F-modules

$$\pi_\flat(L')/L'^{\flat} \xleftarrow{\sim} L'/(L'^{\flat} \oplus L'_x) \xrightarrow{\sim} \pi_x(L')/L'_x.$$

In particular, all three abelian groups are O_F-cyclic modules.

Proof. Consider the map

$$\phi\colon\ L' \longrightarrow \pi_x(L')/L'_x.$$

We show that the kernel of ϕ is $L'^{\flat} \oplus L'_x$; the other assertion can be proved similarly.

Let $u \in L'$ and write $u = u^\flat + u^\natural$ uniquely for $u^\flat \in L^\flat_F$, $u^\natural \in \langle x \rangle_F$. Then $\phi(u) = u^\natural \mod L'_x$. If $u \in \ker(\phi)$, then $u^\natural \in L'$. It follows that $u^\flat = u - u^\natural \in L'$, and hence $u^\flat \in L'^\flat$. Therefore $u \in L'^\flat \oplus L'_x$ and $\ker(\phi) \subset L'^\flat \oplus L'_x$. Conversely, if $u \in L'^\flat \oplus L'_x$, then $u^\flat \in L'^\flat, u^\natural \in L'_x$, and clearly $\phi(u) = 0$. This completes the proof. □

Now we assume that $x \perp L^\flat$. We rename the projection map to the line $\langle x \rangle_F = (L^\flat_F)^\perp$ as $\pi_\perp$. Then we have a formula relating the volume of L' to that of $L'^\flat = L' \cap L^\flat_F$ and of the image of the projection $\pi_\perp$ (the analog of "base × height" formula for the area of a parallelogram)

$$\mathrm{vol}(L') = \mathrm{vol}(L'^\flat)\,\mathrm{vol}(\pi_\perp(L')). \tag{7.2.2.1}$$

In fact, this is clear if $L' = L'^\flat \obot \pi_\perp(L')$ and in general we obtain the formula by Lemma 7.2.2:

$$\frac{\mathrm{vol}(L')}{\mathrm{vol}(L'^\flat \obot L'_x)} = \frac{\mathrm{vol}(\pi_\perp(L'))}{\mathrm{vol}(L'_x)}.$$

7.3. **Local constancy of $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$.** We now resume the convention in §7.1. In particular, $L^\flat$ is now an integral lattice. When $\mathrm{rank}\, L = n$ with $\mathrm{val}(L)$ odd, recall that the derived local density is (Corollary 3.5.3)

$$\partial\mathrm{Den}(L) = \sum_{L \subset L' \subset L'^\vee} \mathfrak{m}(t(L')),$$

where

$$\mathfrak{m}(a) = \begin{cases} (1+q)(1-q^2)\cdots(1-(-q)^{a-1}), & a \geq 2, \\ 1, & a \in \{0,1\}. \end{cases}$$

Definition 7.3.1. For $x \in \mathbb{V} \setminus L^\flat_F$, define

$$\partial\mathrm{Den}_{L^\flat}(x) \coloneqq \partial\mathrm{Den}(L^\flat + \langle x \rangle).$$

Then

$$\partial\mathrm{Den}_{L^\flat}(x) = \sum_{L^\flat \subset L' \subset L'^\vee} \mathfrak{m}(t(L'))\mathbf{1}_{L'}(x), \tag{7.3.1.1}$$

where the sum is over all integral lattices $L' \subset \mathbb{V}$ of rank n. Note that this is a finite sum for x in a compact subset of $\mathbb{V} \setminus L^\flat_F$. However, when varying $x \in \mathbb{V} \setminus L^\flat_F$, infinitely many L' can appear.

Definition 7.3.2. Recall that we have defined the horizontal part $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ in Definition 6.1.2. Now define the *vertical part of the derived local density*

$$\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x) := \partial\mathrm{Den}_{L^\flat}(x) - \partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x), \quad x \in \mathbb{V} \setminus L_F^\flat. \tag{7.3.2.1}$$

Obviously the functions $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ and $\partial\mathrm{Den}_{L^\flat}$ are locally constant on $\mathbb{V} \setminus L_F^\flat$. Hence $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ is also locally constant on $\mathbb{V} \setminus L_F^\flat$.

Definition 7.3.3.

(a) Let $\mathrm{L}^1_{\mathrm{c}}(\mathbb{V})$ be the space of integrable functions that are defined on a dense open subset of $\mathbb{V}$ and vanish outside a compact subset of $\mathbb{V}$.

(b) Let $\mathbb{W}$ be a non-degenerate co-dimension one subspace of $\mathbb{V}$. We say that a smooth function f on $\mathbb{V} \setminus \mathbb{W}$ has *logarithmic singularity along* $\mathbb{W}$ *near* $w \in \mathbb{W}$, there is a neighborhood $\mathcal{U}_w$ of w in $\mathbb{V}$ such that

$$f(u) = C_0 \log |(u^\perp, u^\perp)| + C_1$$

holds for all $u \in \mathcal{U}_w \setminus \mathbb{W}$, where $u^\perp \in \mathbb{W}^\perp$ denotes the orthogonal projection of u to $\mathbb{W}^\perp$, and C_0, C_1 are constants (depending on w). We say that a smooth function f on $\mathbb{V} \setminus \mathbb{W}$ has *logarithmic singularity along* $\mathbb{W}$ if it does so near every $w \in \mathbb{W}$. Notice that such a function f is locally integrable on $\mathbb{V}$ (i.e., $\int_{\mathcal{C}} |f(x)|\,\mathrm{d}x < \infty$ for any compact open $\mathcal{C} \subset \mathbb{V}$).

Proposition 7.3.4.

(a) The functions $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ and $\partial\mathrm{Den}_{L^\flat}$ lie in $\mathrm{L}^1_{\mathrm{c}}(\mathbb{V})$, and they have logarithmic singularity along $L_F^\flat$.

(b) The function $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ extends to a (necessarily unique) element in $C_c^\infty(\mathbb{V})$ (we will still denote this extension by $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$), i.e., there exists an element in $C_c^\infty(\mathbb{V})$ whose restriction to the open dense subset $\mathbb{V} \setminus L_F^\flat$ is equal to $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$.

Proof. Consider the set

$$N(L^\flat) := \{x \in \mathbb{V} \mid \langle x \rangle + L^\flat \text{ is integral}\}. \tag{7.3.4.1}$$

We claim that $N(L^\flat)$ is a *compact open* subset of $\mathbb{V}$. To show the claim, we rewrite the above set as

$$N(L^\flat) = \{x \in \mathbb{V} \mid (x, L^\flat) \subset O_F, \quad (x,x) \in O_F\}.$$

Write $x = x^\flat + x^\perp$ according to the orthogonal direct sum $\mathbb{V} = L_F^\flat \oplus (L_F^\flat)^\perp$. Then the condition $(x, L^\flat) \subset O_F$ is equivalent to $x^\flat \in (L^\flat)^\vee$. Since $(L^\flat)^\vee$ is a compact subset of $L_F^\flat$, $(x^\flat, x^\flat)$ is bounded in F. Together with the integrality of the norm (x,x), it follows that $(x^\perp, x^\perp)$ is also bounded. Therefore $x^\perp$ lies in a bounded subset $\mathcal{L}$ of the F-line $(L_F^\flat)^\perp$. It follows that $N(L^\flat)$ is contained in a bounded set $(L^\flat)^\vee \oplus \mathcal{L}$. Since $N(L^\flat)$ is open and closed in $\mathbb{V}$, it must be compact.

Note that all three functions $\partial\mathrm{Den}_{L^\flat}$, $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ and $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ vanish outside $N(L^\flat)$. It follows that all three vanish outside a compact subset of $\mathbb{V}$, and are smooth functions on $\mathbb{V} \setminus L_F^\flat$. To show part (a) it suffices to show both functions $\partial\mathrm{Den}_{L^\flat}$, $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ have logarithmic singularity along $L_F^\flat$ near each $e \in L_F^\flat$ (then the integrability follows by the consideration of the support). To show part (b), it suffices to show that $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ is a constant near each $e \in L_F^\flat$.

We now consider the behavior of the three functions $\partial\mathrm{Den}_{L^\flat}$, $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ and $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ near each $e \in L^\flat_F$. By the above discussion on the support, we may assume $e \in L^\flat_F \cap N(L^\flat)$.

First we consider the case $e \in L^\flat$, and we consider its neighborhood $M(L^\flat) = L^\flat \oplus \langle u\rangle$, the lattice defined by (7.1.0.3). Obviously the three functions are all invariant under $L^\flat$-translation. By Lemma 7.3.5, both $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ and $\partial\mathrm{Den}_{L^\flat}$ have logarithmic singularity along $L^\flat_F$ near such e.

Again by Lemma 7.3.5, when $x \in \langle \varpi u\rangle$ is non-zero, we have

$$\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x) - \partial\mathrm{Den}_{L^\flat,\mathscr{V}}(\varpi^{-1}x) = \mathrm{Den}(-q, L^\flat) - \frac{1}{\mathrm{vol}(L^\flat)}\mathrm{Den}((-q)^{-1}, L^\flat),$$

which vanishes by the functional equation for $\mathrm{Den}(X, L^\flat)$ evaluated at $X = -q$, cf. (3.6.2.2). It follows that $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x) = \partial\mathrm{Den}_{L^\flat,\mathscr{V}}(\varpi^{-1}x)$ when $x \in \langle\varpi u\rangle$ is non-zero. Therefore $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ is a constant in $M(L^\flat) \setminus L^\flat_F$.

Next we consider the case $e \in L^\flat_F \cap N(L^\flat)$ but $e \notin L^\flat$. We denote $\widetilde{L}^\flat := \langle e\rangle + L^\flat$. Choose an orthogonal basis $e_1, \cdots, e_{n-1}$ of the lattice $L^\flat$ and write $e = \lambda_1 e_1 + \cdots + \lambda_{n-1}e_{n-1}$. Up to re-ordering these basis vectors, we may assume that λ_1 attains the *minimal* valuation among all of the coefficients $\lambda_i, 1 \le i \le n-1$. Since we are assuming $e \notin L^\flat$, we have $\lambda_1 \notin O_F$. Let us denote

$$\lambda := \lambda_1^{-1} \in O_F.$$

Then we have

$$e_1 = \lambda e - \lambda_1^{-1}\lambda_2 e_2 - \cdots - \lambda_1^{-1}\lambda_{n-1}e_{n-1} \in \langle \lambda e, e_2, e_3, \ldots, e_{n-1}\rangle.$$

Fix a basis vector e_n of the line $(L^\flat_F)^\perp$. Since all three functions are invariant under $L^\flat$-translation, it suffices to consider the behavior of the function:

$$t \in O_F \longmapsto \partial\mathrm{Den}_{L^\flat}(e + te_n)$$

when t is near $0 \in O_F$, and the similar functions for $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}, \partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ respectively.

Set $x_t := e + te_n$ and $M_t := \langle e_2, e_3, \cdots, e_{n-1}, x_t\rangle$. Then we have

$$\begin{aligned} L^\flat + \langle x_t\rangle &= \langle e_1, e_2, e_3, \cdots, e_{n-1}, e + te_n\rangle \\ &= \langle \lambda te_n, e_2, e_3, \cdots, e_{n-1}, e + te_n\rangle \\ &= M_t + \langle \lambda te_n\rangle. \end{aligned}$$

Note that the vector space $\mathbb{V}$ has an orthogonal basis $\{e_2, e_3, \cdots, e_{n-1}, e_1 + \lambda te_n, e_n'\}$, where $e_n' := e_n - \sigma(t)\mu e_1$ and $\mu = \sigma(\lambda)\frac{(e_n,e_n)}{(e_1,e_1)} \in F$, where $\sigma(t)$ denotes the Galois conjugation of t. A straightforward computation shows that, when $|t|$ is sufficiently small, with respect to $\mathbb{V} = \langle e_n'\rangle_F \oplus \langle e_n'\rangle_F^\perp$, the projection of e_n to $\langle e_n'\rangle_F^\perp$ lies in M_t, and to the line $\langle e_n'\rangle_F$ is $u_t e_n'$ where $u_t \in O_F^\times$ is a unit (since $u_t \to 1$ as $t \to 0$). It follows that, when $|t|$ is sufficiently small,

$$L^\flat + \langle x_t\rangle = M_t \oplus \langle \lambda te_n'\rangle.$$

Note that, when $|t|$ is sufficiently small, the lattice M_t has the same fundamental invariants as $\widetilde{L}^\flat = L^\flat + \langle e\rangle = \langle e_2, e_3, \cdots, e_{n-1}, e\rangle$, and (e_n', e_n') differs from (e_n, e_n) by a unit, and thus the lattices $M_t \oplus \langle\lambda te_n'\rangle$ and $\widetilde{L}^\flat \oplus \langle \lambda te_n\rangle$ have the same fundamental invariants. Since $\partial\mathrm{Den}_{L^\flat}(x_t) =$

$\partial\mathrm{Den}(L^\flat+\langle x_t\rangle)$ depends only on the fundamental invariants of the lattice $L^\flat+\langle x_t\rangle$, we obtain that, when $|t|$ is sufficiently small,

$$\partial\mathrm{Den}_{L^\flat}(x_t)=\partial\mathrm{Den}_{\widetilde{L}^\flat}(\lambda t e_n). \tag{7.3.4.2}$$

Now by Lemma 7.3.5, the function $\partial\mathrm{Den}_{L^\flat}$ has logarithmic singularity near e.

Next we investigate the behavior of $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x_t)$ when $t\to 0$. By (6.1.2.1), we have

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x_t)-\partial\mathrm{Den}_{\widetilde{L}^\flat,\mathscr{H}}(x_t)=\sum_{\substack{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee\\ e\notin L'^\flat,\, t(L'^\flat)\le 1}}\ \sum_{\substack{L'^\flat+\langle x_t\rangle\subset L'\subset L'^\vee\\ L'\cap L^\flat_F=L'^\flat}}\mathfrak{m}(t(L')). \tag{7.3.4.3}$$

We *claim* that, when $|t|$ is sufficiently small, the right hand side is a constant dependent on $L^\flat$ and e but not on t. The outer sum has only finitely many terms. Therefore, to show the claim, it suffices to show that each of the inner sums is a constant dependent on $L'^\flat$ and e but not on t. Now fix an integral lattice $L'^\flat\supset L^\flat$ such that $e\notin L'^\flat$ and $t(L'^\flat)\le 1$. We may further assume that $L'^\flat+\langle e\rangle$ is integral (otherwise the inner sum for such $L'^\flat$ is empty when $|t|$ is sufficiently small). Then $L'^\flat$ must be an orthogonal sum $M\oplus\langle f\rangle$ for some self-dual lattice M of rank $n-2$, and a rank one lattice $\langle f\rangle$. Denote by e^* the orthogonal projection of e to the line $\langle f\rangle_F$. Since $L'^\flat+\langle e\rangle$ is integral and $e\notin L'^\flat$, we must have $e-e^*\in M$ and $f=\xi e^*$ for some $\xi\in O_F$ but $\xi\notin O_F^\times$. Let $\mathbb{W}=\langle f,e_n\rangle_F$ be the orthogonal complement of M_F. Then the inner sum associated to $L'^\flat$ is equal to

$$\sum_{\substack{\langle f,e^*+te_n\rangle\subset L'\subset L'^\vee\subset\mathbb{W}\\ L'\cap L^\flat_F=\langle f\rangle}}\mathfrak{m}(t(L'))=\partial\mathrm{Den}(\langle f,e^*+te_n\rangle)-\partial\mathrm{Den}(\langle\varpi^{-1}f,e^*+te_n\rangle).$$

Now it is easy to see that, when $|t|$ is sufficiently small, the lattice $\langle f,e^*+te_n\rangle=\langle\xi e^*,e^*+te_n\rangle$ (resp., $\langle\varpi^{-1}f,e^*+te_n\rangle=\langle\varpi^{-1}\xi e^*,e^*+te_n\rangle$) has the same fundamental invariants as $\langle\xi te_n,e^*\rangle$ (resp., $\langle\varpi^{-1}\xi te_n,e^*\rangle$). By Lemma 7.3.5, when $|t|$ is sufficiently small, the difference

$$\begin{aligned}&\partial\mathrm{Den}(\langle f,e^*+te_n\rangle)-\partial\mathrm{Den}(\langle\varpi^{-1}f,e^*+te_n\rangle)\\ =&\partial\mathrm{Den}(\langle\xi te_n,e^*\rangle)-\partial\mathrm{Den}(\langle\varpi^{-1}\xi te_n,e^*\rangle)\\ =&\partial\mathrm{Den}_{\langle e^*\rangle}(\xi te_n)-\partial\mathrm{Den}_{\langle e^*\rangle}(\varpi^{-1}\xi te_n)\\ =&\mathrm{Den}(-q,\langle e^*\rangle)\end{aligned}$$

is a constant independent of t. This proves the claim. Note that $\partial\mathrm{Den}_{\widetilde{L}^\flat,\mathscr{H}}(x_t)=\partial\mathrm{Den}_{\widetilde{L}^\flat,\mathscr{H}}(te_n)$ and it has logarithmic singularity along $\widetilde{L}^\flat_F=L^\flat_F$ by Lemma 7.3.5. It follows from the claim that the function $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ also has logarithmic singularity along $L^\flat_F$ near e. Now we have completed the proof of part (a).

By (7.3.4.2) and (7.3.4.3), let $t\to 0$ and denote by C the constant equal to (7.3.4.3):

$$\begin{aligned}\partial\mathrm{Den}_{L^\flat,\mathscr{V}}(x_t)&=\partial\mathrm{Den}_{L^\flat}(x_t)-\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x_t)\\ &=\partial\mathrm{Den}_{\widetilde{L}^\flat}(\lambda te_n)-\partial\mathrm{Den}_{\widetilde{L}^\flat,\mathscr{H}}(te_n)-C\\ &=(\partial\mathrm{Den}_{\widetilde{L}^\flat}(\lambda te_n)-\partial\mathrm{Den}_{\widetilde{L}^\flat}(te_n))+\partial\mathrm{Den}_{\widetilde{L}^\flat,\mathscr{V}}(te_n)-C.\end{aligned}$$

By Lemma 7.3.5, the term $(\partial\mathrm{Den}_{\widetilde{L}^\flat}(\lambda t e_n) - \partial\mathrm{Den}_{\widetilde{L}^\flat}(te_n))$ is a constant dependent on $L^\flat$ and e but not on t. By the previous case (replacing $L^\flat$ by $\widetilde{L}^\flat$) that we have considered, the term $\partial\mathrm{Den}_{\widetilde{L}^\flat,\mathscr{V}}(te_n)$ is a constant when $t \to 0$. This shows that $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ is a constant near e, and we have completed the proof of part (b). $\square$

Lemma 7.3.5. *Assume that* $x \perp L^\flat$ *and* $\mathrm{val}(x) \geq 1 + e_{\max}(L^\flat)$. *Then*

$$\partial\mathrm{Den}_{L^\flat}(x) - \partial\mathrm{Den}_{L^\flat}(\varpi^{-1}x) = \mathrm{Den}(-q, L^\flat),$$

and

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x) - \partial\mathrm{Den}_{L^\flat,\mathscr{H}}(\varpi^{-1}x) = \frac{1}{\mathrm{vol}(L^\flat)}\mathrm{Den}((-q)^{-1}, L^\flat).$$

Proof. The first part follows from the induction formula in Proposition 3.7.1

$$\mathrm{Den}(X, L) = X^2\mathrm{Den}(X, L') + (1 - X)\mathrm{Den}(-qX, L^\flat),$$

where

$$L' = L^\flat \obot \langle \varpi^{-1}x\rangle, \quad L = L^\flat \obot \langle x\rangle.$$

Now we consider the second part. By the definition (6.1.2.1) of the function $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$, we obtain

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x) - \partial\mathrm{Den}_{L^\flat,\mathscr{H}}(\varpi^{-1}x) = \sum_{\substack{L^\flat\subset L'\subset L'^\vee,\\ t(L'^\flat)\leq 1,\, L'\cap\langle x\rangle_F = \langle x\rangle}} \mathfrak{m}(t(L')),$$

where we recall that $L'^\flat = L' \cap L^\flat_F$, cf. (6.1.2.2). This sum can be rewritten as a double sum, first over all L' with a given $L' \cap L^\flat_F = L'^\flat$ then over all $L'^\flat$

$$\sum_{\substack{L'^\flat\subset (L'^\flat)^\vee\\ L^\flat\subset L'^\flat,\, t(L'^\flat)\leq 1}} \sum_{\substack{L'\subset L'^\vee\\ L'\cap L^\flat_F = L'^\flat, L'\cap\langle x\rangle_F=\langle x\rangle}} \mathfrak{m}(t(L')). \tag{7.3.5.1}$$

Fix $L'^\flat$ with $t(L'^\flat) \leq 1$ and we consider the inner sum. Since $t(L'^\flat) \leq 1$, we may assume that $L'^\flat$ has an orthogonal basis $e'_1, \cdots, e'_{n-1}$ such that $\mathrm{val}(e'_1) = \mathrm{val}(e'_2) = \cdots = \mathrm{val}(e'_{n-2}) = 0$, and $a'_{n-1} := \mathrm{val}(e'_{n-1})$.

By Lemmas 7.2.1 and 7.2.2, each lattice L' in the inner sum is of the form $L'^\flat + \langle u\rangle$ where u satisfies

$$(u, L'^\flat) \subset O_F, \quad (u, u) \in O_F.$$

Write $u = u^\flat + u^\perp$ according to the orthogonal direct sum $\mathbb{V} = L^\flat_F \obot (L^\flat_F)^\perp$. We *claim* that $\mathrm{val}(u^\flat) \geq 0$ and $\mathrm{val}(u^\perp) \geq 1$.

To prove the claim, we first note that the condition $(u, L'^\flat) \subset O_F$ above is equivalent to $u^\flat \in (L'^\flat)^\vee$. Therefore we may write $u^\flat = \lambda_1 e'_1 + \cdots + \lambda_{n-1}e'_{n-1}$ where $\lambda_i \in O_F$ $(1 \leq i \leq n-2)$ and $\lambda_{n-1} \in \varpi^{-a'_{n-1}}O_F$. By Lemma 7.2.2, we have

$$\tfrac{\pi_\perp(L')}{L'\cap\langle x\rangle_F} = \tfrac{\langle u^\perp\rangle}{\langle x\rangle} \xrightarrow{\sim} \tfrac{\pi_\flat(L')}{L'^\flat} = \tfrac{L'^\flat+\langle u^\flat\rangle}{L'^\flat} \simeq \tfrac{O_F+\lambda_{n-1}\cdot O_F}{O_F}. \tag{7.3.5.2}$$

This isomorphism implies that

$$\max\{0, -2\,\mathrm{val}(\lambda_{n-1})\} = -\mathrm{val}(u^\perp) + \mathrm{val}(x). \tag{7.3.5.3}$$

Now if $\mathrm{val}(u^\flat) < 0$, from $(u,u) \in O_F$ it follows that $\mathrm{val}(u^\flat) = \mathrm{val}(u^\perp) < 0$ and $\mathrm{val}(u^\flat) = 2\,\mathrm{val}(\lambda_{n-1}) + a'_{n-1}$. Hence $2\,\mathrm{val}(\lambda_{n-1}) < -a'_{n-1}$ (in particular, $\mathrm{val}(\lambda_{n-1}) < 0$). It follows from (7.3.5.3) that $\mathrm{val}(x) = a'_{n-1}$, which contradicts $\mathrm{val}(x) > a_{n-1} \geq a'_{n-1}$. Therefore we must have $\mathrm{val}(u^\flat) \geq 0$ and $\mathrm{val}(u^\perp) \geq 0$. It then follows that $\mathrm{val}(\lambda_{n-1}e'_{n-1}) \geq 0$, or equivalently $2\,\mathrm{val}(\lambda_{n-1}) + a'_{n-1} \geq 0$. By (7.3.5.3), we have either $\mathrm{val}(u^\perp) = \mathrm{val}(x) \geq 1$ or

$$\mathrm{val}(u^\perp) = \mathrm{val}(x) + 2\,\mathrm{val}(\lambda_{n-1}) \geq (1 + a_{n-1}) - a'_{n-1} \geq 1.$$

Here the last inequality follows from $e_{\max}(L'^\flat) = a'_{n-1} \leq e_{\max}(L^\flat) = a_{n-1}$ by (7.1.0.4) applied to $L^\flat \subset L'^\flat$. We have thus completed the proof of the claim.

Now we define $\widetilde{L}'^\flat := \pi_\flat(L') = L'^\flat + \langle u^\flat \rangle$. Then $\widetilde{L}'^\flat$ is an integral lattice. By $\mathrm{val}(u^\perp) \geq 1$, we obtain

$$t(L') = t(\widetilde{L}'^\flat) + 1.$$

Moreover, for a given integral lattice $\widetilde{L}'^\flat \supset L'^\flat$, the set of desired integral lattices L' is bijective to the set of generators of the cyclic O_F-module $\widetilde{L}'^\flat / L'^\flat$. Therefore the inner sum in (7.3.5.1) is equal to

$$\sum_{L'^\flat \subset \widetilde{L}'^\flat} \mathfrak{m}(t(\widetilde{L}'^\flat) + 1)[\widetilde{L}'^\flat : L'^\flat] \cdot \begin{cases} 1, & \text{if } \widetilde{L}'^\flat = L'^\flat, \\ (1 - q^{-2}), & \text{if } \widetilde{L}'^\flat \neq L'^\flat, \end{cases} \tag{7.3.5.4}$$

where the index $[\widetilde{L}'^\flat : L'^\flat] = \frac{\mathrm{vol}(\widetilde{L}'^\flat)}{\mathrm{vol}(L'^\flat)}$. For the sum (7.3.5.4), we distinguish three cases.

(1) If $t(L'^\flat) = 0$, i.e., $a'_{n-1} = 0$, then the sum is equal to 1.

(2) If $a'_{n-1} > 0$ is odd, then the sum is equal to

$$(1+q)(1 + (q^2 - 1) + \cdots + (q^{a'_{n-1}-1} - q^{a'_{n-1}-3})) = q^{a'_{n-1}-1}(1+q).$$

(3) If $a'_{n-1} > 0$ is even, then the sum is equal to

$$(1+q)(1 + (q^2 - 1) + \cdots + (q^{a'_{n-1}-2} - q^{a'_{n-1}-4})) + (q^{a'_{n-1}} - q^{a'_{n-1}-2}) = q^{a'_{n-1}-1}(1+q).$$

Therefore the inner sum in (7.3.5.1) is equal to

$$\begin{cases} 1, & t(L'^\flat) = 0, \\ (1 + q^{-1}) \frac{1}{\mathrm{vol}(L'^\flat)}, & t(L'^\flat) = 1. \end{cases} \tag{7.3.5.5}$$

We obtain that (7.3.5.1) is equal to

$$\sum_{L^\flat \subset L'^\flat,\, t(L'^\flat)=0} 1 + \sum_{L^\flat \subset L'^\flat,\, t(L'^\flat)=1} (1 + q^{-1}) \frac{1}{\mathrm{vol}(L'^\flat)} = \frac{1}{\mathrm{vol}(L^\flat)} \mathrm{Den}((-q)^{-1}, L^\flat),$$

by (3.6.1.2), and hence

$$\partial\mathrm{Den}_{L^\flat,\mathscr{H}}(x) - \partial\mathrm{Den}_{L^\flat,\mathscr{H}}(\varpi^{-1}x) = \frac{1}{\mathrm{vol}(L^\flat)} \mathrm{Den}((-q)^{-1}, L^\flat).$$

This completes the proof. $\square$

We introduce an auxiliary function on $\mathbb{V} \setminus L^\flat_F$,

$$\widetilde{\partial\mathrm{Den}}_{L^\flat}(x) = \sum_{L^\flat \subset L' \subset L'^\vee} \mathbf{1}_{L'}(x).$$

Similar to Proposition 7.3.4, we have:

Lemma 7.3.6. *The function $\widetilde{\partial\mathrm{Den}}_{L^\flat}$ lies in $\mathrm{L}^1_c(\mathbb{V})$, having logarithmic singularity along $L^\flat_F$.*

Proof. It suffices to show the logarithmic singularity near $0 \in \mathbb{V}$. The behavior of $\widetilde{\partial\mathrm{Den}}_{L^\flat}$ near an arbitrary $e \in \mathbb{V}$ is then reduced to this case by the same argument as the proof of part (a) in Proposition 7.3.4 for $\partial\mathrm{Den}_{L^\flat}$. More precisely, the equality (7.3.4.2) also holds for the function $\widetilde{\partial\mathrm{Den}}_{L^\flat}$, since $\widetilde{\partial\mathrm{Den}}_{L^\flat}(x)$ depends only on the fundamental invariants of the lattice $L^\flat + \langle x\rangle$.

Note that the function $\widetilde{\partial\mathrm{Den}}_{L^\flat}$ is invariant under $L^\flat$-translation. It suffices to show that, when $x \perp L^\flat_F$ and $\mathrm{val}(x) > 2\, e_{\max}(L^\flat)$,

$$\widetilde{\partial\mathrm{Den}}_{L^\flat}(x) = C_0\, \mathrm{val}(x) + C_1$$

for some constants C_0, C_1. Write the function as a double sum:

$$\widetilde{\partial\mathrm{Den}}_{L^\flat}(x) = \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \sum_{\substack{L' \subset L'^\vee \\ L' \cap L^\flat_F = L'^\flat}} \mathbf{1}_{L'}(x).$$

Since the outer sum has only finitely many terms, it suffices to prove the desired logarithmic singularity for the inner sum associated to each integral lattice $L'^\flat \supset L^\flat$. Fix such an $L'^\flat$. It suffices to show that, when $x \perp L^\flat_F$ and $\mathrm{val}(x) > 2e_{\max}(L'^\flat)$ (we remind the reader that $e_{\max}(L'^\flat) \leq e_{\max}(L^\flat)$ by (7.1.0.4)), the cardinality

$$\#\{L' \mid L' \subset L'^\vee,\, L' \cap L^\flat_F = L'^\flat, L' \cap (L^\flat_F)^\perp = \langle x\rangle\} \tag{7.3.6.1}$$

is independent of x.

Following the proof of Lemma 7.3.5, each lattice L' in the above set is of the form $L^\flat + \langle u\rangle$ where

$$(u, L'^\flat) \subset O_F, \quad (u,u) \in O_F.$$

Write $u = u^\flat + u^\perp$ according to the orthogonal direct sum $\mathbb{V} = L^\flat_F \oplus (L^\flat_F)^\perp$. We *claim* that $\mathrm{val}(u^\perp) \geq 1$. In fact, by $(u, L'^\flat) \subset O_F$, we obtain $u^\flat \in (L'^\flat)^\vee$, and hence $\mathrm{length}_{O_F} \frac{L'^\flat + \langle u^\flat\rangle}{L'^\flat} \leq e_{\max}(L'^\flat)$. Comparing the lengths of the O_F-modules in (7.3.5.2), we obtain

$$-\mathrm{val}(u^\perp) + \mathrm{val}(x) = 2\ \mathrm{length}_{O_F} \frac{L'^\flat + \langle u^\flat\rangle}{L'^\flat} \leq 2\, e_{\max}(L'^\flat).$$

The claim follows.

It follows that the cardinality (7.3.6.1) is given by (7.3.5.4) without the weight factor $\mathfrak{m}(t(\widetilde{L}'^\flat)+1)$, hence independent of x. This completes the proof. $\square$

By Proposition 7.3.4, the functions $\partial\mathrm{Den}_{L^\flat}, \partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ and $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ are all in $\mathrm{L}^1(\mathbb{V})$, hence Fourier transforms exist for all of them.

Corollary 7.3.7. *The Fourier transforms of $\partial\mathrm{Den}_{L^\flat}$ and $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$ are given by (pointwise) absolutely convergent sums*

$$\widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{H}}(x) = \sum_{L^\flat\subset L'\subset L'^\vee,\ t(L'^\flat)\le 1} \mathrm{vol}(L')\mathfrak{m}(t(L'))1_{L'^\vee}(x), \tag{7.3.7.1}$$

and

$$\widehat{\partial\mathrm{Den}}_{L^\flat}(x) = \sum_{L^\flat\subset L'\subset L'^\vee} \mathrm{vol}(L')\mathfrak{m}(t(L'))1_{L'^\vee}(x), \tag{7.3.7.2}$$

where $x\in\mathbb{V}$ in both equalities.

Proof. By definition (1.7.0.2) we have

$$\widehat{\partial\mathrm{Den}}_{L^\flat}(x) = \int_{\mathbb{V}} \sum_{L^\flat\subset L'\subset L'^\vee} \mathfrak{m}(t(L'))1_{L'}(y)\psi(\mathrm{tr}_{F/F_0}(x,y))\,\mathrm{d}y. \tag{7.3.7.3}$$

By Lemma 7.3.6 we have

$$\begin{aligned}\int_{\mathbb{V}} \sum_{L^\flat\subset L'\subset L'^\vee} |\mathfrak{m}(t(L'))1_{L'}(y)\psi(\mathrm{tr}_{F/F_0}(x,y))|\,\mathrm{d}y &\le |\mathfrak{m}(n)|\int_{\mathbb{V}} \sum_{L^\flat\subset L'\subset L'^\vee} 1_{L'}(y)\,\mathrm{d}y \\ &= |\mathfrak{m}(n)|\int_{\mathbb{V}} \widetilde{\partial\mathrm{Den}}_{L^\flat}(y)\,\mathrm{d}y<\infty.\end{aligned}$$

Therefore we can interchange the order of the integration and the sum in (7.3.7.3). This proves (7.3.7.2). The equality (7.3.7.1) is proved similarly. □

7.4. **Fourier transform of $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$.**

Theorem 7.4.1. *Assume that $x\perp L^\flat$ and $\mathrm{val}(x)<0$. Then*

$$\widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{V}}(x)=0.$$

Proof. This follows from Lemma 7.4.2 below, and the functional equation (3.6.2.2)

$$\mathrm{Den}(-q,L^\flat+\langle u^\flat\rangle) = \frac{1}{\mathrm{vol}(L^\flat+\langle u^\flat\rangle)}\mathrm{Den}((-q)^{-1},L^\flat+\langle u^\flat\rangle). \quad □$$

Lemma 7.4.2. *Assume that $x\perp L^\flat$ and $\mathrm{val}(x)<0$. Then*

$$\widehat{\partial\mathrm{Den}}_{L^\flat}(x) = (1-q^{-2})^{-1}\,\mathrm{vol}(\langle x\rangle^\vee)\int_{L^\flat_F}\mathrm{Den}(-q,L^\flat+\langle u^\flat\rangle)\,du^\flat,$$

and

$$\widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{H}}(x) = (1-q^{-2})^{-1}\,\mathrm{vol}(\langle x\rangle^\vee)\int_{L^\flat_F}\frac{1}{\mathrm{vol}(L^\flat+\langle u^\flat\rangle)}\mathrm{Den}((-q)^{-1},L^\flat+\langle u^\flat\rangle)\,du^\flat.$$

Recall that $\langle x\rangle^\vee$ denotes the dual lattice of $\langle x\rangle$ in the line $\langle x\rangle_F$. Here we use the self-dual measures on $L^\flat_F$ and on $\langle x\rangle_F$ respectively, cf. §1.7.

Proof. First we consider the Fourier transform of $\partial\mathrm{Den}_{L^\flat}$. By (7.3.7.2), it is equal to the (pointwise) absolutely convergent sum

$$\widehat{\partial\mathrm{Den}}_{L^\flat}(x) = \sum_{L^\flat \subset L' \subset L'^\vee,\, x \in L'^\vee} \mathrm{vol}(L')\mathfrak{m}(t(L')).$$

For an integral lattice $L'^\flat \supset L^\flat$, define

$$\Sigma(L'^\flat, x) = \{L' \subset \mathbb{V} \mid x \in L'^\vee, L' \subset L'^\vee, L'^\flat = L' \cap L^\flat_F\}. \tag{7.4.2.1}$$

Then

$$\widehat{\partial\mathrm{Den}}_{L^\flat}(x) = \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \sum_{L' \in \Sigma(L'^\flat, x)} \mathrm{vol}(L')\mathfrak{m}(t(L')). \tag{7.4.2.2}$$

By Lemmas 7.2.1 and Lemma 7.2.2, we have a bijection

$$\begin{aligned} [((\langle x\rangle + L'^\flat)^{\vee,\circ}/L'^\flat) \setminus (L^\flat_F/L'^\flat)]/O_F^\times &\xrightarrow{\sim} \Sigma(L'^\flat, x) \\ u &\longmapsto L'^\flat + \langle u\rangle. \end{aligned} \tag{7.4.2.3}$$

Here, though $(\langle x\rangle + L'^\flat)^{\vee,\circ}$ is not necessarily a lattice, it is invariant under $L'^\flat$-translation and $O_F^\times$-multiplication. Hence the quotient on the left hand side makes sense.

Now we follow the same argument as in the proof of Lemma 7.3.5. Write $u = u^\flat + u^\perp$ according to the orthogonal direct sum $\mathbb{V} = L^\flat_F \obot (L^\flat_F)^\perp$. Then the condition $x \in L'^\vee$ is equivalent to the projection $\pi_\perp(L') \subset \langle x\rangle^\vee$ (inside the line $(L^\flat_F)^\perp = \langle x\rangle_F$), or equivalently, $(x, u^\perp) \in O_F$. Since $\mathrm{val}(x) < 0$, we must have $\mathrm{val}(u^\perp) > 0$ (due to $2\,\mathrm{val}((x, u^\perp)) = \mathrm{val}(x) + \mathrm{val}(u^\perp)$). It follows from the integrality of the norm (u, u) and $(u^\perp, u^\perp)$ that $u^\flat$ also has integral norm and hence $u^\flat \in (L'^\flat)^{\vee,\circ}$. Thus we have

$$(\langle x\rangle + L'^\flat)^{\vee,\circ} = (L'^\flat)^{\vee,\circ} \obot \langle x\rangle^\vee$$

and a bijection with the left hand side of (7.4.2.3)

$$(L'^\flat)^{\vee,\circ}/L'^\flat \times \frac{\langle x\rangle^\vee \setminus \{0\}}{O_F^\times} \xrightarrow{\sim} [(L'^\flat)^{\vee,\circ}/L'^\flat \times (\langle x\rangle^\vee \setminus \{0\})]/O_F^\times$$

sending $(u^\flat, O_F^\times \cdot \varpi^m x)$ to the $O_F^\times$-orbit of $(u^\flat, \varpi^m x)$ (with the diagonal $O_F^\times$-action)[3]. We have the resulting bijection

$$(L'^\flat)^{\vee,\circ}/L'^\flat \times \frac{\langle x\rangle^\vee \setminus \{0\}}{O_F^\times} \xrightarrow{\sim} \Sigma(L'^\flat, x).$$

The second factor $\frac{\langle x\rangle^\vee \setminus \{0\}}{O_F^\times}$ can be further identified with the set of lattices contained in $\langle x\rangle^\vee$ (corresponding to $\langle u^\perp\rangle = \pi_\perp(L')$). We write $\widetilde{L}'^\flat := \pi_\flat(L') = L'^\flat + \langle u^\flat\rangle$. Then $\widetilde{L}'^\flat$ is an integral lattice. By $\mathrm{val}(u^\perp) \geq 1$, we obtain

$$t(L') = t(\widetilde{L}'^\flat) + 1,$$

and by (7.2.2.1),

$$\mathrm{vol}(L') = \mathrm{vol}(L'^\flat)\,\mathrm{vol}(\pi_\perp(L')).$$

[3]The bijection depends on the choice of a basis vector of $\langle x\rangle$, and here we have simply chosen x.

Therefore the inner sum in (7.4.2.2) is equal to

$$\begin{aligned}
&\mathrm{vol}(L'^\flat) \sum_{u^\flat \in \frac{(L'^\flat)^{\vee,\circ}}{L'^\flat}} \mathfrak{m}(t(\widetilde{L}'^\flat)+1) \sum_{N \subset \langle x\rangle^\vee} \mathrm{vol}(N) \\
&= \mathrm{vol}(L'^\flat)\,\mathrm{vol}(\langle x\rangle^\vee)\big(\sum_{i\ge 0} q^{-2i}\big) \sum_{u^\flat \in \frac{(L'^\flat)^{\vee,\circ}}{L'^\flat}} \mathfrak{m}(t(\widetilde{L}'^\flat)+1) \\
&= \mathrm{vol}(L'^\flat)\,\mathrm{vol}(\langle x\rangle^\vee)(1-q^{-2})^{-1} \sum_{u^\flat \in \frac{(L'^\flat)^{\vee,\circ}}{L'^\flat}} \mathfrak{m}(t(\widetilde{L}'^\flat)+1).
\end{aligned}$$

We now return to the sum (7.4.2.2), which is now equal to

$$\widehat{\partial\mathrm{Den}}_{L^\flat}(x) = \mathrm{vol}(\langle x\rangle^\vee) \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat)(1-q^{-2})^{-1} \sum_{u^\flat \in \frac{(L'^\flat)^{\vee,\circ}}{L'^\flat}} \mathfrak{m}(t(\widetilde{L}'^\flat)+1). \tag{7.4.2.4}$$

For a given integral lattice $\widetilde{L}'^\flat$ such that $\widetilde{L}'^\flat/L'^\flat$ is a cyclic O_F-module, the number of $u^\flat \in \frac{(L'^\flat)^{\vee,\circ}}{L'^\flat}$ such that $L'^\flat + \langle u^\flat\rangle = \widetilde{L}'^\flat$ is

$$\begin{cases} [\widetilde{L}'^\flat : L'^\flat](1-q^{-2}) = \frac{\mathrm{vol}(\widetilde{L}'^\flat)}{\mathrm{vol}(L'^\flat)}(1-q^{-2}), & \text{if } \widetilde{L}'^\flat \neq L'^\flat, \\ 1, & \text{if } \widetilde{L}'^\flat = L'^\flat. \end{cases}$$

We thus obtain

$$\begin{aligned}
\widehat{\partial\mathrm{Den}}_{L^\flat}(x) =& \mathrm{vol}(\langle x\rangle^\vee) \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat) \sum_{L'^\flat \subset \widetilde{L}'^\flat,\, \widetilde{L}'^\flat/L'^\flat \text{ cyclic}} \frac{\mathrm{vol}(\widetilde{L}'^\flat)}{\mathrm{vol}(L'^\flat)} \mathfrak{m}(t(\widetilde{L}'^\flat)+1) \\
&+ q^{-2}(1-q^{-2})^{-1}\,\mathrm{vol}(\langle x\rangle^\vee) \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat)\mathfrak{m}(t(L'^\flat)+1).
\end{aligned}$$

Here we split the contribution of the factor corresponding to $\widetilde{L}'^\flat = L'^\flat$ into two pieces $q^{-2}+(1-q^{-2})$. Interchanging the sum over $L'^\flat$ and $\widetilde{L}'^\flat$, we obtain

$$\begin{aligned}
\widehat{\partial\mathrm{Den}}_{L^\flat}(x) =& \mathrm{vol}(\langle x\rangle^\vee) \sum_{L^\flat \subset \widetilde{L}'^\flat \subset (\widetilde{L}'^\flat)^\vee} \mathrm{vol}(\widetilde{L}'^\flat)\mathfrak{m}(t(\widetilde{L}'^\flat)+1) \sum_{L^\flat \subset L'^\flat \subset \widetilde{L}'^\flat,\, \widetilde{L}'^\flat/L'^\flat \text{ cyclic}} 1 \\
&+ q^{-2}(1-q^{-2})^{-1}\,\mathrm{vol}(\langle x\rangle^\vee) \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat)\mathfrak{m}(t(L'^\flat)+1).
\end{aligned} \tag{7.4.2.5}$$

Next we consider the integral

$$\int_{L^\flat_F} \mathrm{Den}(-q, L^\flat + \langle u^\flat\rangle)du^\flat.$$

This can be written as a weighted sum over integral lattices $M \subset L^\flat_F$ such that $L^\flat \subset M$ and $M/L^\flat$ is a cyclic O_F-module, with the weight factor

$$\begin{cases} \mathrm{vol}(M)(1-q^{-2}), & \text{if } M \neq L^\flat, \\ \mathrm{vol}(L^\flat), & \text{if } M = L^\flat. \end{cases}$$

Therefore we obtain

$$\int_{L^\flat_F} \mathrm{Den}(-q, L^\flat + \langle u^\flat \rangle) du^\flat = q^{-2}\,\mathrm{vol}(L^\flat)\mathrm{Den}(-q, L^\flat) + (1-q^{-2}) \sum_{L^\flat \subset M \subset M^\vee,\, M/L^\flat \text{ cyclic}} \mathrm{vol}(M)\mathrm{Den}(-q, M). \tag{7.4.2.6}$$

Again here we split the contribution of the factor corresponding to $M = L^\flat$ into two pieces $q^{-2} + (1-q^{-2})$. By the formula (3.6.2.1), the first term is equal to

$$q^{-2}\,\mathrm{vol}(L^\flat)\mathrm{Den}(-q, L^\flat) = q^{-2} \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat)\mathfrak{m}(t(L'^\flat)+1). \tag{7.4.2.7}$$

Again by (3.6.2.1), the second term in (7.4.2.6) is equal to

$$\begin{aligned}
&\sum_{L^\flat \subset M \subset M^\vee,\, M/L^\flat \text{ cyclic}} \mathrm{vol}(M)\mathrm{Den}(-q, M) \\
=& \sum_{L^\flat \subset M \subset L'^\flat \subset (L'^\flat)^\vee,\, M/L^\flat \text{ cyclic}} \mathrm{vol}(M)\frac{\mathrm{vol}(L'^\flat)}{\mathrm{vol}(M)}\mathfrak{m}(t(L'^\flat)+1) \\
=& \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat)\mathfrak{m}(t(L'^\flat)+1) \cdot \#\{M \mid L^\flat \subset M \subset L'^\flat,\, M/L^\flat \text{ cyclic}\}.
\end{aligned}$$

Now note that we have an equality

$$\#\{M \mid L^\flat \subset M \subset L'^\flat,\, M/L^\flat \text{ cyclic}\} = \#\{M \mid L^\flat \subset M \subset L'^\flat,\, L'^\flat/M \text{ cyclic}\}.$$

In fact, the right hand side is the same as

$$\#\{M^\vee \mid L'^{\flat,\vee} \subset M^\vee \subset L^{\flat,\vee},\, M^\vee/L'^{\flat,\vee} \text{ cyclic}\}.$$

and this is equal to the left hand side, due to the (non-canonical) isomorphism of finite O_F-modules

$$L'^\flat/L^\flat \xrightarrow{\sim} (L^\flat)^\vee/(L'^\flat)^\vee.$$

It follows that

$$\begin{aligned}
&\sum_{L^\flat \subset M \subset L'^\flat \subset (L'^\flat)^\vee,\, M/L^\flat \text{ cyclic}} \mathrm{vol}(M)\mathrm{Den}(-q, M) \\
=& \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat)\mathfrak{m}(t(L'^\flat)+1) \cdot \#\{M \mid L^\flat \subset M \subset L'^\flat,\, L'^\flat/M \text{ cyclic}\}.
\end{aligned} \tag{7.4.2.8}$$

By (7.4.2.6), (7.4.2.7) and (7.4.2.8), we obtain

(7.4.2.9)

$$\begin{aligned}
\int_{L^\flat_F} \mathrm{Den}(-q, L^\flat + \langle u^\flat \rangle) du^\flat =& (1-q^{-2}) \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat)\mathfrak{m}(t(L'^\flat)+1) \cdot \sum_{L^\flat \subset M \subset L'^\flat,\, L'^\flat/M \text{ cyclic}} 1 \\
&+ q^{-2} \sum_{L^\flat \subset L'^\flat \subset (L'^\flat)^\vee} \mathrm{vol}(L'^\flat)\mathfrak{m}(t(L'^\flat)+1).
\end{aligned}$$

Comparing (7.4.2.9) with (7.4.2.5) we obtain

$$\widehat{\partial\mathrm{Den}}_{L^\flat}(x) = (1-q^{-2})^{-1}\,\mathrm{vol}(\langle x\rangle^\vee)\int_{L_F^\flat}\mathrm{Den}(-q, L^\flat+\langle u^\flat\rangle)du^\flat,$$

and this completes the proof of the first part concerning $\widehat{\partial\mathrm{Den}}_{L^\flat}$.

Similarly, let us consider the horizontal part $\partial\mathrm{Den}_{L^\flat,\mathscr{H}}$. By (7.3.7.1), we have a (pointwise) absolutely convergent sum

$$\widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{H}}(x) = \sum_{\substack{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee\\ t(L'^\flat)\le 1}}\ \sum_{L'\in\Sigma(L'^\flat,x)}\mathfrak{m}(t(L'))\,\mathrm{vol}(L'). \tag{7.4.2.10}$$

Here $\Sigma(L'^\flat,x)$ is the set defined by (7.4.2.1). Similar to the equation (7.4.2.4) for $\widehat{\partial\mathrm{Den}}_{L^\flat}$, we obtain

$$\widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{H}}(x) = \mathrm{vol}(\langle x\rangle^\vee)\sum_{\substack{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee\\ t(L'^\flat)\le 1}}\mathrm{vol}(L'^\flat)(1-q^{-2})^{-1}\sum_{u^\flat\in\frac{(L'^\flat)^{\vee,\circ}}{L'^\flat}}\mathfrak{m}(t(\widetilde{L}'^\flat)+1).$$

The inner sum is equal to (7.3.5.4), hence equal to (7.3.5.5). We obtain

$$\begin{aligned}\widehat{\partial\mathrm{Den}}_{L^\flat,\mathscr{H}}(x) &= (1-q^{-2})^{-1}\,\mathrm{vol}(\langle x\rangle^\vee)\sum_{\substack{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee\\ t(L'^\flat)\le 1}}\mathrm{vol}(L'^\flat)\begin{cases}1, & t(L'^\flat)=0,\\ q^{-1}\mathfrak{m}(t(L'^\flat)+1)\frac{1}{\mathrm{vol}(L'^\flat)}, & t(L'^\flat)=1\end{cases}\\ &= (1-q^{-2})^{-1}\,\mathrm{vol}(\langle x\rangle^\vee)\sum_{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee,\ t(L'^\flat)\le 1}\begin{cases}1, & t(L'^\flat)=0,\\ 1+q^{-1}, & t(L'^\flat)=1.\end{cases}\end{aligned}$$

From the formula (3.6.1.2), it follows that

$$\begin{aligned}&\int_{L_F^\flat}\frac{1}{\mathrm{vol}(L^\flat+\langle u^\flat\rangle)}\mathrm{Den}((-q)^{-1}, L^\flat+\langle u^\flat\rangle)\,du^\flat\\ =&\sum_{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee,\ t(L'^\flat)=0}\int_{L_F^\flat}\mathbf{1}_{L'^\flat}(u^\flat)\,du^\flat\\ &+\sum_{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee,\ t(L'^\flat)=1} q^{-1}\mathfrak{m}(t(L'^\flat)+1)\frac{1}{\mathrm{vol}(L'^\flat)}\int_{L_F^\flat}\mathbf{1}_{L'^\flat}(u^\flat)\,du^\flat\\ =&\sum_{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee,\ t(L'^\flat)=0}1+\sum_{L^\flat\subset L'^\flat\subset(L'^\flat)^\vee,\ t(L'^\flat)=1}(1+q^{-1}).\end{aligned}$$

This completes the proof of the second part concerning the horizontal part. $\square$

8. Uncertainty principle and the proof of the main theorem

8.1. **Uncertainty principle.**

8.1.1. *Quadratic case.* In this subsection we first let $\mathbb{V}$ be a (non-degenerate) quadratic space of dimension d over a non-archimedean local field F with characteristic not equal to 2. Here we allow the residue characteristic to be $p=2$. We denote by $(\ ,\ )$ the symmetric bi-linear form on $\mathbb{V}$. Let $\mathbb{V}^\circ$ (resp. $\mathbb{V}^{\circ\circ}$) denote the "positive cone" (resp. "strictly positive cone"), defined by

$$\mathbb{V}^\circ=\{x\in\mathbb{V}\mid \mathrm{val}((x,x)/2)\geq 0\},\quad \mathbb{V}^{\circ\circ}=\{x\in\mathbb{V}\mid \mathrm{val}((x,x)/2)>0\}. \tag{8.1.1.1}$$

Fix an unramified additive character $\psi:F\to\mathbb{C}^\times$ and, similar to the hermitian case (cf. §1.7), we define the Fourier transform on $C_c^\infty(\mathbb{V})$ by

$$\widehat{f}(x):=\int_{\mathbb{V}} f(y)\psi((x,y))\mathrm{d}y,\quad x\in\mathbb{V}. \tag{8.1.1.2}$$

Proposition 8.1.2. *Let $\phi\in C_c^\infty(\mathbb{V})$ satisfy*

- $\mathrm{supp}(\phi)\subset\mathbb{V}^{\circ\circ}$, *and*
- $\mathrm{supp}(\widehat{\phi})\subset\mathbb{V}^{\circ}$.

Then $\phi=0$.

Proof. If $\dim\mathbb{V}$ is odd, we consider the "doubling" quadratic space $\mathbb{V}\oplus\mathbb{V}$ and the function $\phi\otimes\phi\in C_c^\infty(\mathbb{V}\oplus\mathbb{V})$. It is easy to see that $\mathrm{supp}(\phi\otimes\phi)=\mathrm{supp}(\phi)\times\mathrm{supp}(\phi)$, $\widehat{\phi\otimes\phi}=\widehat{\phi}\otimes\widehat{\phi}$, $\mathbb{V}^\circ\times\mathbb{V}^\circ\subset(\mathbb{V}\oplus\mathbb{V})^\circ$, and $\mathbb{V}^{\circ\circ}\times\mathbb{V}^{\circ\circ}\subset(\mathbb{V}\oplus\mathbb{V})^{\circ\circ}$. Therefore it suffices to consider the case when $\dim\mathbb{V}$ is even, which we assume from now on. We use the the Weil representation ω of $\mathrm{SL}_2(F)$. The group $\mathrm{SL}_2(F)$ acts on $C_c^\infty(\mathbb{V})$ by the following formula

$$\begin{aligned}\omega\begin{pmatrix}a&\\&a^{-1}\end{pmatrix}\phi(x)&=\chi_{\mathbb{V}}(a)|a|^{d/2}\phi(ax),\\ \omega\begin{pmatrix}1&b\\&1\end{pmatrix}\phi(x)&=\psi\left(\frac{1}{2}b\,(x,x)\right)\phi(x),\\ \omega\begin{pmatrix}&1\\-1&\end{pmatrix}\phi(x)&=\gamma_{\mathbb{V}}\,\widehat{\phi}(x),\end{aligned} \tag{8.1.2.1}$$

where $\chi_{\mathbb{V}}$ is a quadratic character of $F^\times$ associated to the quadratic space $\mathbb{V}$, and $\gamma_{\mathbb{V}}$ is the Weil constant (see [Kud97b, p.642]).

By the assumption on the support, the functions ϕ and $\widehat{\phi}$ are fixed by $N(\varpi^{-1}O_F)$ and $N(O_F)$ respectively, where N denotes the unipotent subgroup of the standard Borel of SL_2 of upper triangular matrices. Therefore ϕ is fixed by $N(\varpi^{-1}O_F)$ and $N_-(O_F)$ (the transpose of $N(O_F)$). However, $N(\varpi^{-1}O_F)$ and $N_-(O_F)$ generate $\mathrm{SL}_2(F)$. We sketch a proof of this well-known fact. Using the following identity in $\mathrm{SL}_2(F)$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1&a/c\\&1\end{pmatrix}\begin{pmatrix}&-1/c\\c&\end{pmatrix}\begin{pmatrix}1&d/c\\&1\end{pmatrix},\quad c\neq 0,$$

it is easy to show that the group $\mathrm{SL}_2(F)$ is generated by $N(F)$ and any single element in $\mathrm{SL}_2(F)\setminus B(F)$. Now we first apply the above equality to $\begin{pmatrix}1&\\1&1\end{pmatrix}$ (resp. $\begin{pmatrix}1&\\\varpi&1\end{pmatrix}$) to generate $\begin{pmatrix}&-1\\1&\end{pmatrix}$

(resp. $\begin{pmatrix} & -1/\varpi \\ \varpi & \end{pmatrix}$). Then we note that $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} & -1/\varpi \\ \varpi & \end{pmatrix} = \begin{pmatrix} -\varpi & \\ & -1/\varpi \end{pmatrix}$ and this element together with $N(\varpi^{-1}O_F)$ generate $N(F)$.

It follows that ϕ is fixed by $\mathrm{SL}_2(F)$ and therefore $\mathrm{supp}(\phi)$ is contained in the null cone $\{x \in \mathbb{V} : (x,x) = 0\}$ (e.g., by using the invariance under the diagonal torus, or $N(F)$). Since ϕ is locally constant, it must vanish identically. $\square$

Remark 8.1.3. The uncertainty principle is also used in the new proof by Beuzart-Plessis [BP21] of the Jacquet–Rallis fundamental lemma.

Corollary 8.1.4. *Let $\phi \in C_c^\infty(\mathbb{V})$ satisfy*

- $\mathrm{supp}(\phi) \subset \mathbb{V}^{\circ\circ}$*, and*
- $\widehat{\phi}$ *is a multiple of* ϕ*.*

Then $\phi = 0$.

8.1.5. *Hermitian case.* Now we return to the case of hermitian space with respect to a (possibly ramified) quadratic extension F/F_0 where F_0 is non-archimedean local field with characteristic not equal to 2. Define $\mathbb{V}^\circ$ and $\mathbb{V}^{\circ\circ}$

(8.1.5.1) $$\mathbb{V}^\circ = \{x \in \mathbb{V} \mid \mathrm{val}(x) \geq 0\}, \quad \mathbb{V}^{\circ\circ} = \{x \in \mathbb{V} \mid \mathrm{val}(x) > 0\},$$

where we recall that $\mathrm{val}(x) = \mathrm{val}((x,x))$ for the hermitian form $(\ ,\ )$ on $\mathbb{V}$, cf. §1.7.

Proposition 8.1.6. *Let $\phi \in C_c^\infty(\mathbb{V})$ satisfy*

- $\mathrm{supp}(\phi) \subset \mathbb{V}^{\circ\circ}$*, and*
- $\mathrm{supp}(\widehat{\phi}) \subset \mathbb{V}^\circ$*.*

Then $\phi = 0$.

Proof. Consider the symmetric bilinear form on the underlying F_0-vector space of $\mathbb{V}$,

$$(x,y)_{F_0} := \mathrm{tr}_{F/F_0}(x,y) \in F_0, \quad x, y \in \mathbb{V}.$$

Then the Fourier transform on $C_c^\infty(\mathbb{V})$ defined in §1.7 using the hermitian form $(\ ,\ )$ on $\mathbb{V}$ is the same as the one in (8.1.1.2) using $(\ ,\ )_{F_0}$ on the underlying F_0-vector space of $\mathbb{V}$. Since $(x,x)_{F_0}/2 = (x,x)$ for any $x \in \mathbb{V}$, the cones defined by (8.1.1.1) and (8.1.5.1) coincide. Therefore the desired assertion follows from Proposition 8.1.2. $\square$

The uncertainty principle implies that, by Lemma 6.3.1, the function $\mathrm{Int}_{L^\flat,\mathscr{V}}$ is determined by its restriction to

$$\mathbb{V}^\circ \setminus \mathbb{V}^{\circ\circ} = \{x \in \mathbb{V} \mid \mathrm{val}(x) = 0\}.$$

Ideally one would like to prove the same conclusion as Lemma 6.3.1 holds for the function $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$. Then, by induction on $\dim \mathbb{V}$, we can prove the main Theorem 3.4.1. However, we have not succeeded finding a direct proof the analog of Corollary 6.3.3 for $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$. Nevertheless, a weaker version of the uncertainty principle suffices to prove the identity $\mathrm{Int}_{L^\flat,\mathscr{V}} = \partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ and this is what we will actually do in the next subsection. A posteriori we can deduce that the function $\partial\mathrm{Den}_{L^\flat,\mathscr{V}}$ also satisfies the same identity as $\mathrm{Int}_{L^\flat,\mathscr{V}}$ does in Corollary 6.3.3.

8.2. **The proof of Theorem 3.4.1.** We now prove the main Theorem 3.4.1. Fix a rank $n-1$ lattice $L^\flat \subset \mathbb{V}$ such that $L^\flat_F$ is non-degenerate. We want to prove an identity of functions on $\mathbb{V} \setminus L^\flat_F$

$$\mathrm{Int}_{L^\flat} = \partial\mathrm{Den}_{L^\flat}.$$

By Theorem 6.1.3, equivalently we need to show

Theorem 8.2.1. *Let $L^\flat \subset \mathbb{V}$ be a rank $n-1$ lattice such that $L^\flat_F$ is non-degenerate. Then*

$$\mathrm{Int}_{L^\flat,\mathscr{V}} = \partial\mathrm{Den}_{L^\flat,\mathscr{V}} \tag{8.2.1.1}$$

as elements in $C_c^\infty(\mathbb{V})$.

Proof. We prove the assertion by induction on $\mathrm{val}(L^\flat)$. Let $(a_1, a_2, \cdots, a_{n-1})$ be the fundamental invariants of the lattice $L^\flat$, cf. §7.1. Let $M = M(L^\flat) = L^\flat \ominus \langle u\rangle$ be the lattice defined by (7.1.0.3).

Lemma 8.2.2. *Let $x \in \mathbb{V} \setminus L^\flat_F$ and let $(a'_1, a'_2, \cdots, a'_n)$ be the fundamental invariants of the lattice $L^\flat + \langle x\rangle$. Then the inequality*

$$a'_1 + \cdots + a'_{n-1} \geq a_1 + \cdots + a_{n-1} \tag{8.2.2.1}$$

holds if and only if $x \in M$.

Proof. For $x \in M$, we write $x = x^\flat + x^\perp$ where $x^\flat \in L^\flat$ and $x^\perp \perp L^\flat$. Then $L^\flat + \langle x\rangle = L^\flat + \langle x^\perp\rangle$. Therefore we may assume that $x \perp L^\flat$. It follows that $\mathrm{val}(x) \geq a_{n-1}$ by the definition of the lattice M, and $a'_i = a_i$ for $1 \leq i \leq n-1$. Hence $a'_1 + \cdots + a'_{n-1} = a_1 + \cdots + a_{n-1}$, and the equation (8.2.2.1) holds.

We now assume that the inequality (8.2.2.1) holds. We start with a special case. If $x \perp L^\flat$, the fundamental invariants of the lattice $L^\flat + \langle x\rangle$ is a re-ordering of $(a_1, a_2, \cdots, a_{n-1}, \mathrm{val}(x))$. From the inequality (8.2.2.1), it follows that $\mathrm{val}(x) \geq a_{n-1}$, and therefore $x \in M$.

Now we consider the general case. Let $\{e_1, \cdots, e_{n-1}\}$ be an orthogonal basis of $L^\flat$ such that $(e_i, e_i) = \varpi^{a_i}$. Write

$$x = \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} + x^\perp,$$

where $\lambda_i \in F, 1 \leq i \leq n-1$ and $x^\perp \perp L^\flat$. The fundamental matrix of the basis $\{e_1, \cdots, e_{n-1}, x\}$ of $L^\flat + \langle x\rangle$ is of the form

$$T = \begin{pmatrix} \varpi^{a_1} & & & (e_1, x) \\ & \ddots & & \vdots \\ & & \varpi^{a_{n-1}} & (e_{n-1}, x) \\ (x, e_1) & \cdots & (x, e_{n-1}) & (x, x) \end{pmatrix}.$$

We now use the characterization of the sum $a'_1 + \cdots + a'_{n-1}$ as the minimum among the valuations of the determinants of all $(n-1) \times (n-1)$-minors of T. The set of such minors is bijective to the set of (i,j)-th entry: removing i-th row and j-th column to get such a minor. The valuation of the determinant of the (n,i)-th minor is

$$\mathrm{val}((e_i, x)) - a_i + (a_1 + \cdots + a_{n-1}).$$

From the inequality (8.2.2.1), it follows that

$$\mathrm{val}((e_i, x)) \geq a_i,$$

or equivalently $\lambda_i \in O_F$, for all $1 \leq i \leq n-1$. Therefore $x - x^{\perp} \in L^{\flat}$, and $L^{\flat} + \langle x \rangle = L^{\flat} + \langle x^{\perp} \rangle$. Now we can assume that $x \perp L^{\flat}$ and by the special case above we complete the proof. $\square$

Now we assume that the equation (8.2.1.1)

$$\mathrm{Int}_{L'^{\flat}, \mathscr{V}} = \partial\mathrm{Den}_{L'^{\flat}, \mathscr{V}}$$

holds for $L'^{\flat}$ such that $\mathrm{val}(L'^{\flat}) < \mathrm{val}(L^{\flat})$. We may further assume that $L^{\flat} + \langle x \rangle$ is integral and has a basis $(e'_1, e'_2, \cdots, e'_n)$ such that $\mathrm{val}(e'_i) = a'_i$. Let $L'^{\flat} = \langle e'_1, \cdots, e'_{n-1} \rangle$. Then we have

$$\mathrm{Int}_{L^{\flat}}(x) = \mathrm{Int}_{L'^{\flat}}(x'), \text{ and } \quad \partial\mathrm{Den}_{L^{\flat}}(x) = \partial\mathrm{Den}_{L'^{\flat}}(x'),$$

where $x' = e'_n$. By Lemma 8.2.2, if $x \notin M$, then we have a strict inequality

$$a'_1 + \cdots + a'_{n-1} < a_1 + \cdots + a_{n-1}.$$

And so $\mathrm{val}(L'^{\flat}) < \mathrm{val}(L^{\flat})$. By induction hypothesis, we have

$$\mathrm{Int}_{L'^{\flat}, \mathscr{V}}(x') = \partial\mathrm{Den}_{L'^{\flat}, \mathscr{V}}(x').$$

It follows that the support of the difference

$$\phi = \mathrm{Int}_{L^{\flat}, \mathscr{V}} - \partial\mathrm{Den}_{L^{\flat}, \mathscr{V}} \in C_c^{\infty}(\mathbb{V})$$

is contained in the lattice M.

By Corollary 6.3.3, we know

$$\widehat{\mathrm{Int}}_{L^{\flat}, \mathscr{V}}(x) = -\,\mathrm{Int}_{L^{\flat}, \mathscr{V}}(x).$$

In particular, if $\mathrm{val}(x) < 0$, then

$$\widehat{\mathrm{Int}}_{L^{\flat}, \mathscr{V}}(x) = 0.$$

We know a little less about $\partial\mathrm{Den}_{L^{\flat}, \mathscr{V}}$: by Theorem 7.4.1, the vanishing $\widehat{\partial\mathrm{Den}}_{L^{\flat}, \mathscr{V}}(x) = 0$ holds when $\mathrm{val}(x) < 0$ *and* $x \perp L^{\flat}$. It follows that, when $\mathrm{val}(x) < 0$ and $x \perp L^{\flat}$,

$$\widehat{\phi}(x) = 0.$$

Obviously the function ϕ is invariant under $L^{\flat}$. By the constraints imposed by the support of ϕ (being contained in M), it is of the form

$$\phi = \mathbf{1}_{L^{\flat}} \otimes \phi_{\perp},$$

where $\phi_{\perp} \in C_c^{\infty}((L_F^{\flat})^{\perp})$ is supported on the (rank one) lattice $M_{\perp} = \langle u \rangle$. Then

$$\widehat{\phi} = \mathrm{vol}(L^{\flat}) \mathbf{1}_{L^{\flat, \vee}} \otimes \widehat{\phi}_{\perp}.$$

Here $\widehat{\phi}_{\perp}$ is invariant under the translation by the dual lattice $M_{\perp}^{\vee} = \langle u^{\vee} \rangle$, where $u^{\vee} = \varpi^{-a_n} u$. Note that $\mathrm{val}(u^{\vee}) = -a_n < 0$. Now the Fourier transform $\widehat{\phi}_{\perp}$ vanishes at every $x \perp L^{\flat}$ such that $\mathrm{val}(x) < 0$. It follows that $\widehat{\phi}_{\perp}$ vanishes identically. Therefore $\phi = 0$. This completes the proof. $\square$

Part 2. Local Kudla–Rapoport conjecture: the almost self-dual case

9. Local density for an almost self-dual lattice

9.1. **Local density for an almost self-dual lattice.** In this section we allow F_0 to be a non-archimedean local field of characteristic not equal to 2 (but possibly with residue characteristic 2), and F an unramified quadratic extension.

Recall that we have defined the local density for two hermitian O_F-lattices L and M

$$\mathrm{Den}(M,L)=\lim_{N\to+\infty}\frac{\#\mathrm{Rep}_{M,L}(O_{F_0}/\varpi^N)}{q^{N\cdot\dim(\mathrm{Rep}_{M,L})_{F_0}}}$$

in terms of the scheme $\mathrm{Rep}_{M,L}$, cf. (3.1.0.1) in Section 3.1.

Let L be a hermitian O_F-lattice of rank n. For $k\geq 0$, set

$$M=\langle 1\rangle^{n-1+k}\obot\langle\varpi\rangle,\quad \widetilde{M}=\langle 1\rangle^{n+1+k},$$

and

$$L^\sharp=L\obot\ell,\quad \ell=\langle u_0\rangle,\quad (u_0,u_0)=\varpi. \tag{9.1.0.2}$$

We then have the following “cancellation law”.

Lemma 9.1.1. *Let $k\geq 0$. Then*

$$\mathrm{Den}(M,L)=\frac{\mathrm{Den}(\widetilde{M},L^\sharp)}{\mathrm{Den}(\widetilde{M},\ell)}.$$

Proof. For any hermitian O_F-lattice L, we denote

$$L_i=L\otimes_{O_F}O_F/\varpi^i,$$

endowed with the reduction of the hermitian form.

Then the restriction to ℓ_i defines a map

$$\begin{aligned}\mathrm{Res}\colon\ \mathrm{Herm}(L_i^\sharp,\widetilde{M}_i)&\longrightarrow\mathrm{Herm}(\ell_i,\widetilde{M}_i)\\ \varphi&\longmapsto\varphi|_{\ell_i}.\end{aligned}$$

Let $\varphi\in\mathrm{Herm}(L_i^\sharp,\widetilde{M}_i)$. Denote by $\varphi(\ell_i)^\perp$ the orthogonal complement in $\widetilde{M}_i$ of the image $\varphi(\ell_i)$, i.e.,

$$\varphi(\ell_i)^\perp=\{x\in\widetilde{M}_i\mid (x,\varphi(\ell_i))=0\}.$$

Now let $i\geq 2$. We *claim* that there is an isomorphism of hermitian modules over O_F/ϖ^i:

$$\varphi(\ell_i)^\perp\xrightarrow{\ \sim\ }M_i\ .$$

Since the norm of u_0 has valuation one, so does its image $w_0:=\varphi(u_0)\in\widetilde{M}_i$ (this makes sense when $i\geq 2$). Hence $w_0\notin\varpi\widetilde{M}_i$, i.e., $w_0 \bmod \varpi\neq 0\in\widetilde{M}_1=\widetilde{M}_i\otimes_{O_F/\varpi^i}O_F/\varpi$. By the non-degeneracy of the hermitian form on the reduction $\widetilde{M}_i$, the map

$$\begin{aligned}\widetilde{M}_i&\longrightarrow O_F/\varpi^i\\ x&\longmapsto(x,w_0)\end{aligned}$$

is surjective, and its kernel is $\varphi(\ell_i)^\perp$ by definition. The kernel is a free module over O_F/ϖ^i (since it must be flat, being the kernel of a surjective morphism between finite free modules; alternatively, look at the reduction mod ϖ and apply Nakayama's lemma).

Now there exists $w_0' \in \widetilde{M}_i$ such that $(w_0', w_0) = 1$. Then $\{w_0, w_0'\}$ span a self-dual submodule of rank two, which must be an orthogonal direct summand of $\widetilde{M}_i$, again by non-degeneracy of the hermitian form on $\widetilde{M}_i$. This reduces the assertion $\varphi(\ell_i)^\perp \simeq M_i$ to the case $\operatorname{rank} \widetilde{M}_i = 2$. In the rank two case, it is easy to verify the desired isomorphism, e.g., using the basis $\{w_0, w_0'\}$. This proves the claim.

Note that the fiber of the map Res above $\varphi|_{\ell_i}$ is the set $\mathrm{Herm}(L_i, \varphi(\ell_i)^\perp)$ (and $\varphi(\ell_i)^\perp$ depends only on the restriction $\varphi|_{\ell_i}$). It follows from the claim that the fiber has a constant cardinality (in particular, the map Res is surjective), namely that of $\mathrm{Herm}(L_i, M_i)$. Hence,

$$\#\mathrm{Herm}(L_i^\sharp, \widetilde{M}_i) = \#\mathrm{Herm}(L_i, M_i) \cdot \#\mathrm{Herm}(\ell_i, \widetilde{M}_i).$$

The result then follows from

$$r(L^\sharp)(2r(\widetilde{M}) - r(L^\sharp)) = r(L)(2r(M) - r(L)) + r(\ell)(2r(\widetilde{M}) - r(\ell)),$$

where r denotes the rank, cf. (3.1.0.2). □

Recall that by (3.2.0.3)

$$\mathrm{Den}(\langle 1\rangle^{n-1+k}, \langle 1\rangle^{n-1}) = \prod_{i=1}^{n-1}(1-(-q)^{-i}X)\Big|_{X=(-q)^{-k}}.$$

Theorem 9.1.2. *Let $\Lambda = \langle 1\rangle^{n-1} \obot \langle \varpi\rangle$. Let $k \geq 0$ and L be a hermitian O_F-lattice of rank n. Then*

$$\frac{\mathrm{Den}(\Lambda \obot \langle 1\rangle^k, L)}{\mathrm{Den}(\langle 1\rangle^{n-1+k}, \langle 1\rangle^{n-1})} = \mathrm{Den}(X, L^\sharp)\Big|_{X=(-q)^{-k}}.$$

Proof. By (3.2.0.3), we have

$$\mathrm{Den}(\langle 1\rangle^{n+1+k}, \langle 1\rangle^{1}) = (1-(-q)^{-1}X)\Big|_{X=(-q)^{-n-k}}.$$

and

$$\mathrm{Den}(\langle 1\rangle^{n+1+k}, \langle 1\rangle^{n+1}) = \prod_{i=1}^{n+1}(1-(-q)^{-i}X)\Big|_{X=(-q)^{-k}}.$$

It follows that

$$\begin{aligned}\frac{\mathrm{Den}(\langle 1\rangle^{n+1+k}, \langle 1\rangle^{n+1})}{\mathrm{Den}(\langle 1\rangle^{n+1+k}, \langle 1\rangle^{1})} &= \prod_{i=1}^{n}(1-(-q)^{-i}X)\Big|_{X=(-q)^{-k}} \\ &= \mathrm{Den}(\langle 1\rangle^{n+k}, \langle 1\rangle^{n}).\end{aligned}$$

(Alternatively, repeat the proof of Lemma 9.1.1 in the case ℓ a self-dual lattice of rank one.)

By Example 3.5.2, we have $\mathrm{Den}(X, \ell) = 1 - X$, and hence

$$\frac{\mathrm{Den}(\langle 1\rangle^{n+1+k}, \ell)}{\mathrm{Den}(\langle 1\rangle^{n+1+k}, \langle 1\rangle^{1})} = \mathrm{Den}((-q)^{-n-k}, \ell) = (1-(-q)^{-n}X)\Big|_{X=(-q)^{-k}}.$$

It follows that

$$\begin{aligned}\frac{\mathrm{Den}(\langle 1\rangle^{n+1+k},\langle 1\rangle^{n+1})}{\mathrm{Den}(\langle 1\rangle^{n+1+k},\ell)} &= \prod_{i=1}^{n-1}(1-(-q)^{-i}X)\Big|_{X=(-q)^{-k}} \\ &= \mathrm{Den}(\langle 1\rangle^{n-1+k},\langle 1\rangle^{n-1}).\end{aligned}$$

Finally, by Lemma 9.1.1, we obtain

$$\begin{aligned}\frac{\mathrm{Den}(\langle 1\rangle^{n-1+k}\oplus\langle\varpi\rangle, L)}{\mathrm{Den}(\langle 1\rangle^{n-1+k},\langle 1\rangle^{n-1})} &= \frac{\mathrm{Den}(\langle 1\rangle^{n+1+k},L^\sharp)/\mathrm{Den}(\langle 1\rangle^{n+1+k},\ell)}{\mathrm{Den}(\langle 1\rangle^{n+1+k},\langle 1\rangle^{n+1})/\mathrm{Den}(\langle 1\rangle^{n+1+k},\ell)} \\ &= \mathrm{Den}(X,L^\sharp)\Big|_{X=(-q)^{-k}}.\end{aligned}$$

This completes the proof. □

Example 9.1.3 (The case $\operatorname{rank} L = 2$)**.** If $\operatorname{rank} L = 2$, Theorem 9.1.2 above specializes to Sankaran's formula [San17, Proposition 3.1] which we recall now. Let $L = \langle\varpi^a\rangle \oplus \langle\varpi^b\rangle, a \le b$, $a+b$ even. Define

$$\epsilon = \begin{cases} 0, & \text{if } b \text{ is even} \\ 1, & \text{if } b \text{ is odd.}\end{cases}$$

Then the formula *loc. cit.* asserts that the LHS of Theorem 9.1.2 is equal to

$$\begin{aligned}(1-X)(X^2-(q^2-q)X+1)^\epsilon + \frac{1-X}{1-q^{-1}X}\Bigg\{ qX(1-q)\frac{(qX)^b-(qX)^\epsilon}{qX-1} \\ + X^2(q-q^{-1}X)\frac{X^{2b}-X^{2\epsilon}}{X^2-1} + \left[-q^{b+1}(X-1)+qX^{b+1}-q^{-1}X^{b+2}\right]\frac{X^{a+1}-X^{b+1}}{X^2-1}\Bigg\}.\end{aligned} \tag{9.1.3.1}$$

On the other hand, this is consistent with the explicit formula for $\mathrm{Den}(X, L^\sharp)$ given by [Ter13a, proof of Theorem 5.2].

$$\mathrm{Den}(X,L^\sharp) = \frac{1}{1+X}\left\{\sum_{l=0}^{b+1}X^l(q^l-q^{1+b-l}X^{a+1}) - \sum_{l=0}^{b-1}X^{1+l}(q^{2+l}-q^{1+b-l}X^{a+1})\right\}. \tag{9.1.3.2}$$

In fact, two functions on $(a,b)\in(\mathbb{Z}_{\ge 0})^2$ (not only for $(a+b)$ such that $2 \mid a+b$) are characterized by the following properties:

- The value at $(0,0)$ (resp., $(1,1)$) is $1-X$ (resp., $(1-X)(X^2-(q^2-q)X+1)$).
- The term involving a is

$$\begin{aligned}&\frac{1-X}{1-q^{-1}X}\cdot\left[-q^{b+1}(X-1)+qX^{b+1}-q^{-1}X^{b+2}\right]\cdot\frac{X^{a+1}}{X^2-1} \\ =&\frac{1}{1+X}X^{a+1}\left\{-\sum_{l=0}^{b+1}X^l q^{1+b-l}+\sum_{l=0}^{b-1}X^{1+l}q^{1+b-l}\right\}.\end{aligned}$$

  The two expressions come from (9.1.3.1) and (9.1.3.2) respectively.
- The term not involving a is a function ϕ in one variable $b \in \mathbb{Z}_{\ge 0}$, which satisfies a difference equation

$$\phi(b)-\phi(b-1) = \frac{1}{1+X}q^{b+1}X^b(X-1).$$

The difference equation is easy to see from (9.1.3.2), and from (9.1.3.1) by a straightforward calculation.

Definition 9.1.4. After Theorem 9.1.2, define the (normalized) *local Siegel series relative to* $\Lambda = \langle 1\rangle^{n-1} \oplus \langle \varpi\rangle$ as the polynomial $\mathrm{Den}_\Lambda(X, L) \in \mathbb{Z}[X]$ such that

$$\mathrm{Den}_\Lambda((-q)^{-k}, L) = \frac{\mathrm{Den}(\Lambda \oplus \langle 1\rangle^k, L)}{\mathrm{Den}(\langle 1\rangle^{n-1} \oplus \langle 1\rangle^k, \langle 1\rangle^{n-1})}. \tag{9.1.4.1}$$

Then by Theorem 9.1.2,

$$\mathrm{Den}_\Lambda(X, L) = \mathrm{Den}(X, L^\sharp) \in \mathbb{Z}[X]. \tag{9.1.4.2}$$

In particular, if $\mathrm{val}(L)$ is even, then $\mathrm{Den}_\Lambda(1, L) = 0$. In this case, we denote the central derivative of local density by

$$\partial\mathrm{Den}_\Lambda(L) := -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}_\Lambda(X, L).$$

9.2. **Relation with local Whittaker functions.** Let $\Lambda = \langle 1\rangle^{n-1} \oplus \langle\varpi\rangle$ be an almost self-dual hermitian O_F-lattice. Let L be a hermitian O_F-lattice of rank n. Let $T = ((x_i, x_j))_{1\le i,j\le n}$ be the fundamental matrix of an O_F-basis $\{x_1, \dots, x_n\}$ of L, an $n\times n$ hermitian matrix over F. Associated to the standard Siegel–Weil section of the characteristic function $\varphi_1 = \mathbf{1}_{\Lambda^n}$ and the unramified additive character $\psi : F_0 \to \mathbb{C}^\times$, there is a local (generalized) Whittaker function $W_T(g, s, \varphi_1)$ (see §12.2, §12.3 for the precise definition). By [KR14, Proposition 10.1], when $g = 1$, it satisfies the interpolation formula for integers $s = k \ge 0$ (notice $\gamma_p(V) = -1$ in the notation there),

$$W_T(1, k, \varphi_1) = (-q)^n \cdot \mathrm{Den}(\Lambda \oplus \langle 1\rangle^{2k}, L).$$

So its value at $s = 0$ is

$$W_T(1, 0, \varphi_1) = (-q)^{-n} \cdot \mathrm{Den}(\Lambda, L) = (-q)^{-n} \cdot \mathrm{Den}_\Lambda(L) \cdot \mathrm{Den}(\langle 1\rangle^{n-1}, \langle 1\rangle^{n-1}),$$

and its derivative at $s = 0$ is

$$W_T'(1, 0, \varphi_1) = (-q)^{-n} \cdot \partial\mathrm{Den}_\Lambda(L) \cdot \mathrm{Den}(\langle 1\rangle^{n-1}, \langle 1\rangle^{n-1}) \cdot \log q^2.$$

Plugging in (3.2.0.3), we obtain

$$W_T(1, 0, \varphi_1) = \mathrm{Den}_\Lambda(L) \cdot (-q)^{-n} \prod_{i=1}^{n-1} (1 - (-q)^{-i}), \tag{9.2.0.3}$$

$$W_T'(1, 0, \varphi_1) = \partial\mathrm{Den}_\Lambda(L) \cdot (-q)^{-n} \prod_{i=1}^{n-1} (1 - (-q)^{-i}) \cdot \log q^2. \tag{9.2.0.4}$$

10. Kudla–Rapoport cycles in the almost principally polarized case

10.1. **Rapoport–Zink spaces $\mathcal{N}_n^1$ with almost self-dual level.** We recall the construction from [RSZ18, §5]. For a $\mathrm{Spf}\, O_{\breve F}$-scheme S, we consider triples (Y, ι, λ) over S as in §2.1, except that λ is no longer principal; instead, it is required that $\ker\lambda \subseteq Y[\iota(\varpi)]$ and has order q^2. Up to O_F-linear quasi-isogeny compatible with polarizations, there is a unique such triple $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ over $\mathrm{Spec}\, \bar k$. Let $\mathcal{N}^1 = \mathcal{N}_n^1 = \mathcal{N}_{F/F_0,n}^1$ be the formal scheme over $\mathrm{Spf}\, O_{\breve F}$ which represents the functor sending each S to the set of isomorphism classes of tuples $(Y, \iota, \lambda, \rho)$, where the *framing* $\rho : Y \times_S \bar{S} \to \mathbb{Y} \times_{\operatorname{Spec} \bar{k}} \bar{S}$ is an O_F-linear quasi-isogeny of height 0 such that $\rho^*((\lambda_{\mathbb{Y}})_{\bar{S}}) = \lambda_{\bar{S}}$.

The Rapoport–Zink space $\mathcal{N}^1 = \mathcal{N}_n^1$ is a formal scheme, locally formally of finite type, regular, of relative dimension $n-1$ and has semi-stable reduction over Spf $O_{\breve{F}}$ ([RSZ18, Theorem 5.1], [Cho18, Theorem 1.2]). Denote

$$\mathbb{W}_n = \operatorname{Hom}^\circ_{O_F}(\overline{\mathbb{E}}, \mathbb{Y}),$$

and endow it with the hermitian form by the formula similar to $\mathbb{V}_n$ (cf. §2.2). Then $\mathbb{W}_n$ is a *split* hermitian space of dimension n. Similar to §2.3, for every non-zero $x \in \mathbb{W}_n$ we can define the special divisor on $\mathcal{N}_n^1$, denoted by $\mathcal{Y}(x)$ (resp. $\mathcal{Y}'(x)$), over which the special homomorphism x (resp. $\lambda_{\mathbb{Y}} \circ x$) extends to a homomorphism $\bar{\mathcal{E}}_S \to Y$ (resp. $\bar{\mathcal{E}}_S \to Y^\vee$). Then, by a similar argument to the self-dual case in [KR11, Proposition 3.5], the special divisors $\mathcal{Y}(x)$ and $\mathcal{Y}'(x)$ are Cartier (cf. [Cho18, Proposition 5.9], denoted by $\mathcal{Z}(x)$ and $\mathcal{Y}(x)$ in *loc. cit.*).

For the later use, we recall from [Cho18, Proposition 5.10]

$$(10.1.0.5) \qquad \begin{cases} \mathcal{Y}(x) \simeq \mathcal{N}_{n-1}^1, & \text{when } \operatorname{val}(x) = 0, \\ \mathcal{Y}'(x) \simeq \mathcal{N}_{n-1}, & \text{when } \operatorname{val}(x) = -1. \end{cases}$$

We only indicate the construction of the second isomorphism $\mathcal{Y}'(x) \simeq \mathcal{N}_{n-1}$, since the first one is rather close to the self-dual case (cf. (2.11.0.1)). We may assume that $(x, x) = \varpi^{-1}$. Fix an O_F-linear isomorphism

$$(10.1.0.6) \qquad \beta \colon \mathbb{X}_{n-1} \times \overline{\mathbb{E}} \longrightarrow \mathbb{Y},$$

such that $\beta^*(\lambda_{\mathbb{Y}}) = \lambda_{\mathbb{X}_{n-1}} \times \varpi\lambda_{\overline{\mathbb{E}}}$ and the restriction of β to the second factor is $\varpi x \in \mathbb{W}_n$. Then we have an orthogonal decomposition $\mathbb{W}_n = \mathbb{V}_{n-1} \oplus \langle x \rangle_F$. Then we define a map $\delta : \mathcal{N}_{n-1} \to \mathcal{N}_n^1$ sending $(X^\flat, \iota_{X^\flat}, \lambda_{X^\flat}, \rho_{X^\flat}) \in \mathcal{N}_{n-1}(S)$ to $(X^\flat \times \overline{\mathcal{E}}, \iota_{X^\flat} \times \iota_{\overline{\mathcal{E}}}, \lambda_{X^\flat} \times \varpi\lambda_{\overline{\mathcal{E}}}, \rho_{X^\flat} \times \rho_{\overline{\mathcal{E}}}) \in \mathcal{N}_n^1(S)$. Then the homomorphism $(0, \lambda_{\overline{\mathcal{E}}}) : \overline{\mathcal{E}} \to (X^\flat)^\vee \times \overline{\mathcal{E}}^\vee$ extends $\lambda_{\mathbb{Y}} \circ x \in \mathbb{W}_n$ and the map δ defines an isomorphism $\delta : \mathcal{N}_{n-1} \simeq \mathcal{Y}'(x)$.

Definition 10.1.1. Let $L \subset \mathbb{W}_n$ be an O_F-lattice of rank n and let $x_1, \cdots, x_n$ be an O_F-basis of L. Then we define

$$(10.1.1.1) \qquad \operatorname{Int}'(L) \coloneqq \chi\big(\mathcal{N}_n^1, \mathcal{O}_{\mathcal{Y}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Y}(x_n)}\big).$$

We have not justified the independence of the choice of the basis, which will be proved under a conjectural relation between $\mathcal{N}_n^1$ and some auxiliary Rapoport–Zink spaces. It turns out that $\operatorname{Int}'(L)$ is not equal to the derived local density $\partial\operatorname{Den}_\Lambda(L)$ (cf. Theorem 10.5.1 below). This is a typical phenomenon in the presence of bad reductions, cf. [KR00b, San17, RSZ17, RSZ18]. Therefore, we will instead define a variant $\operatorname{Int}(L)$ of $\operatorname{Int}'(L)$, which will give an exact identity $\operatorname{Int}(L) = \partial\operatorname{Den}_\Lambda(L)$ (Theorem 10.3.1).

10.2. **Auxiliary Rapoport–Zink spaces.** Before we present our variant, we need an auxiliary moduli space (cf. [LRZ21]). Let $(\mathbb{X}_{n+1}, \iota_{\mathbb{X}_{n+1}}, \lambda_{\mathbb{X}_{n+1}})$ be as in §2.1. Fix an O_F-linear isogeny

$$(10.2.0.2) \qquad \alpha \colon \mathbb{Y} \times \overline{\mathbb{E}} \longrightarrow \mathbb{X}_{n+1},$$

such that $\ker\alpha \subset (\mathbb{Y}\times\overline{\mathbb{E}})[\varpi]$ and $\alpha^*(\lambda_{\mathbb{X}_{n+1}}) = \lambda_{\mathbb{Y}}\times\varpi\lambda_{\overline{\mathbb{E}}}$. Let $x_0 \in \mathbb{V}_{n+1} = \mathrm{Hom}^\circ_{O_F}(\overline{\mathbb{E}}, \mathbb{X}_{n+1})$ be the restriction of α to the second factor. Then the assumption implies that the norm of x_0 is $(x_0, x_0) = \varpi$, and we have an orthogonal decomposition

$$\mathbb{V}_{n+1} = \mathbb{W}_n \oplus \langle x_0\rangle_F.$$

We denote by

$$\widetilde{\mathcal{N}}^1_n \hookrightarrow \mathcal{N}^1_n \times_{\mathrm{Spf}\, O_{\breve F}} \mathcal{N}_{n+1} \tag{10.2.0.3}$$

the closed formal subscheme consisting of tuples $(Y, \iota_Y, \lambda_Y, \rho_Y, X, \iota_X, \lambda_X, \rho_X)$ such that α lifts to an isogeny $\widetilde{\alpha}\colon Y\times\overline{\mathcal{E}} \to X$. If α lifts, then $\widetilde{\alpha}$ is unique and satisfies $\ker\widetilde{\alpha} \subset (Y\times\overline{\mathcal{E}})[\varpi]$ and $\widetilde{\alpha}^*(\lambda_X) = \lambda_Y\times\varpi\lambda_{\overline{\mathcal{E}}}$.

We therefore obtain a diagram

$$\begin{tikzcd} & \widetilde{\mathcal{N}}^1_n \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ \mathcal{N}^1_n & & \mathcal{Z}(x_0) \arrow[r] & \mathcal{N}_{n+1}, \end{tikzcd} \tag{10.2.0.4}$$

where π_1 and π_2 are induced by the two projection maps in (10.2.0.3). Recall from [Ter13b] that the formal scheme $\mathcal{Z}(x_0)$ is regular.

Remark 10.2.1. Let $\Lambda = \langle 1\rangle^{n-1}\oplus\langle\varpi\rangle$ be as before. Let $\Lambda^\sharp$ be a self-dual lattice of rank $n+1$ containing $\Lambda\oplus\langle\varpi\rangle$; there are $q+1$ such lattices in the vector space $\Lambda_F\oplus\langle\varpi\rangle_F$. Then we have a natural embedding of hermitian spaces

$$W_n \coloneqq \Lambda\otimes_{O_F} F \hookrightarrow V_{n+1} \coloneqq \Lambda^\sharp\otimes_{O_F} F$$

and their isometry groups $\mathrm{U}(W_n)\hookrightarrow\mathrm{U}(V_{n+1})$. Let $K = \mathrm{Aut}(\Lambda)$ be the stabilizer of Λ, and similarly let $K^\sharp = \mathrm{Aut}(\Lambda^\sharp)$. Define $\widetilde{K} \coloneqq K\cap K^\sharp$ where the intersection is taken inside the unitary group $\mathrm{U}(V_{n+1})$:

$$\begin{tikzcd} & \widetilde{K} = K\cap K^\sharp \arrow[dl] \arrow[dr] & \\ K = \mathrm{Aut}(\Lambda) & & K^\sharp = \mathrm{Aut}(\Lambda^\sharp). \end{tikzcd}$$

Intuitively, the Rapoport–Zink spaces $\mathcal{N}^1_n, \widetilde{\mathcal{N}}^1_n$, and $\mathcal{N}_{n+1}$ correspond to the level structure $K, \widetilde{K}$, and $K^\sharp$ respectively. It is easy to see that the generic fiber of the map $\pi_1 : \widetilde{\mathcal{N}}^1_n \to \mathcal{N}^1_n$ is finite étale of degree $[K:\widetilde{K}] = q+1$, and the generic fiber of the map $\pi_2 : \widetilde{\mathcal{N}}^1_n \to \mathcal{Z}(x_0)$ is an isomorphism. Therefore, $\mathcal{Z}(x_0)$ is a regular integral model of a finite étale covering of the generic fiber of $\mathcal{N}^1_n$.

Let $x\in\mathbb{W}_n\subset\mathbb{V}_{n+1}$. Denote by $\mathcal{Z}^\flat(x)$ the restriction of the special divisor $\mathcal{Z}(x)$ (on $\mathcal{N}_{n+1}$) to $\mathcal{Z}(x_0)$, i.e.,

$$\mathcal{Z}^\flat(x) \coloneqq \mathcal{Z}(x_0)\cap\mathcal{Z}(x)$$

viewed as a formal subscheme on $\mathcal{Z}(x_0)$.

Remark 10.2.2. It will be clear (cf. Theorem 10.4.3) that the generic fiber of $\mathcal{Z}^\flat(x)$ (viewed as a divisor on the generic fiber of $\widetilde{\mathcal{N}}_n^1$ since π_2 is an isomorphism on the generic fibers) is equal to the pull back along π_1 of the generic fiber of $\mathcal{Y}(x)$ on $\mathcal{N}_n^1$. Therefore, we may use $\mathcal{Z}^\flat(x)$ as an integral model of the pull-back of the generic fiber of $\mathcal{Y}(x)$.

Motivated by Remark 10.2.1, we now define a variant of $\mathrm{Int}'(L)$.

Definition 10.2.3. Let $L \subset \mathbb{W}_n$ be an O_F-lattice of rank n and let $x_1, \cdots, x_n$ be a basis of L. Then we define

$$\mathrm{Int}(L; x_1, \cdots, x_n) = \frac{1}{q+1}\chi\big(\mathcal{Z}(x_0), \mathcal{Z}^\flat(x_1) \overset{\mathbb{L}}{\cap} \cdots \overset{\mathbb{L}}{\cap} \mathcal{Z}^\flat(x_n)\big), \tag{10.2.3.1}$$

where the derived tensor product is taken as $\mathcal{O}_{\mathcal{Z}(x_0)}$-sheaves.

10.3. **The** $\mathrm{Int} = \partial\mathrm{Den}$ **theorem.** The following theorem justifies our definition of the variant of intersection numbers.

Theorem 10.3.1. *Let $L \subseteq \mathbb{W}_n$ be an O_F-lattice of full rank n. Then, for any basis $x_1, \cdots, x_n$ of L, we have*

$$\mathrm{Int}(L; x_1, \cdots, x_n) = \frac{1}{q+1}\partial\mathrm{Den}_\Lambda(L).$$

In particular, $\mathrm{Int}(L; x_1, \cdots, x_n)$ *is independent of the choice of the basis and we therefore denote it by* $\mathrm{Int}(L)$.

Proof. Let $x \in \mathbb{W}_n$ be non-zero. Then $x \perp x_0$. By Lemma 2.8.1, we have

$$\mathcal{O}_{\mathcal{Z}^\flat(x)} = \mathcal{O}_{\mathcal{Z}(x)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_0)}$$

as elements in $K_0'(\mathcal{Z}(x_0))$. Therefore,

$$\chi\big(\mathcal{Z}(x_0), \mathcal{Z}^\flat(x_1) \overset{\mathbb{L}}{\cap} \cdots \overset{\mathbb{L}}{\cap} \mathcal{Z}^\flat(x_n)\big) = \chi\big(\mathcal{N}_{n+1}, \mathcal{Z}(x_0) \overset{\mathbb{L}}{\cap} \mathcal{Z}(x_1) \overset{\mathbb{L}}{\cap} \cdots \overset{\mathbb{L}}{\cap} \mathcal{Z}(x_n)\big),$$

which is $\mathrm{Int}(L^\sharp)$. By Theorem 3.4.1, this is equal to $\partial\mathrm{Den}(L^\sharp)$. By (9.1.4.2) we obtain $\partial\mathrm{Den}(L^\sharp) = \partial\mathrm{Den}_\Lambda(L)$ and the proof is complete. $\square$

Remark 10.3.2. In the notation of §9.2, it follows immediately from Theorem 10.3.1 and (9.2.0.4) that

$$\mathrm{Int}(L) = \frac{W_T'(1, 0, \varphi_1)}{\log q^2} \cdot \frac{(-q)^n - 1}{q+1} \cdot \prod_{i=1}^n (1 - (-q)^{-i})^{-1}.$$

10.4. **The comparison of two divisors.** We compare the two divisors $\mathcal{Y}(x)$ and $\mathcal{Z}^\flat(x)$ after pulling-back to $\widetilde{\mathcal{N}}_n^1$ along the diagram (10.2.0.4). The result is conditional on a conjectural relation between $\mathcal{N}_n^1$, $\widetilde{\mathcal{N}}_n^1$ and $\mathcal{Z}(x_0)$.

Recall from (10.2.0.4) that there are two projections π_1 and π_2. Let $\mathrm{Vert}^0(\mathbb{W}_n)$ be the set of self-dual lattices Λ in $\mathbb{W}_n$. For each type 1 lattice in $\mathbb{V}_{n+1}$ of the form $\Lambda \oplus \langle x_0 \rangle$ where $\Lambda \in \mathrm{Vert}^0(\mathbb{W}_n)$, there is a closed stratum $\mathcal{V}(\Lambda \oplus \langle x_0 \rangle) \subset \mathcal{N}_{n+1}^{\mathrm{red}}$ which consists of a superspecial point (cf. §2.7) contained in $\mathcal{Z}(x_0)$. Let $\mathcal{Z}(x_0)^{\mathrm{ss}} \subset \mathcal{Z}(x_0)$ be the union of all of such superspecial points. Note that $\mathcal{Z}(x_0)^{\mathrm{ss}}$ does not contain *all* superspecial points on $\mathcal{Z}(x_0)$. Define $\mathcal{Z}(x_0)^{\mathrm{ns}} := \mathcal{Z}(x_0) \setminus \mathcal{Z}(x_0)^{\mathrm{ss}}$ (as an open formal subscheme of $\mathcal{Z}(x_0)$).

By the Bruhat–Tits stratification of the reduced locus of $\mathcal{N}_n^1$ in [Cho18], there exist a family of (disjoint) closed formal subschemes $\mathbb{P}_\Lambda = \mathbb{P}^{n-1} \hookrightarrow \mathcal{N}_n^1$ indexed by $\Lambda \in \mathrm{Vert}^0(\mathbb{W}_n)$ (cf. Remark 2.15 of *loc. cit.*). Denote by $\mathcal{N}_n^{1,\mathrm{ss}}$ the (disjoint) union of them, and define $\mathcal{N}_n^{1,\mathrm{ns}} := \mathcal{N}_n^1 \setminus \mathcal{N}_n^{1,\mathrm{ss}}$.

Define $\widetilde{\mathcal{N}}_n^{1,\mathrm{ss}} := \pi_2^{-1}(\mathcal{Z}(x_0)^{\mathrm{ss}})$ and $\widetilde{\mathcal{N}}_n^{1,\mathrm{ns}} := \widetilde{\mathcal{N}}_n^1 \setminus \widetilde{\mathcal{N}}_n^{1,\mathrm{ss}}$. The following conjecture was observed by Kudla and Rapoport in an unpublished manuscript.

Conjecture 10.4.1.

(i) The morphism π_1 is finite flat of degree $q+1$, étale away from $\mathcal{N}_n^{1,\mathrm{ss}}$, and totally ramified along $\mathcal{N}_n^{1,\mathrm{ss}}$.

(ii) The formal scheme $\widetilde{\mathcal{N}}_n^1$ is regular. The morphism π_2 is proper and its restriction to $\widetilde{\mathcal{N}}_n^{1,\mathrm{ns}}$ induces an isomorphism $\widetilde{\mathcal{N}}_n^{1,\mathrm{ns}} \simeq \mathcal{Z}(x_0)^{\mathrm{ns}}$.

(iii) The closed formal subscheme $\widetilde{\mathcal{N}}_n^{1,\mathrm{ss}}$ of $\widetilde{\mathcal{N}}_n^1$ is a Cartier divisor and isomorphic to $\mathcal{N}_n^{1,\mathrm{ss}}$ under π_1. In particular, we may and will identify $\widetilde{\mathcal{N}}_n^{1,\mathrm{ss}}$ with $\mathcal{N}_n^{1,\mathrm{ss}} = \bigsqcup_{\Lambda \in \mathrm{Vert}^0(\mathbb{W}_n)} \mathbb{P}_\Lambda$.

(iv) For $\Lambda_1, \cdots, \Lambda_n \in \mathrm{Vert}^0(\mathbb{W}_n)$, we have

$$\chi\big(\widetilde{\mathcal{N}}_n^1, \mathbb{P}_{\Lambda_1} \overset{\mathbb{L}}{\cap} \cdots \overset{\mathbb{L}}{\cap} \mathbb{P}_{\Lambda_n}\big) = \begin{cases} (-1)^{n-1}, & \Lambda_1 = \cdots = \Lambda_n, \\ 0, & \textit{otherwise.} \end{cases} \tag{10.4.1.1}$$

In [LRZ21] the authors plan to prove this conjecture, which from now on we assume to hold.

We remind the reader of our convention in §1.8.

Lemma 10.4.2. *Let $n \geq 1$. Let $x \in \mathbb{W}_n$ be non-zero vector. Then both $\pi_1^{-1}(\mathcal{Y}(x))$ and $\pi_2^{-1}(\mathcal{Z}^\flat(x))$ are Cartier divisors on $\widetilde{\mathcal{N}}_n^1$. Moreover, we have $\pi_1^*(\mathcal{Y}(x)) = \pi_1^{-1}(\mathcal{Y}(x))$ (in $K_0^{\pi_1^{-1}(\mathcal{Y}(x))}(\widetilde{\mathcal{N}}_n^1) \simeq K_0'(\pi_1^{-1}(\mathcal{Y}(x)))$) and $\pi_2^*(\mathcal{Z}^\flat(x)) = \pi_2^{-1}(\mathcal{Z}^\flat(x))$ (in $K_0^{\pi_2^{-1}(\mathcal{Z}^\flat(x))}(\widetilde{\mathcal{N}}_n^1) \simeq K_0'(\pi_2^{-1}(\mathcal{Z}^\flat(x)))$).*

Proof. We have the following observation which applies to both π_1 and π_2. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism between regular formal schemes such that for every $z \in \mathcal{X}^{\mathrm{red}}$ the induced map on local rings $\pi_z^\sharp : \mathcal{O}_{\mathcal{Y},\pi(z)} \to \mathcal{O}_{\mathcal{X},z}$ is injective. Let D be a Cartier divisor on $\mathcal{Y}$. Then $\pi^{-1}(D)$ is a Cartier divisor.

Now that both $\pi_1^{-1}(\mathcal{Y}(x))$ and $\pi_2^{-1}(\mathcal{Z}^\flat(x))$ are Cartier divisors on $\widetilde{\mathcal{N}}_n^1$, it follows that $\pi_1^*(\mathcal{O}_{\mathcal{Y}(x)}) = \mathcal{O}_{\pi_1^{-1}(\mathcal{Y}(x))}$ and $\pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x)}) = \mathcal{O}_{\pi_2^{-1}(\mathcal{Z}^\flat(x))}$ (e.g., by the argument of [Zha21, Lemma B.2 (i)], which applies even if the vertical map in *loc. cit.* is not a closed immersion). $\square$

Theorem 10.4.3. *Let $n \geq 1$. Let $x \in \mathbb{W}_n$ be a non-zero vector and define* $\mathrm{Vert}(x) := \{\Lambda \in \mathrm{Vert}^0(\mathbb{W}_n) \mid x \in \Lambda\}$. *Define a locally finite Cartier divisor on $\widetilde{\mathcal{N}}_n^1$*

$$\mathrm{Exp}(x) := \sum_{\Lambda \in \mathrm{Vert}(x)} \mathbb{P}_\Lambda.$$

(Here, Exp *stands for "exceptional divisor" since the map π_2 behaves like a blow-up map with center at $\mathcal{Z}(x_0)^{\mathrm{ss}}$.) Then there is an equality of Cartier divisors on $\widetilde{\mathcal{N}}_n^1$*

$$\pi_1^{-1}(\mathcal{Y}(x)) = \pi_2^{-1}(\mathcal{Z}^\flat(x)) - \mathrm{Exp}(x).$$

Proof. We first note that a point $\mathcal{V}(\Lambda \oplus \langle x_0 \rangle)$ in $\mathcal{Z}(x_0)^{\mathrm{ss}}$ corresponding to $\Lambda \in \mathrm{Vert}^0(\mathbb{W}_n)$ lies on $\mathcal{Z}^\flat(x)$ if and only if $x \in \Lambda$.

From the moduli interpretations, it is clear that $\pi_1^{-1}(\mathcal{Y}(x)) \subset \pi_2^{-1}(\mathcal{Z}^\flat(x))$. Denote

$$\mathfrak{D} := \pi_2^{-1}(\mathcal{Z}^\flat(x)) - \pi_1^{-1}(\mathcal{Y}(x)).$$

Let $\mathrm{mult}_\Lambda(x) \in \mathbb{Z}_{\geq 0}$ be the multiplicity of $\mathbb{P}_\Lambda$ (as a Cartier divisor on $\widetilde{\mathcal{N}}_n^1$) in $\mathfrak{D}$. Set

$$\mathfrak{D}' := \mathfrak{D} - \sum_{\Lambda \in \mathrm{Vert}(x)} \mathrm{mult}_\Lambda(x)\, \mathbb{P}_\Lambda,$$

Then $\mathfrak{D}'$ is an effective Cartier divisor and it does not contain any of the $\mathbb{P}_\Lambda$. It suffices to show that $\mathfrak{D}' = 0$ and $\mathrm{mult}_\Lambda(x) = 1$ for every $\Lambda \in \mathrm{Vert}(x)$.

We proceed by induction on $n \geq 1$. When $n = 1$, we know that $\mathcal{N}_1^1 \simeq \mathrm{Spf}\, O_{\breve{F}}$, π_2 is an isomorphism $\widetilde{\mathcal{N}}_1^1 \simeq \mathcal{Z}(x_0)$ where both $\widetilde{\mathcal{N}}_1^1$ and $\mathcal{Z}(x_0)$ are isomorphic to a quasi-canonical lifting, the degree $q+1$ ramified cover $\mathrm{Spf}\, O_{\breve{F},1}$ of $\mathrm{Spf}\, O_{\breve{F}}$. Then $\mathcal{Y}(x)$ is non-empty unless $\mathrm{val}(x) \geq 2$ (note that $\mathrm{val}(x)$ is even), in which case it has $O_{\breve{F}}$-length $\frac{\mathrm{val}(x)}{2}$ by the theory of canonical lifting. By [KR11], we also know that the divisor $\mathcal{Z}^\flat(x) = \mathcal{Z}(x) \cap \mathcal{Z}(x_0)$ has $O_{\breve{F},1}$-length $1 + (q+1)\frac{\mathrm{val}(x)}{2}$. Therefore the desired equality of Cartier divisors on $\widetilde{\mathcal{N}}_1^1$ follows by comparing the $O_{\breve{F},1}$-lengths.

Now let $n \geq 2$. We note that the case $n = 2$ is slightly different (and in fact easier). We *claim* that, when $n = 2$, the divisor $\mathfrak{D}'$ is vertical (cf. §2.9). To show the claim, we first consider the case: $(x, x) \in O_F$ and $(x, x) \neq 0$. By [San17, Theorem 2.9] and the finite flatness of π_1 (Conjecture 10.4.1 (i)), the horizontal part of $\pi_1^{-1}(\mathcal{Y}(x))$ has degree $q+1$ (over $O_{\breve{F}}$); similarly by Theorem 4.2.1 and Conjecture 10.4.1, one can show that the horizontal part of $\pi_2^{-1}(\mathcal{Z}^\flat(x))$ also has degree $q+1$. By $\pi_1^{-1}(\mathcal{Y}(x)) \subset \pi_2^{-1}(\mathcal{Z}^\flat(x))$, their horizontal parts must cancel out in $\mathfrak{D}'$. We then consider the remaining case: $(x, x) = 0$ and $x \neq 0$. We apply Lemma 4.4.1 to deduce that $\mathcal{Z}^\flat(x) = \mathcal{Z}(x_0) \cap \mathcal{Z}(x) \subset \mathcal{N}_3$ has no horizontal part (otherwise the rank two lattice $\langle x, x_0 \rangle$ is embedded into the self-dual lattice L in Lemma 4.4.1; however, the orthogonal complement of $\langle x_0 \rangle_F$ in L_F is a two dimensional non-split hermitian space which has no non-zero isotropic vector). This proves the claim. Now by [Ter13a, Theorem 0.2], the special fiber of $\mathcal{Z}(x_0)$ (as a Cartier divisor on $\mathcal{Z}(x_0)$) is the sum of $\mathcal{V}(\Lambda^\sharp)$ for all $\Lambda^\sharp \in \mathrm{Vert}^3(\mathbb{V}_3)$ containing x_0. Let $\widetilde{\mathcal{V}}(\Lambda^\sharp)$ be the irreducible component of $\pi_2^{-1}(\mathcal{V}(\Lambda^\sharp))$ that is *not* contained in $\widetilde{\mathcal{N}}_2^{1,\mathrm{ss}}$. Then we may write $\mathfrak{D}'$ as a (locally finite) sum of Cartier divisors

$$\mathfrak{D}' = \sum_{\Lambda^\sharp \in \mathrm{Vert}^3(\mathbb{V}_3),\, x_0 \in \Lambda^\sharp} \mathrm{mult}_{\Lambda^\sharp}(x)\, \widetilde{\mathcal{V}}(\Lambda^\sharp), \quad \mathrm{mult}_{\Lambda^\sharp}(x) \in \mathbb{Z}_{\geq 0}. \tag{10.4.3.1}$$

We now return to $n \geq 2$. Now the basic idea is to intersect the given divisors with (many) well-positioned special divisors that are isomorphic to $\mathcal{N}_{n-1}^1$ or $\mathcal{N}_{n-1}$ (cf. (10.1.0.5)). We first determine $\mathrm{mult}_\Lambda(x)$ and show that, when $n \geq 3$, the divisor $\mathfrak{D}'$ does not intersect any of the $\mathbb{P}_\Lambda$ for $\Lambda \in \mathrm{Vert}(x)$. We fix a $\Lambda_0 \in \mathrm{Vert}(x)$. Since Λ_0 is self-dual of rank $n \geq 2$, there exists a vector $e \in \Lambda_0$ such that $\mathrm{val}(e) = 0$ and that e is linearly independent of x. We have an orthogonal decomposition $\mathbb{W}_n = \mathbb{W}_{n-1} \oplus \langle e \rangle$, and let $x^\flat \in \mathbb{W}_{n-1}$ be the projection of x to $\mathbb{W}_{n-1}$. Then $x^\flat \neq 0$. By (10.1.0.5), the special divisor $\mathcal{Y}(e)$ on $\mathcal{N}_n^1$ is isomorphic to $\mathcal{N}_{n-1}^1$. We consider the commutative diagrams with the obvious maps

$$
\begin{tikzcd}
 & \widetilde{\mathcal{N}}^1_{n-1} \arrow[dl, "\pi_1^\flat"'] \arrow[d, "\widetilde{\delta}"'] \arrow[dr, "\pi_2^\flat"] & & \\
\mathcal{N}^1_{n-1} \simeq \mathcal{Y}(e) \arrow[d] & \widetilde{\mathcal{N}}^1_n \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \mathcal{Z}(\langle e, x_0\rangle) \arrow[r] \arrow[d] & \mathcal{Z}(e) \simeq \mathcal{N}_n \arrow[d] \\
\mathcal{N}^1_n & & \mathcal{Z}(x_0) \arrow[r] & \mathcal{N}_{n+1}.
\end{tikzcd}
\tag{10.4.3.2}
$$

(In fact the leftmost and the rightmost squares are cartesian; but the middle square is not.) All the vertical maps are closed immersions. The pull-back of the divisor $\mathfrak{D}$ along the map $\widetilde{\delta}$ is the analogous Cartier divisor $\mathfrak{D}^\flat = (\pi_2^\flat)^{-1}(\mathcal{Z}^\flat(x^\flat)) - (\pi_1^\flat)^{-1}(\mathcal{Y}(x^\flat))$ on $\widetilde{\mathcal{N}}^1_{n-1}$ (cf. [Cho18, Proposition 5.11 (1, 2)]). The pull-back of $\mathfrak{D}'$ along $\widetilde{\delta}$ is then a Cartier divisor

$$
\widetilde{\delta}^{-1}(\mathfrak{D}') = \mathfrak{D}^\flat - \sum_{\substack{\Lambda \in \mathrm{Vert}(x) \\ e \in \Lambda}} \mathrm{mult}_\Lambda(x)\, \mathbb{P}_{\Lambda^\flat}, \tag{10.4.3.3}
$$

where $\Lambda^\flat \in \mathrm{Vert}^0(\mathbb{W}_{n-1})$ denotes the orthogonal complement of e in Λ. By induction hypothesis, we have $\mathfrak{D}^\flat = \mathrm{Exp}(x^\flat)$, which is a sum over $\Lambda \in \mathrm{Vert}(x^\flat)$ (and $\mathrm{Vert}(x^\flat)$ is bijective to the set of $\Lambda = \Lambda^\flat \obot \langle e \rangle$ in the sum (10.4.3.3)), but with known multiplicity $\mathrm{mult}_{\Lambda^\flat}(x^\flat) = 1$. When $n = 2$, by (10.4.3.1) and the fact that $\widetilde{\mathcal{V}}(\Lambda^\sharp)$ does not intersect the image of $\widetilde{\delta}$, we have $\widetilde{\delta}^{-1}(\mathfrak{D}') = 0$, and hence we can already deduce that $\mathrm{mult}_{\Lambda_0}(x) = 1$ for every $\Lambda_0 \in \mathrm{Vert}(x)$.

When $n \geq 3$, we can only deduce that $\mathrm{mult}_{\Lambda_0}(x) \leq 1$. To see that $\mathrm{mult}_{\Lambda_0}(x) \neq 0$, we look at the intersection number between $\mathfrak{D}'$ and a certain line $\mathbb{P}^1$ in $\mathbb{P}_{\Lambda_0}$

$$
\chi(\widetilde{\mathcal{N}}^1_n, \mathbb{P}^1 \overset{\mathbb{L}}{\cap} \mathfrak{D}') = \chi(\widetilde{\mathcal{N}}^1_n, \mathbb{P}^1 \overset{\mathbb{L}}{\cap} \mathfrak{D}) - \mathrm{mult}_{\Lambda_0}(x)\, \chi(\widetilde{\mathcal{N}}^1_n, \mathbb{P}^1 \overset{\mathbb{L}}{\cap} \mathbb{P}_{\Lambda_0}).
$$

Here we choose the line $\mathbb{P}^1$ to be $\mathbb{P}_M$ for some rank two self-dual sublattice $M \subset \Lambda_0$ such that $x \notin M_F^\perp$. Since $\mathfrak{D}'$ does not contain $\mathbb{P}_{\Lambda_0}$, its restriction to $\mathbb{P}_{\Lambda_0}$ is an effective Cartier divisor and therefore the left hand side is non-negative. On the other hand, $\chi(\widetilde{\mathcal{N}}^1_n, \mathbb{P}^1 \overset{\mathbb{L}}{\cap} \mathfrak{D}) = -1$ is strictly negative (e.g., by repeating (10.4.3.2) $n-2$ times to reduce to the case $n = 2$). Therefore $\mathrm{mult}_{\Lambda_0}(x) \neq 0$ and we deduce that $\mathrm{mult}_{\Lambda_0}(x) = 1$. This is true for every $\Lambda_0 \in \mathrm{Vert}(x)$. It follows that $\widetilde{\delta}^{-1}(\mathfrak{D}') = 0$, i.e., $\mathfrak{D}'$ does not intersect $\widetilde{\mathcal{N}}^1_{n-1}$ in (10.4.3.2). This then implies that $\mathfrak{D}'$ does not intersect $\mathbb{P}_{\Lambda_0}$ for every $\Lambda_0 \in \mathrm{Vert}(x)$. (Otherwise the intersection $\mathfrak{D}' \cap \mathbb{P}_{\Lambda_0}$ would be a non-zero Cartier divisor on $\mathbb{P}_{\Lambda_0}$, which necessarily intersects with the hyperplane $\mathbb{P}_{\Lambda_0} \cap \widetilde{\mathcal{N}}^1_{n-1}$ in $\mathbb{P}_{\Lambda_0} \simeq \mathbb{P}^{n-1}$ when $n \geq 3$; hence $\mathfrak{D}'$ must intersect $\widetilde{\mathcal{N}}^1_{n-1}$, a contradiction!)

It remains to show that $\mathfrak{D}'$ is locally trivial at every point $z \in \widetilde{\mathcal{N}}^{1,\mathrm{ns}}_n(\bar{k})$ (i.e., the local equations are all units). Suppose that there exists a point $z \in \widetilde{\mathcal{N}}^{1,\mathrm{ns}}_n(\bar{k}) = \mathcal{Z}(x_0)^{\mathrm{ns}}(\bar{k})$ where the local equation defining $\mathfrak{D}$ is not a unit. By (2.7.0.1), there exists a unique *maximal* vertex lattice $\Lambda^\sharp \subset \mathbb{V}_{n+1}$ such that $z \in \mathcal{V}(\Lambda^\sharp) \subset \mathcal{Z}(\langle x_0, x\rangle)^{\mathrm{red}}$.

We first assume that z is a non-super-general point (as a point on $\mathcal{N}_{n+1}$, §2.7). Then the type $t(\Lambda^\sharp) \leq n$. Therefore there exists $e \in \Lambda^\sharp$ such that $\mathrm{val}(e) = 0$ and e is linearly independent of x. Then $\langle e, x_0\rangle$ is a vertex lattice of rank two. There are exactly two cases:

(i) $\langle e, x_0 \rangle$ has type 1.

(ii) $\langle e, x_0 \rangle$ has type 0.

In the first case, we may assume that $e \perp x_0$. Then we again consider the commutative diagrams (10.4.3.2) and retain the notation there. Now the point z belongs to $\widetilde{\mathcal{N}}_{n-1}^{1,\mathrm{ns}}(\overline{k})$. We have assumed that locally at z the divisor $\mathfrak{D}$ is not defined by a unit. It follows that $\mathfrak{D}^\flat$ is not defined by a unit locally at z, contradicting the induction hypothesis. In the second case, let $e^\flat \in \mathbb{W}_n$ be the orthogonal projection of e to $\mathbb{W}_n$. Then $\mathrm{val}(e^\flat) = -1$, and we may assume that $(e^\flat, e^\flat) = \varpi^{-1}$. Then by (10.1.0.5), the special divisor $\mathcal{Y}'(e^\flat)$ on $\mathcal{N}_n^1$ is isomorphic to $\mathcal{N}_{n-1}$. We consider the commutative diagrams

(10.4.3.4)
$$\begin{tikzcd}
 & \mathcal{N}_{n-1} \arrow[dl] \arrow[d, "\widetilde{\delta}"'] \arrow[dr] & & \\
\mathcal{N}_{n-1} \simeq \mathcal{Y}'(e^\flat) \arrow[d] & \widetilde{\mathcal{N}}_n^1 \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathcal{Z}(\langle e, x_0 \rangle) \simeq \mathcal{N}_{n-1} \arrow[r] \arrow[d] & \mathcal{Z}(e) \simeq \mathcal{N}_n \arrow[d] \\
\mathcal{N}_n^1 & & \mathcal{Z}(x_0) \arrow[r] & \mathcal{N}_{n+1}
\end{tikzcd}$$

where the only non-obvious map $\widetilde{\delta} : \mathcal{N}_{n-1} \to \widetilde{\mathcal{N}}_n^1$ is defined as follows. The natural morphisms $\mathcal{N}_{n-1} \simeq \mathcal{Y}'(e^\flat) \hookrightarrow \mathcal{N}_n^1$ and $\mathcal{N}_{n-1} \simeq \mathcal{Z}(\langle e, x_0 \rangle) \hookrightarrow \mathcal{Z}(x_0) \hookrightarrow \mathcal{N}_{n+1}$ induced a morphism $\mathcal{N}_{n-1} \to \mathcal{N}_n^1 \times \mathcal{N}_{n+1}$, which factors through $\widetilde{\mathcal{N}}_n^1$ (10.2.0.3) and defines the morphism $\widetilde{\delta}$.

The rest of the proof is similar to the first case (using [Cho18, Proposition 5.11 (3, 4)] instead), and we omit the detail. When $n = 2$, this already implies $\mathfrak{D}' = 0$ because every curve $\widetilde{\mathcal{V}}(\Lambda^\sharp)$ in (10.4.3.1) must pass through some non-super-general point $z \in \mathcal{Z}(x_0)^{\mathrm{ns}}(\overline{k})$ (there are q^3+1 of them on $\mathcal{V}(\Lambda^\sharp)$ and only $q+1$ lie in $\mathcal{Z}(x_0)^{\mathrm{ss}}$).

Finally, we assume that $n \geq 3$ and $z \in \mathfrak{D}'(\overline{k})$ is a super-general point. Then $n+1$ is odd and $t(\Lambda^\sharp) = n+1$. Consider $\mathcal{V}(\Lambda^\sharp)^{\mathrm{ns}} := \mathcal{V}(\Lambda^\sharp) \setminus \mathcal{Z}(x_0)^{\mathrm{ss}}$ (the complement of a finite scheme in a projective scheme of dimension $\frac{n}{2}$). The restriction of $\mathfrak{D}'$ to $\mathcal{V}(\Lambda^\sharp)^{\mathrm{ns}}$ is a projective scheme and is a Cartier divisor. We have shown that this Cartier divisor is locally trivial at all non-super-general points. Therefore it must be contained in the open subscheme $\mathcal{V}(\Lambda^\sharp)^\circ$. Then this Cartier divisor on $\mathcal{V}(\Lambda^\sharp)^{\mathrm{ns}}$ must be trivial due to the affineness of $\mathcal{V}(\Lambda^\sharp)^\circ$ [Lus76, Corollary 2.8] and its dimension $\frac{n}{2} \geq 2$. This completes the induction. $\square$

10.5. **The intersection number** $\mathrm{Int}'(L)$**.** We are now ready to complete the computation of the intersection number $\mathrm{Int}'(L)$ defined by (10.1.1.1). Note that the result is conditional on Conjecture 10.4.1.

Theorem 10.5.1. *Let $L \subseteq \mathbb{W}_n$ be an O_F-lattice. Then*

$$\mathrm{Int}'(L) = \frac{1}{q+1} \left(\partial \mathrm{Den}_\Lambda(L) - \mathrm{Den}(L) \right).$$

In particular, the definition (10.1.1.1) *is independent of the choice of the basis.*

Remark 10.5.2. The case $n = 2$ is due to [San17].

Example 10.5.3 (The case $n=1$)**.** When $n=1$, let $L=\langle x\rangle\subset\mathbb{W}_1$. It is easy to see that

$$\mathrm{Int}'(L)=\begin{cases}\frac{\mathrm{val}(x)}{2}, & \mathrm{val}(x)\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, the local density formula shows that

$$\partial\mathrm{Den}_\Lambda(L)=\begin{cases}1+(q+1)\frac{\mathrm{val}(x)}{2}, & \mathrm{val}(x)\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

This verifies the theorem in the case $n=1$.

Proof. By the projection formula for the finite flat map π_1, we obtain an equality in $K_0'(\mathcal{Y}(x_1)\cap\cdots\cap\mathcal{Y}(x_n))$

$$\pi_{1*}(\pi_1^*(\mathcal{O}_{\mathcal{Y}(x_1)})\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\pi_1^*(\mathcal{O}_{\mathcal{Y}(x_n)}))=\deg(\pi_1)\,\mathcal{O}_{\mathcal{Y}(x_1)}\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Y}(x_n)},$$

and hence

$$\begin{aligned}\mathrm{Int}'(L)=&\chi\left(\mathcal{N}_n^1,\mathcal{O}_{\mathcal{Y}(x_1)}\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Y}(x_n)}\right)\\ =&\frac{1}{\deg(\pi_1)}\chi\left(\widetilde{\mathcal{N}}_n^1,\pi_1^*(\mathcal{O}_{\mathcal{Y}(x_1)})\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\pi_1^*(\mathcal{O}_{\mathcal{Y}(x_n)})\right).\end{aligned}$$

For two Cartier divisors D_1 and D_2 on a regular formal scheme $\mathcal{X}$, we have $\mathcal{O}_{D_1+D_2}=\mathcal{O}_{D_1}+\mathcal{O}_{D_2}\in K_0^{D_1\cup D_2}(\mathcal{X})/\mathrm{F}^2K_0^{D_1\cap D_2}(\mathcal{X})$. This allows us to apply Lemma 10.4.2 and the equality of Cartier divisors in Theorem 10.4.3 and to obtain an equality in $K_0'(\pi_2^{-1}(\mathcal{Z}^\flat(x)))/\mathrm{F}^1K_0'(\mathrm{Exp}(x))$

$$\pi_1^*(\mathcal{O}_{\mathcal{Y}(x)})=\pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x)})-\mathcal{O}_{\mathrm{Exp}(x)}.$$

For $\Lambda\in\mathrm{Vert}(x)$, let $i_\Lambda:\mathbb{P}_\Lambda\to\widetilde{\mathcal{N}}_n^1$ denote the closed immersion. For any $\mathcal{F}\in\mathrm{F}^1K_0(\mathbb{P}_\Lambda)$ and $n-1$ Cartier divisors $D_1,\cdots,D_{n-1}$ on $\widetilde{\mathcal{N}}_n^1$, we have an equality in $K_0(\mathbb{P}_\Lambda)$

$$\mathcal{F}\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}}\mathcal{O}_{D_1}\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}}\cdots\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}}\mathcal{O}_{D_{n-1}}=\mathcal{F}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathbb{P}_\Lambda}}i_\Lambda^*(\mathcal{O}_{D_1})\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathbb{P}_\Lambda}}\cdots\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathbb{P}_\Lambda}}i_\Lambda^*(\mathcal{O}_{D_{n-1}}).$$

Since $i_\Lambda^*(\mathcal{O}_{D_i})\in\mathrm{F}^1K_0(\mathbb{P}_\Lambda)$, the above product belongs to $\mathrm{F}^nK_0(\mathbb{P}_\Lambda)$ by [Zha21, B.3] (applied to the scheme $\mathbb{P}_\Lambda$). Since $\dim\mathbb{P}_\Lambda=n-1$, we conclude that $\mathcal{F}\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}}\mathcal{O}_{D_1}\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}}\cdots\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}}\mathcal{O}_{D_{n-1}}=0$. It follows that

(10.5.3.1)

$$\mathrm{Int}'(L)=\frac{1}{\deg(\pi_1)}\chi\left(\widetilde{\mathcal{N}}_n^1,\left(\pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x_1)})-\mathcal{O}_{\mathrm{Exp}(x_1)}\right)\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\left(\pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x_n)})-\mathcal{O}_{\mathrm{Exp}(x_n)}\right)\right).$$

We apply the projection formula to the proper morphism $\pi_2:\widetilde{\mathcal{N}}_n^1\to\mathcal{Z}(x_0)$

$$\pi_{2*}\big(\mathcal{O}_{\mathbb{P}_\Lambda}\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}}\pi_2^*(\mathcal{F})\big)=\pi_{2*}(\mathcal{O}_{\mathbb{P}_\Lambda})\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{Z}(x_0)}}\mathcal{F},$$

where $\Lambda\in\mathrm{Vert}^0(\mathbb{W}_n)$ and $\mathcal{F}\in K_0(\mathcal{Z}(x_0))$. Since the first factor $\pi_{2*}(\mathcal{O}_{\mathbb{P}_\Lambda})$ is supported on a zero-dimensional subscheme of $\mathcal{Z}(x_0)$, we have

$$\chi\big(\widetilde{\mathcal{N}}_n^1,\mathcal{O}_{\mathbb{P}_\Lambda}\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}}\pi_2^*(\mathcal{F})\big)=0,\tag{10.5.3.2}$$

for any $\mathcal{F}\in\mathrm{F}^1K_0(\mathcal{Z}(x_0))$.

Next, we recall (10.4.1.1). Together with (10.5.3.1) and (10.5.3.2) we obtain

(10.5.3.3)
$$\deg(\pi_1)\,\mathrm{Int}'(L) = \chi\left(\widetilde{\mathcal{N}}_n^1, \pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x_1)}) \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x_n)})\right) + (-1)^n \sum_{\substack{\Lambda\in \mathrm{Vert}^0(\mathbb{W}_n)\\ L\subset\Lambda}} (-1)^{n-1}.$$

Apply again the projection formula for π_2,

$$\begin{aligned}\pi_{2*}(\pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x_1)})) &= \pi_{2*}(\mathcal{O}_{\widetilde{\mathcal{N}}_n^1} \otimes_{\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}} \pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x_1)}))\\ &= \pi_{2*}(\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{Z}(x_0)}} \mathcal{O}_{\mathcal{Z}^\flat(x_1)} \quad \in K_0'(\mathcal{Z}(x_0)).\end{aligned}$$

Note that $\pi_{2*}(\mathcal{O}_{\widetilde{\mathcal{N}}_n^1}) - \mathcal{O}_{\mathcal{Z}(x_0)}$ is supported on $\mathcal{Z}(x_0)^{\mathrm{ss}}$ which is zero-dimensional. We obtain

$$\begin{aligned}&\chi\left(\widetilde{\mathcal{N}}_n^1, \pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x_1)}) \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \pi_2^*(\mathcal{O}_{\mathcal{Z}^\flat(x_n)})\right)\\ =&\chi\left(\mathcal{Z}(x_0), \mathcal{Z}^\flat(x_1) \overset{\mathbb{L}}{\cap} \cdots \overset{\mathbb{L}}{\cap} \mathcal{Z}^\flat(x_n)\right)\\ =&\partial\mathrm{Den}_\Lambda(L),\end{aligned}$$

where the last equality follows from Theorem 10.3.1.

Finally, by (3.6.1.1) we have

$$\#\left\{\Lambda \in \mathrm{Vert}^0(\mathbb{W}_n) \mid L \subset \Lambda\right\} = \mathrm{Den}(L).$$

By (10.5.3.3) and $\deg(\pi_1) = q+1$, the proof is complete. □

Remark 10.5.4. In the notation of §3.3 and §9.2, it follows immediately from Theorem 10.5.1, (3.3.0.2), (9.2.0.4) that

$$\mathrm{Int}'(L) = \left(\frac{W_T'(1,0,\varphi_1)}{\log q^2}\cdot\frac{(-q)^n-1}{q+1} - W_T(1,0,\varphi_0)\cdot\frac{1}{q+1}\right)\cdot\prod_{i=1}^n(1-(-q)^{-i})^{-1}.$$

Part 3. Semi-global and global applications: arithmetic Siegel–Weil formula

In this part we apply our main Theorem 3.4.1 to prove an identity between the local intersection number of Kudla–Rapoport cycles on (integral models of) unitary Shimura varieties at an inert prime with hyperspecial level and the derivative of a Fourier coefficient of Siegel–Eisenstein series on unitary groups (also known as the *local arithmetic Siegel–Weil formula*). This is achieved by relating the Kudla–Rapoport cycles on unitary Shimura varieties to those on unitary Rapoport–Zink spaces via the p-adic uniformization, and by relating the Fourier coefficients to local representation densities. This deduction is more or less standard (see [KR14] and [Ter13a]), and we will state the results for more general totally real base fields and level structures, making use of the recent advance on integral models of unitary Shimura varieties ([RSZ20]). We will also apply the main Theorem 10.3.1 in the almost self-dual case to deduce a similar identity at an inert prime with almost self-dual level. Finally, combining these semi-global identities with archimedean identities of Liu [Liu11a] and Garcia–Sankaran [GS19] will allow us to deduce the *arithmetic Siegel–Weil formula* for Shimura varieties with minimal levels at inert primes, at least when the quadratic extension is unramified at all finite places.

11.1. **Shimura varieties.** We will closely follow [RSZ20] and [RSZ21]. In this part we switch to global notations. Let F be a CM number field, with F_0 its totally real subfield of index 2. We fix a CM type $\Phi \subseteq \mathrm{Hom}(F, \overline{\mathbb{Q}})$ of F and a distinguished element $\phi_0 \in \Phi$. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and identify the CM type Φ with the set of archimedean places of F, and also with the set of archimedean places of F_0. Let V be an F/F_0-hermitian space of dimension $n \geq 2$. Let $V_\phi = V \otimes_{F,\phi} \mathbb{C}$ be the associated $\mathbb{C}/\mathbb{R}$-hermitian space for $\phi \in \Phi$. Assume the signature of V_ϕ is given by

$$(r_\phi, r_{\bar{\phi}}) = \begin{cases} (n-1, 1), & \phi = \phi_0, \\ (n, 0), & \phi \in \Phi \setminus \{\phi_0\}. \end{cases}$$

Define a variant $G^{\mathbb{Q}}$ of the unitary similitude group $\mathrm{GU}(V)$ by

$$G^{\mathbb{Q}} := \{g \in \mathrm{Res}_{F_0/\mathbb{Q}} \mathrm{GU}(V) : c(g) \in \mathbb{G}_m\},$$

where c denotes the similitude character. Define a cocharacter

$$h_{G^{\mathbb{Q}}} : \mathbb{C}^\times \to G^{\mathbb{Q}}(\mathbb{R}) \subseteq \prod_{\phi \in \Phi} \mathrm{GU}(V_\phi)(\mathbb{R}) \simeq \prod_{\phi \in \Phi} \mathrm{GU}(r_\phi, r_{\bar{\phi}})(\mathbb{R}),$$

where its ϕ-component is given by

$$h_{G^{\mathbb{Q}},\phi}(z) = \mathrm{diag}\{z \cdot 1_{r_\phi}, \bar{z} \cdot 1_{r_{\bar{\phi}}}\}$$

under the decomposition of V_ϕ into positive definite and negative definite parts. Then its $G^{\mathbb{Q}}(\mathbb{R})$-conjugacy class defines a Shimura datum $(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$. Let $E_r = E(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$ be the reflex field, i.e., the subfield of $\overline{\mathbb{Q}}$ fixed by $\{\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma^*(r) = r\}$, where $r : \mathrm{Hom}(F, \overline{\mathbb{Q}}) \to \mathbb{Z}$ is the function defined by $r(\phi) = r_\phi$.

We similarly define the group $Z^{\mathbb{Q}}$ (a torus) associated to a totally positive definite F/F_0-hermitian space of dimension 1 (i.e., of signature $\{(1,0)_{\phi\in\Phi}\}$) and a cocharacter $h_{Z^{\mathbb{Q}}}$ of $Z^{\mathbb{Q}}$. The reflex field $E_\Phi = E(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ is equal to the reflex field of the CM type Φ, i.e., the subfield of $\overline{\mathbb{Q}}$ fixed by $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma \circ \Phi = \Phi\}$.

Now define a Shimura datum $(\widetilde{G}, \{h_{\widetilde{G}}\})$ by

$$\widetilde{G} := Z^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}} = \{(z, g) \in Z^{\mathbb{Q}} \times G^{\mathbb{Q}} : \mathrm{Nm}_{F/F_0}(z) = c(g)\}, \quad h_{\widetilde{G}} = (h_{Z^{\mathbb{Q}}}, h_{G^{\mathbb{Q}}}).$$

Its reflex field E is equal to the composite $E_r E_\Phi$, and the CM field F becomes a subfield of E via the embedding ϕ_0. Let $K \subseteq \widetilde{G}(\mathbb{A}_f)$ be a compact open subgroup. Then the associated Shimura variety $\mathrm{Sh}_K = \mathrm{Sh}_K(\widetilde{G}, \{h_{\widetilde{G}}\})$ is of dimension $n-1$ and has a canonical model over $\mathrm{Spec}\, E$. We remark that $E = F$ when $F/\mathbb{Q}$ is Galois, or when $F = F_0 K$ for some imaginary quadratic $K/\mathbb{Q}$ and the CM type Φ is induced from a CM type of $K/\mathbb{Q}$ (e.g., when $F_0 = \mathbb{Q}$).

11.2. **Semi-global integral models at hyperspecial levels.** Let p be a prime number. If $p = 2$, then we assume all places v of F_0 above p are unramified in F. Fix an embedding $\tilde{\nu} : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Let ν be the place of E above p induced by $\tilde{\nu}$. It determines places v_0 of F_0 and w_0 of F via the embedding ϕ_0. To specify the level K, notice that for $G := \mathrm{Res}_{F_0/\mathbb{Q}} \mathrm{U}(V)$ we have an isomorphism

(11.2.0.1) $$\widetilde{G} \simeq Z^{\mathbb{Q}} \times G, \quad (z, g) \longmapsto (z, z^{-1} g).$$

We consider the open compact subgroup of the form

$$K \simeq K_{Z^{\mathbb{Q}}} \times K_G$$

under the decomposition (11.2.0.1). We assume that $K_{Z^{\mathbb{Q}}}$ is the unique maximal open compact subgroup of $Z^{\mathbb{Q}}(\mathbb{A}_f)$ and

$$K_G = \prod_{v|p} K_{G,v} \times K_G^p.$$

In this subsection, we assume

(H1) v_0 is inert in F (possibly ramified over p).

(H2) V_{v_0} is split and we take K_{G,v_0} to be the stabilizer of a self-dual lattice $\Lambda_{v_0} \subseteq V_{v_0}$, a hyperspecial subgroup of $\mathrm{U}(V)(F_{0,v_0})$.

(H3) For each place $v \neq v_0$ of F_0 above p, let $K_{G,v}^\circ$ be the maximal compact subgroup of $\mathrm{U}(V)(F_{0,v})$ given by the stabilizer of a vertex lattice $\Lambda_v \subseteq V_v$. We take $K_{G,v} = K_{G,v}^\circ$ if v is nonsplit in F. We take $K_{G,v} \subseteq K_{G,v}^\circ$ to be any open compact subgroup if v is split in F.

(H4) $K_G^p \subseteq G(\mathbb{A}_f^p)$ is any open compact subgroup.

(H5) For each place $v \neq v_0$ of F_0 above p such that v is split in F and $K_{G,v} \neq K_{G,v}^\circ$, there exists a place w of F above v matching with the CM type Φ: if $\phi \in \mathrm{Hom}(F, \overline{\mathbb{Q}})$ induces the place w (via $\tilde{\nu} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$), then $\phi \in \Phi$.

(H6) If v_0 is ramified over p, then the subset $\{\phi \in \Phi : \phi \text{ induces } w_0\} \subseteq \mathrm{Hom}(F_{w_0}, \overline{\mathbb{Q}}_p)$ is the pullback of a CM type $\Phi^{\mathrm{ur}} \subseteq \mathrm{Hom}(F_{w_0}^{\mathrm{ur}}, \overline{\mathbb{Q}}_p)$ of $F_{w_0}^{\mathrm{ur}}$. Here $F_{w_0}^{\mathrm{ur}}$ is the maximal subfield of F_{w_0} unramified over $\mathbb{Q}_p$.

Under these conditions, Rapoport–Smithling–Zhang [RSZ20, §4.1] and [RSZ21, §4–5] (see also [Liu21, Proposition C.20]) construct a smooth integral model $\mathcal{M}_K$ of Sh_K over $O_{E,(\nu)}$. More precisely, for a locally noetherian $O_{E,(\nu)}$-scheme S, we consider $\mathcal{M}_K(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0})$, where

(M1) A_0 (resp. A) is an abelian scheme over S.

(M2) ι_0 (resp. ι) is an action of $O_F \otimes \mathbb{Z}_{(p)}$ on A_0 (resp. A) satisfying the Kottwitz condition of signature $\{(1,0)_{\phi\in\Phi}\}$ (resp. signature $\{(r_\phi, r_{\bar{\phi}})_{\phi\in\Phi}\}$).

(M3) λ_0 (resp. λ) is a polarization of A_0 (resp. A) whose Rosati involution induces the automorphism given by the nontrivial Galois automorphism of F/F_0 via ι_0 (resp. ι).

(M4) $\bar{\eta}^p$ is a K_G^p-orbit of $\mathbb{A}_{F,f}^p$-linear isometries between lisse $\mathbb{A}_{F,f}^p$-sheaves

$$\eta^p : \mathrm{Hom}_F(\hat{V}^p(A_0), \hat{V}^p(A)) \simeq V \otimes_F \mathbb{A}_{F,f}^p.$$

Here $\hat{V}^p(\cdot)$ denotes the $\mathbb{A}_{F,f}^p$-Tate module.

(M5) $\bar{\eta}_p^{v_0}$ is a collection $\{\bar{\eta}_v\}$, where $v \neq v_0$ runs over places of F_0 above p such that v is split in F and $K_{G,v} \neq K_{G,v}^\circ$, and each $\bar{\eta}_v$ is a $K_{G,v}$-orbit of $F_{\overline{w}}$-linear isomorphisms between lisse $F_{\overline{w}}$-sheaves

$$\eta_v : \mathrm{Hom}_{O_{F_{\overline{w}}}}(A_0[\overline{w}^\infty], A[\overline{w}^\infty]) \otimes_{O_{F_{\overline{w}}}} F_{\overline{w}} \simeq V \otimes_F F_{\overline{w}}.$$

Here $v = w\overline{w}$ and w is the place above v matching with the CM type Φ by (H5), and we view $K_{G,v}$ as an open subgroup of $\mathrm{GL}(V_{\overline{w}}) \cong \mathrm{U}(V_v)$ under the decomposition $V_v \cong V_w \oplus V_{\overline{w}}$. Notice that by the Kottwitz signature condition, both $A_0[\overline{w}^\infty]$ and $A[\overline{w}^\infty]$ are étale $O_{F_{\overline{w}}}$-modules (cf. [Liu21, Definition C.19]).

Such a tuple is required to satisfy the following extra conditions:

(M6) $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_0^{\mathfrak{a},\xi}(S)$. Here $\mathcal{M}_0^{\mathfrak{a},\xi}$ is an integral model of $\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ coming from an axillary moduli problem depending on a choice of an nonzero coprime-to-p ideal $\mathfrak{a}$ of O_{F_0} and ξ a certain similarity class of 1-dimensional hermitian F/F_0-hermitian spaces ([RSZ20, §3.2]). These axillary choices are made to ensure that the unitary group in 1-variable with $\mathfrak{a}$-level structure exists and so $\mathcal{M}_0^{\mathfrak{a},\xi}$ is non-empty. In particular, the polarization λ_0 is coprime-to-p. We remark that when F/F_0 is ramified at some finite place, one may choose $\mathfrak{a}$ to be the trivial ideal. Moreover, when $F_0 = \mathbb{Q}$, there is only one choice of ξ, and the condition $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_0^{\mathfrak{a},\xi}(S)$ is nothing but requiring λ_0 to be principal.

(M7) For each place v of F_0 above p, λ induces a polarization λ_v on the p-divisible group $A[v^\infty]$. We require $\ker \lambda_v \subseteq A[\iota(\varpi_v)]$ of rank equal to the size of $\Lambda_v^\vee/\Lambda_v$, where ϖ_v is a uniformizer of $F_{0,v}$. In particular, we require λ_{v_0} to be principal.

(M8) For the place v_0, we further require the *Eisenstein condition* in [RSZ21, §5.2, case (2)]. We remark the Eisenstein condition holds automatically when v_0 is unramified over p,

(M9) For each place $v \neq v_0$ of F_0 above p, we further require the *sign condition* and *Eisenstein condition* as explained in [RSZ20, §4.1]. We remark that the sign condition holds automatically when v is split in F, and the Eisenstein condition holds automatically when the places of F above v are unramified over p.

A morphism $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0}) \to (A_0', \iota_0', \lambda_0', A', \iota', \lambda', \bar{\eta}^{p\prime}, \bar{\eta}_p^{v_0\prime})$ in this groupoid is an isomorphism $(A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A_0', \iota_0', \lambda_0')$ in $\mathcal{M}_0^{\mathfrak{a},\xi}(S)$ and an $O_{F,(p)}$-linear quasi-isogeny $A \to A'$ inducing an isomorphism $A[p^\infty] \xrightarrow{\sim} A'[p^\infty]$, pulling λ' back to λ, pulling $\bar{\eta}^{p\prime}$ back to $\bar{\eta}^p$ and pulling $\bar{\eta}_p^{v_0\prime}$ back to $\bar{\eta}_p^{v_0}$.

By [RSZ20, Theorem 4.1], [RSZ21, Theorem 5.4 (c)], the functor $S \mapsto \mathcal{M}_K(S)$ is represented by a Deligne–Mumford stack $\mathcal{M}_K$ smooth over $\operatorname{Spec} O_{E,(\nu)}$. For K_G^p small enough, $\mathcal{M}_K$ is relatively representable over $\mathcal{M}_0^{\mathfrak{a},\xi}$, with generic fiber naturally isomorphic to the canonical model of Sh_K over $\operatorname{Spec} E$.

11.3. **Semi-global integral models at almost self-dual parahoric levels.** With the same set-up as §11.2, but replace the assumptions (H1) and (H2) by

(A1) v_0 is inert in F and unramified over p.

(A2) V_{v_0} is nonsplit and we take K_{G,v_0} to be the stabilizer of an almost self-dual lattice $\Lambda_{v_0} \subseteq V_{v_0}$, a maximal parahoric subgroup of $\mathrm{U}(V)(F_{0,v_0})$.

For a locally noetherian $O_{E,(\nu)}$-scheme S, we consider $\mathcal{M}_K(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0})$ satisfying (M1)— (M9). In particular, λ_{v_0} is almost principal instead of principal in (M7).

By [RSZ20, Theorem 4.7], the functor $S \mapsto \mathcal{M}_K(S)$ is represented by a Deligne–Mumford stack $\mathcal{M}_K$ flat over $\operatorname{Spec} O_{E,(\nu)}$. For K_G^p small enough, $\mathcal{M}_K$ is relatively representable over $\mathcal{M}_0^{\mathfrak{a},\xi}$, with generic fiber naturally isomorphic to the canonical model of Sh_K over $\operatorname{Spec} E$. Moreover, when ν is unramified over p (e.g., all p-adic places of F are unramified over p), $\mathcal{M}_K$ has semi-stable reduction over $\operatorname{Spec} O_{E,(\nu)}$ by [RSZ20, Theorem 4.7] and [Cho18, Proposition 1.4].

11.4. **Semi-global integral models at split primes.** With the same set-up as §11.2, but replace the assumption (H1) by

(S) v_0 is split in F (possibly ramified over p).

For a locally noetherian $O_{E,(\nu)}$-scheme S, we consider $\mathcal{M}_K(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0})$ satisfying (M1)— (M9). We further require

(MS) when p is locally nilpotent on S, the p-divisible group $A[w_0^\infty]$ is a Lubin–Tate group of type $r|_{w_0}$ ([RZ17, §8]). We remark that this condition holds automatically when v_0 is unramified over p.

By [RSZ20, Theorem 4.2], as in the hyperspecial case, the functor $S \mapsto \mathcal{M}_K(S)$ is represented by a Deligne–Mumford stack $\mathcal{M}_K$ smooth over $\operatorname{Spec} O_{E,(\nu)}$. For K_G^p small enough, $\mathcal{M}_K$ is relatively representable over $\mathcal{M}_0^{\mathfrak{a},\xi}$, with generic fiber naturally isomorphic to the canonical model of Sh_K over $\operatorname{Spec} E$.

11.5. **Semi-global integral models with Drinfeld levels at split primes.** With the same set-up as §11.4, we may consider semi-global integral models with Drinfeld levels by further requiring

(D) (a) the place ν of E matches the CM type Φ (in the sense of [RSZ20, §4.3]): if $\phi \in \operatorname{Hom}(F, \overline{\mathbb{Q}})$ induces the p-adic place w_0 of F (via $\tilde{\nu} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$), then $\phi \in \Phi$. We remark that this matching condition holds automatically when $F = F_0 K$ for some imaginary quadratic $K/\mathbb{Q}$ and the CM type Φ is induced from a CM type of $K/\mathbb{Q}$ (e.g., when $F_0 = \mathbb{Q}$), or when v_0 is of degree one over p.

(b) the extension $E_\nu / E_{r|_{v_0}}$ is unramified, where $E_{r|_{v_0}}$ is the local reflex field as defined in [RSZ20, §4.1]. We remark that this condition holds automatically if all p-adic places of F are unramified over p.

For $m \geq 0$, we consider the open compact subgroup $K_G^m \subseteq K_G$ such that $K_{G,v_0}^m \subseteq K_{G,v_0}$ is the principal congruence subgroup modulo $\varpi_{v_0}^m$, and $K_{G,v}^m = K_{G,v}$ for $v \neq v_0$. Write $K^m = K_{Z^\mathbb{Q}} \times K_G^m$. Notice that $K^0 = K$. We define a semi-global integral model $\mathcal{M}_{K^m}$ of Sh_{K^m} over $O_{E,(\nu)}$ as follows. For a locally noetherian $O_{E,(\nu)}$-scheme S, we consider $\mathcal{M}_{K^m}(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0}, \eta_{\overline{w}_0})$, where $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0}) \in \mathcal{M}_K(S)$ and the additional datum $\eta_{\overline{w}_0}$ is a Drinfeld level structure:

(MD) when p is locally nilpotent on S, $\eta_{\overline{w}_0}$ is an $O_{F,\overline{w}_0}$-linear homomorphism of finite flat group schemes

$$\eta_{\overline{w}_0} : \varpi_{\overline{w}_0}^{-m} \Lambda_{\overline{w}_0} / \Lambda_{\overline{w}_0} \to \underline{\mathrm{Hom}}_{O_{F,\overline{w}_0}}(A_0[\overline{w}_0^m], A[\overline{w}_0^m]),$$

which is a Drinfeld $\overline{w}_0^m$-level structure on the target (cf. [HT01, §II.2]).

By [RSZ20, Theorem 4.5], [RSZ21, Remark 5.6] (where the second condition in (D) should be added to ensure regularity), the functor $S \mapsto \mathcal{M}_{K^m}(S)$ is represented by a regular Deligne–Mumford stack $\mathcal{M}_K$, flat over $\operatorname{Spec} O_{E,(\nu)}$ and finite flat over $\mathcal{M}_K$, with generic fiber naturally isomorphic to the canonical model of Sh_{K^m} over $\operatorname{Spec} E$.

11.6. **Semi-global integral models at ramified primes.** With the same set-up as §11.2, but replace the assumption (H1) by

(R) v_0 is ramified in F (so $p \neq 2$) and unramified over p.

For a locally noetherian $O_{E,(\nu)}$-scheme S, we consider $\mathcal{M}_K(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0})$ satisfying (M1)— (M9). We further require

(MR) when p is locally nilpotent on S, the p-divisible group $A[w_0^\infty]$ satisfies the Pappas wedge condition ([KR14, Definition 2.4], [RSZ21, §5.2, case (3)]).

By [RSZ21, Theorem 5.4 (d)], the functor $S \mapsto \mathcal{M}_K(S)$ is represented by a Deligne–Mumford stack $\mathcal{M}_K$ flat over $\operatorname{Spec} O_{E,(\nu)}$. For K_G^p small enough, $\mathcal{M}_K$ is relatively representable over $\mathcal{M}_0^{\mathfrak{a},\xi}$, with generic fiber naturally isomorphic to the canonical model of Sh_K over $\operatorname{Spec} E$. By [RSZ21, Theorem 6.7], it has isolated singularities and we may further obtain a regular model by blowing up (the *Krämer model*, see [RSZ21, Definition 6.10]) which we still denote by $\mathcal{M}_K$.

12. Incoherent Eisenstein series

12.1. **Siegel Eisenstein series.** Let W be the standard split F/F_0-skew-hermitian space of dimension $2n$. Let $G_n = \mathrm{U}(W)$. Write $G_n(\mathbb{A}) = G_n(\mathbb{A}_{F_0})$ for short. Let $P_n(\mathbb{A}) = M_n(\mathbb{A})N_n(\mathbb{A})$ be the standard Siegel parabolic subgroup of $G_n(\mathbb{A})$, where

$$M_n(\mathbb{A}) = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} : a \in \mathrm{GL}_n(\mathbb{A}_F) \right\},$$
$$N_n(\mathbb{A}) = \left\{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} : b \in \mathrm{Herm}_n(\mathbb{A}_F) \right\}.$$

Let $\eta : \mathbb{A}_{F_0}^\times / F_0^\times \to \mathbb{C}^\times$ be the quadratic character associated to F/F_0. Fix $\chi : \mathbb{A}_F^\times \to \mathbb{C}^\times$ a character such that $\chi|_{\mathbb{A}_{F_0}^\times} = \eta^n$. We may view χ as a character on $M_n(\mathbb{A})$ by $\chi(m(a)) = \chi(\det(a))$ and extend it to $P_n(\mathbb{A})$ trivially on $N_n(\mathbb{A})$. Define the *degenerate principal series* to be the unnormalized smooth induction

$$I_n(s, \chi) \coloneqq \mathrm{Ind}_{P_n(\mathbb{A})}^{G_n(\mathbb{A})}(\chi \cdot |\cdot|_F^{s+n/2}), \quad s \in \mathbb{C}.$$

For a standard section $\Phi(-, s) \in I_n(s, \chi)$ (i.e., its restriction to the standard maximal compact subgroup of $G_n(\mathbb{A})$ is independent of s), define the associated *Siegel Eisenstein series*

$$E(g, s, \Phi) \coloneqq \sum_{\gamma \in P_n(F_0) \backslash G_n(F_0)} \Phi(\gamma g, s), \quad g \in G_n(\mathbb{A}),$$

which converges for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$. Notice that $E(g, s, \Phi)$ depends on the choice of χ.

12.2. **Fourier coefficients and derivatives.** By class field theory[4], we may and do choose an additive character $\psi : \mathbb{A}_{F_0}/F_0 \to \mathbb{C}^\times$ such that ψ is unramified outside $\mathrm{Spl}(F/F_0)$ (the set of finite places of F_0 split in F). We have a Fourier expansion

$$E(g, s, \Phi) = \sum_{T \in \mathrm{Herm}_n(F)} E_T(g, s, \Phi),$$

where

$$E_T(g, s, \Phi) = \int_{N_n(F_0)\backslash N_n(\mathbb{A})} E(n(b)g, s, \Phi)\psi(-\operatorname{tr}(Tb))\,\mathrm{d}n(b),$$

and the Haar measure $\mathrm{d}n(b)$ is normalized to be self-dual with respect to ψ. When T is nonsingular, for factorizable $\Phi = \otimes_v \Phi_v$ we have a factorization of the Fourier coefficient into a product

$$E_T(g, s, \Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v),$$

where the *local (generalized) Whittaker function* is defined by

$$W_{T,v}(g_v, s, \Phi_v) = \int_{N_n(F_{0,v})} \Phi_v(w_n^{-1} n(b) g, s)\psi(-\operatorname{tr}(Tb))\,\mathrm{d}n(b), \quad w_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

and has analytic continuation to $s \in \mathbb{C}$. Thus we have a decomposition of the derivative of a nonsingular Fourier coefficient at $s = s_0$,

$$E'_T(g, s_0, \Phi) = \sum_v E'_{T,v}(g, s_0, \Phi), \tag{12.2.0.2}$$

where

$$E'_{T,v}(g, s, \Phi) = W'_{T,v}(g_v, s, \Phi_v) \cdot \prod_{v' \neq v} W_{T,v'}(g_{v'}, s, \Phi_{v'}). \tag{12.2.0.3}$$

12.3. **Incoherent Eisenstein series.** Let $\mathbb{V}$ be an $\mathbb{A}_F/\mathbb{A}_{F_0}$-hermitian space of rank n. Let $\mathscr{S}(\mathbb{V}^n)$ be the space of Schwartz functions on $\mathbb{V}^n$. The fixed choice of χ and ψ gives a *Weil representation* $\omega = \omega_{\chi,\psi}$ of $G_n(\mathbb{A}) \times \mathrm{U}(\mathbb{V})$ on $\mathscr{S}(\mathbb{V}^n)$. Explicitly, for $\varphi \in \mathscr{S}(\mathbb{V}^n)$ and $\mathbf{x} \in \mathbb{V}^n$,

$$\begin{aligned} \omega(m(a))\varphi(\mathbf{x}) &= \chi(m(a))|\det a|_F^{n/2}\varphi(\mathbf{x}\cdot a), && m(a) \in M_n(\mathbb{A}),\\ \omega(n(b))\varphi(\mathbf{x}) &= \psi(\operatorname{tr} b\, T(\mathbf{x}))\varphi(\mathbf{x}), && n(b) \in N_n(\mathbb{A}),\\ \omega(w_n)\varphi(\mathbf{x}) &= \gamma_{\mathbb{V}}^n \cdot \widehat{\varphi}(\mathbf{x}), && w_n = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right),\\ \omega(h)\varphi(\mathbf{x}) &= \varphi(h^{-1}\cdot\mathbf{x}), && h \in \mathrm{U}(\mathbb{V}). \end{aligned}$$

[4]This should be well-known, but we include the argument for the convenience of the readers. Let $\psi_0 = \psi_{\mathbb{Q}} \circ \operatorname{tr}_{F_0/\mathbb{Q}} : \mathbb{A}_{F_0}/F_0 \to \mathbb{C}^\times$, where $\psi_{\mathbb{Q}} : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \mathbb{C}^\times$ is the standard additive character (so $\psi_{\mathbb{Q},\infty}(x) = e^{2\pi i x}$). Then the conductor of ψ_0 is the different ideal $\delta_{F_0/\mathbb{Q}}$ of $F_0/\mathbb{Q}$. Let H be the Hilbert class field of F_0. Since F/F_0 is ramified at infinite places, we know that H and F are linearly disjoint over F_0. It follows that $\mathrm{Gal}(H/F_0)$ is generated by the Frobenii associated to $\mathrm{Spl}(F/F_0)$. Hence by class field theory, the ideal class group of F_0 is generated by $\mathrm{Spl}(F/F_0)$. In particular, we may find $a \in F_0^\times$ such that the ideal $a\delta_{F_0}$ is supported on $\mathrm{Spl}(F/F_0)$. Then the character $\psi(x) = \psi_0(ax)$ works. Moreover, by a theorem of Hecke ([Hec54, Theorem 176], see also [Arm67]), the ideal class of $\delta_{F_0/\mathbb{Q}}$ is a square. Hence we may further choose $a \in F_0^\times$ to be a square.

Here $T(\mathbf{x}) = ((x_i, x_j))_{1\le i,j\le n}$ is the *fundamental matrix* of $\mathbf{x}$, $\gamma_{\mathbb{V}}$ is the Weil constant (see [KR14, (10.3)]), and $\widehat{\varphi}$ is the Fourier transform of φ using the self-dual Haar measure on $\mathbb{V}^n$ with respect to $\psi \circ \mathrm{tr}_{F/F_0}$.

For $\varphi \in \mathscr{S}(\mathbb{V}^n)$, define a function

$$\Phi_\varphi(g) \coloneqq \omega(g)\varphi(0), \quad g \in G_n(\mathbb{A}).$$

Then $\Phi_\varphi \in I_n(0,\chi)$. Let $\Phi_\varphi(-,s) \in I_n(s,\chi)$ be the associated standard section, known as *the standard Siegel–Weil section* associated to φ. For $\varphi \in \mathscr{S}(\mathbb{V}^n)$, we write

$$E(g,s,\varphi) \coloneqq E(g,s,\Phi_\varphi), \quad E_T(g,s,\varphi) \coloneqq E_T(g,s,\Phi_\varphi), \quad E'_{T,v}(g,s,\varphi) \coloneqq E'_{T,v}(g,s,\Phi_\varphi),$$

and similarly for $W_{T,v}(g_v,s,\varphi_v)$. We say $\mathbb{V}$ (resp. Φ_φ, $E(g,s,\varphi)$) is *coherent* if $\mathbb{V} = V \otimes_{F_0} \mathbb{A}_{F_0}$ for some F/F_0-hermitian space V, and *incoherent* otherwise. When $E(g,s,\varphi)$ is incoherent, its central value $E(g,0,\varphi)$ automatically vanishes (cf. [KR14, §9]). In this case, we write the central derivatives as

$$\partial\mathrm{Eis}(g,\varphi) \coloneqq E'(g,0,\varphi), \quad \partial\mathrm{Eis}_T(g,\varphi) \coloneqq E'_T(g,0,\varphi), \quad \partial\mathrm{Eis}_{T,v}(g,\varphi) \coloneqq E'_{T,v}(g,0,\varphi).$$

Let $T \in \mathrm{Herm}_n(F)$ be nonsingular. Then $W_{T,v}(g_v,0,\varphi_v) \neq 0$ only if $\mathbb{V}_v$ represents T (i.e., there exists $\mathbf{x} \in \mathbb{V}_v^n$ such that $T(\mathbf{x}) = T$), hence $\partial\mathrm{Eis}_{T,v}(g,\varphi) \neq 0$ only if $\mathbb{V}_{v'}$ represents T for all $v' \neq v$. Let $\mathrm{Diff}(T,\mathbb{V})$ be the set of finite places v such that $\mathbb{V}_v$ does not represent T. Since $\mathbb{V}$ is incoherent, by (12.2.0.2) we know that $\partial\mathrm{Eis}_T(g,\varphi) \neq 0$ only if $\mathrm{Diff}(T,\mathbb{V}) = \{v\}$ is a singleton, and in this case v is necessarily nonsplit in F (cf. [KR14, Lemma 9.1]). Thus

(12.3.0.4) $$\partial\mathrm{Eis}_T(g,\varphi) \neq 0 \Rightarrow \mathrm{Diff}(T,\mathbb{V}) = \{v\}, \quad \partial\mathrm{Eis}_T(g,\varphi) = \partial\mathrm{Eis}_{T,v}(g,\varphi).$$

We say $\varphi_v \in \mathscr{S}(\mathbb{V}_v^n)$ is *nonsingular* if its support lies in $\{\mathbf{x} \in \mathbb{V}_v^n : \det T(\mathbf{x}) \neq 0\}$. By [Liu11b, Proposition 2.1], we have

(12.3.0.5) $$\varphi \text{ is nonsingular at two finite places} \Longrightarrow \partial\mathrm{Eis}_T(g,\varphi) = 0 \text{ for any singular } T.$$

12.4. **Classical incoherent Eisenstein series associated to the Shimura datum.** Assume that we are in the situation in §11.1. Let $\mathbb{V}$ be the incoherent hermitian space obtained from V so that $\mathbb{V}$ has signature $(n,0)_{\phi\in\Phi}$ and $\mathbb{V}_v \cong V_v$ for all finite places V_v.

The hermitian symmetric domain for $G_n = \mathrm{U}(W)$ is the *hermitian upper half space*

$$\begin{aligned}\mathbb{H}_n &= \{\mathsf{z} \in \mathrm{Mat}_n(F_\infty) : \frac{1}{2i}\left(\mathsf{z} - {}^t\bar{\mathsf{z}}\right) > 0\} \\ &= \{\mathsf{z} = \mathsf{x} + i\mathsf{y} :\ \mathsf{x} \in \mathrm{Herm}_n(F_\infty),\ \mathsf{y} \in \mathrm{Herm}_n(F_\infty)_{>0}\},\end{aligned}$$

where $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^\Phi$. Define the *classical incoherent Eisenstein series* to be

$$E(\mathsf{z},s,\varphi) \coloneqq \chi_\infty(\det(a))^{-1}\det(\mathsf{y})^{-n/2}\cdot E(g_{\mathsf{z}},s,\varphi), \quad g_{\mathsf{z}} \coloneqq n(\mathsf{x})m(a) \in G_n(\mathbb{A}),$$

where $a \in \mathrm{GL}_n(F_\infty)$ such that $\mathsf{y} = a{}^t\bar{a}$. Notice that $E(\mathsf{z},s,\Phi)$ does not depend on the choice of χ. We write the central derivatives as

$$\partial\mathrm{Eis}(\mathsf{z},\varphi) \coloneqq E'(\mathsf{z},0,\varphi), \quad \partial\mathrm{Eis}_T(\mathsf{z},\varphi) \coloneqq E'_T(\mathsf{z},0,\varphi), \quad \partial\mathrm{Eis}_{T,v}(\mathsf{z},\varphi) \coloneqq E'_{T,v}(\mathsf{z},0,\varphi).$$

Then we have a Fourier expansion

$$\partial\mathrm{Eis}(\mathsf{z},\varphi)=\sum_{T\in\mathrm{Herm}_n(F)}\partial\mathrm{Eis}_T(\mathsf{z},\varphi) \tag{12.4.0.6}$$

By (12.3.0.4) we know that

$$\partial\mathrm{Eis}_T(\mathsf{z},\varphi)\neq 0\Rightarrow \mathrm{Diff}(T,\mathbb{V})=\{v\},\quad \partial\mathrm{Eis}_T(\mathsf{z},\varphi)=\partial\mathrm{Eis}_{T,v}(\mathsf{z},\varphi). \tag{12.4.0.7}$$

For the fixed open compact subgroup $K\subseteq\widetilde{G}(\mathbb{A}_f)$, we will choose

$$\varphi=\varphi_K\otimes\varphi_\infty\in\mathscr{S}(\mathbb{V}^n)$$

such that $\varphi_K\in\mathscr{S}(\mathbb{V}_f^n)$ is K-invariant (where K acts on $\mathbb{V}$ via the second factor K_G) and φ_∞ is the Gaussian function

$$\varphi_\infty(\mathbf{x})=\psi_\infty(i\,\mathrm{tr}\,T(\mathbf{x})).$$

For our fixed choice of Gaussian φ_∞, we write

$$E(\mathsf{z},s,\varphi_K)=E(\mathsf{z},s,\varphi_K\otimes\varphi_\infty),\quad \partial\mathrm{Eis}(\mathsf{z},\varphi_K)=\partial\mathrm{Eis}(\mathsf{z},\varphi_K\otimes\varphi_\infty)$$

and so on for short. When $T>0$ is totally positive definite, by [Liu11a, Proposition 4.5 (2)] the archimedean Whittaker function is

$$W_{T,\infty}(\mathsf{z},0,\varphi_\infty)=c_\infty\cdot q^T,\quad q^T:=\psi_\infty(\mathrm{tr}(T\mathsf{z}))$$

for some constant c_∞ independent of T. It follows from (12.2.0.3) that we have a factorization

$$\partial\mathrm{Eis}_{T,v}(\mathsf{z},\varphi_K)=c_\infty\cdot W'_{T,v}(1,0,\varphi_{K,v})\cdot\prod_{v'\neq v,v'\nmid\infty}W_{T,v'}(1,0,\varphi_{K,v'})\cdot q^T. \tag{12.4.0.8}$$

13. The semi-global identity at inert primes

In this section we assume that we are in the situation of §11.2 (hyperspecial level) or §11.3 (almost self-dual level). We fix the level K as above and write $\mathcal{M}=\mathcal{M}_K$ for short.

13.1. **p-adic uniformization of the supersingular locus of $\mathcal{M}$.** Let $\widehat{\mathcal{M}^{\mathrm{ss}}}$ be the completion of the base change $\mathcal{M}_{O_{\breve{E}_\nu}}$ along the supersingular locus $\mathcal{M}^{\mathrm{ss}}_{k_\nu}$ of its special fiber $\mathcal{M}_{k_\nu}$. Here E_ν is the completion of E at ν and k_ν is its residue field. Assume $p>2$. Then we have a p-adic uniformization theorem ([RZ96], [Cho18, Theorem 4.3], see also the proof of [RSZ20, Theorem 8.15]),

$$\widehat{\mathcal{M}^{\mathrm{ss}}}\simeq\widetilde{G}'(\mathbb{Q})\backslash[\mathcal{N}'\times\widetilde{G}(\mathbb{A}_f^p)/K^p]. \tag{13.1.0.9}$$

Here $\widetilde{G}'=Z^{\mathbb{Q}}\times_{\mathbb{G}_m}G'^{\mathbb{Q}}$ is the group associated to a F/F_0-hermitian space V' obtained from V by changing the signature at ϕ_0 from $(n-1,1)$ to $(n,0)$ and the invariant at v_0 from $+1$ (resp. -1) to -1 (resp. $+1$) (i.e., V'_{v_0} is a non-split (resp. split) $F_{w_0}/F_{0,v_0}$-hermitian space) in the hyperspecial case (resp. the almost self-dual case). The relevant Rapoport–Zink space $\mathcal{N}'$ associated to $\widetilde{G}'$ is given by

$$\mathcal{N}'\simeq(Z^{\mathbb{Q}}(\mathbb{Q}_p)/K_{Z^{\mathbb{Q}},p})\times\mathcal{N}_{O_{\breve{E}_\nu}}\times\prod_{v\neq v_0}\mathrm{U}(V)(F_{0,v})/K_{G,v},$$

where the product is over places $v \neq v_0$ of F_0 over p, and $\mathcal{N}$ is isomorphic to $\mathcal{N}_{F_{w_0}/F_{0,v_0},n}$, the Rapoport–Zink space defined in §2.1 in the hyperspecial case[5], or isomorphic to $\mathcal{N}^1_{F_{w_0}/F_{0,v_0},n}$, the Rapoport–Zink space defined in §10.1 in the almost self-dual case.

13.2. **The hermitian lattice** $\mathbb{V}(A_0, A)$**.** For a locally noetherian $O_{E,(\nu)}$-scheme S and a point $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p) \in \mathcal{M}(S)$, define the *space of special homomorphisms* to be

$$\mathbb{V}(A_0, A) := \mathrm{Hom}_{O_F}(A_0, A) \otimes \mathbb{Z}_{(p)},$$

a free $O_{F,(p)} := O_F \otimes \mathbb{Z}_{(p)}$-module of finite rank. Then $\mathbb{V}(A_0, A)$ carries a $O_{F,(p)}$-valued hermitian form: for $x, y \in \mathbb{V}(A_0, A)$, the pairing $(x, y) \in O_{F,(p)}$ is given by

$$(A_0 \xrightarrow{x} A \xrightarrow{\lambda} A^\vee \xrightarrow{y^\vee} A_0^\vee \xrightarrow{\lambda_0^{-1}} A_0) \in \mathrm{End}_{O_F}(A_0) \otimes \mathbb{Z}_{(p)} = \iota_0(O_{F,(p)}) \simeq O_{F,(p)}.$$

Notice that λ_0^{-1} makes sense as the polarization λ_0 is coprime-to-p by (M6).

Let $m \geq 1$. Given an m-tuple $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{V}(A_0, A)^m$, define its *fundamental matrix* to be

$$T(\mathbf{x}) := ((x_i, x_j))_{1 \leq i,j \leq m} \in \mathrm{Herm}_m(O_{F,(p)}),$$

an $m \times m$ hermitian matrix over $O_{F,(p)}$.

13.3. **Semi-global Kudla–Rapoport cycles** $\mathcal{Z}(T, \varphi_K)$**.** We say a Schwartz function $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$ is v_0-*admissible* if it is K-invariant and $\varphi_{K,v} = \mathbf{1}_{(\Lambda_v)^m}$ for all v above p such that v is nonsplit in F. First we consider a special v_0-admissible Schwartz function of the form

$$\varphi_K = (\varphi_i) \in \mathscr{S}(\mathbb{V}_f^m), \quad \varphi_i = \mathbf{1}_{\Omega_i}, \quad i = 1, \ldots, m, \tag{13.3.0.10}$$

where $\Omega_i \subseteq \mathbb{V}_f$ is a K-invariant open compact subset such that $\Omega_{i,v} = \Lambda_v$ for all v above p such that v is nonsplit in F. Given such a special Schwartz function φ_K and $T \in \mathrm{Herm}_m(O_{F,(p)})$, define a semi-global *Kudla–Rapoport cycle* $\mathcal{Z}(T, \varphi_K)$ over $\mathcal{M}$ as follows. For a locally noetherian $O_{E,(\nu)}$-scheme S, define $\mathcal{Z}(T, \varphi_K)(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0}, \mathbf{x})$ where

(1) $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p, \bar{\eta}_p^{v_0}) \in \mathcal{M}(S)$,

(2) $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{V}(A_0, A)^m$ with fundamental matrix $T(\mathbf{x}) = T$.

(3) $\eta^p(\mathbf{x}_*) \in (\Omega_i^{(p)}) \subseteq (\mathbb{V}_f^{(p)})^m$. Here $\mathbf{x}_* \in \mathrm{Hom}_F(\hat{V}^p(A_0), \hat{V}^p(A))^m$ is the element induced by $\mathbf{x}$.

(4) $\eta_v(\mathbf{x}_*) \in (\Omega_{i,\overline{w}}) \subseteq \mathbb{V}_{\overline{w}}^m$ for all $\bar{\eta}_v \in \bar{\eta}_p^{v_0}$ (cf. (M5)). Here $\mathbf{x}_* \in \mathrm{Hom}_{O_{F_{\overline{w}}}}(A_0[\overline{w}^\infty], A[\overline{w}^\infty])^m \otimes_{O_{F_{\overline{w}}}} F_{\overline{w}}$ is the element induced by $\mathbf{x}$.

The functor $S \mapsto \mathcal{Z}(T, \varphi_K)(S)$ is represented by a (possibly empty) Deligne–Mumford stack which is finite and unramified over $\mathcal{M}$ ([KR14, Proposition 2.9]), and thus defines a cycle $\mathcal{Z}(T, \varphi_K) \in \mathrm{Z}^*(\mathcal{M})$. For a general v_0-admissible Schwartz function $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$, by extending $\mathbb{C}$-linearly we obtain a cycle $\mathcal{Z}(T, \varphi_K) \in \mathrm{Z}^*(\mathcal{M})_{\mathbb{C}}$.

[5]We use the convention $(1, n-1)$ for the signature of Rapoport–Zink spaces while the convention $(n-1, 1)$ for Shimura varieties; each of these two conventions is more preferable in its respective setting. Strictly speaking, [RSZ20, Theorem 8.15] assumes that v_0 is unramified over p. This assumption can be dropped (in the hyperspecial case) due to the Eisenstein condition in (M8). In fact, under the assumption (H6), [Mih, Definition 2.8 (ii)] specializes to the Eisenstein condition in (M8) for signature $(1, n-1)$, so Mihatsch's theorem [Mih, Theorem 3.1] is still applicable even when v_0 is ramified over p.

13.4. **Variants of semi-global Kudla–Rapoport cycles $\mathcal{Z}^\flat(T,\varphi_K)$ at almost self-dual level.** Assume that we are in the situation of §13.3 and Λ_{v_0} is almost self-dual. We will define a variant $\mathcal{Z}^\flat(T,\varphi_K)$ of the semi-global Kudla–Rapoport cycle. To do so, consider a diagram of Shimura varieties

$$\begin{tikzcd} & \mathrm{Sh}_{K\cap K^\sharp} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Sh}_K & & \mathrm{Sh}_{K^\sharp}, \end{tikzcd} \tag{13.4.0.11}$$

where the level at v_0 is modified as in Remark 10.2.1.

More precisely, consider a F/F_0-hermitian space of dimension $n+1$,

$$V^\sharp := V \oplus \langle x_0\rangle,$$

where $u_0=(x_0,x_0)$ is totally positive, has valuation 1 at v_0 and valuation 0 for all places $v\neq v_0$ of F_0 above p. We take the level $K^\sharp\subseteq \widetilde{G}^\sharp(\mathbb{A}_f)$ of the form as in §11.2 such that

(1) $K^\sharp_{G,v_0}$ is the stabilizer of a self-dual lattice $\Lambda^\sharp_{v_0}\subseteq V^\sharp_{v_0}$,

(2) for $v\neq v_0$ a place of F_0 above p, $K^\sharp_{G,v}$ is the stabilizer of the lattice $\Lambda_v\oplus\langle x_0\rangle\subseteq V^\sharp_v$,

(3) $K^p\supseteq K^{\sharp,p}\cap\widetilde{G}(\mathbb{A}_f^p)$.

Denote by

$$\mathrm{Sh}_{K^\sharp}=\mathrm{Sh}_{K^\sharp}(\widetilde{G}^\sharp,\{h_{\widetilde{G}^\sharp}\}),\quad \mathrm{Sh}_{K\cap K^\sharp}=\mathrm{Sh}_{K\cap K^\sharp}(\widetilde{G},\{h_{\widetilde{G}}\})$$

the Shimura varieties defined in §11.1.

Let $\mathcal{M}_{K^\sharp}$ be the semi-global model of $\mathrm{Sh}_{K^\sharp}$ over $O_{E,(\nu)}$ as defined in §11.2. Define the semi-global integral model $\mathcal{M}_{K\cap K^\sharp}$ of $\mathrm{Sh}_{K\cap K^\sharp}$ over $O_{E,(\nu)}$ as follows. For a locally noetherian $O_{E,(\nu)}$-scheme S, define $\mathcal{M}_{K\cap K^\sharp}(S)$ to be the groupoid of tuples $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\bar\eta^p,\bar\eta_p^{v_0},A^\sharp,\iota^\sharp,\lambda^\sharp,\bar\eta^{\sharp,p},\bar\eta_p^{\sharp,v_0},\alpha)$, where

(1) $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\bar\eta^p,\bar\eta_p^{v_0})\in\mathcal{M}_{K_p(K^{\sharp,p}\cap K^p)}(S)$,

(2) $(A_0,\iota_0,\lambda_0,A^\sharp,\iota^\sharp,\lambda^\sharp,\bar\eta^{\sharp,p},\bar\eta_p^{\sharp,v_0})\in\mathcal{M}_{K^\sharp}(S)$,

(3) $\alpha: A\times A_0\to A^\sharp$ is an isogeny of degree q_{v_0} such that
   (a) $\ker\alpha\subseteq (A\times A_0)[u_0]$,
   (b) $\alpha^*(\lambda^\sharp)=\lambda\times u_0\lambda_0$,
   (c) The $K_G^{\sharp,p}$-orbit of $\alpha_*(\eta^p\times\eta_0^p)$ coincides with $\bar\eta^{\sharp,p}$, where η_0^p is the $\mathbb{A}^p_{F,f}$-linear map between lisse $\mathbb{A}^p_{F,f}$-sheaves given by

$$\eta_0^p:\mathrm{Hom}_F(\hat V^p(A_0),\hat V^p(A_0))\to\langle x_0\rangle_F\otimes_F\mathbb{A}^p_{F,f},\quad \mathrm{id}_{\hat V^p(A_0)}\longmapsto x_0.$$

   (d) For all $\bar\eta_v\in\bar\eta_p^{v_0}$, the $K^\sharp_{G,v}$-orbit of $\alpha_*(\eta_v\times\eta_{0,v})$ coincides with $\bar\eta^\sharp_v$, where $\eta_{0,v}$ is the $F_{\overline{w}}$-linear map between lisse $F_{\overline{w}}$-sheaves

$$\eta_{0,v}:\mathrm{Hom}_{O_{F_{\overline{w}}}}(A_0[\overline{w}^\infty],A_0[\overline{w}^\infty])\otimes_{O_{F_{\overline{w}}}}F_{\overline{w}}\to\langle x_0\rangle_F\otimes_F F_{\overline{w}},\quad \mathrm{id}_{A_0[\overline{w}^\infty]}\longmapsto x_0.$$

Then the diagram (13.4.0.11) extends to semi-global integral models

$$\begin{tikzcd} & \mathcal{M}_{K\cap K^\sharp} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_K & & \mathcal{M}_{K^\sharp}. \end{tikzcd}$$

The p-adic uniformization theorem (13.1.0.9) of [RZ96] then holds for $\mathcal{M}_{K\cap K^\sharp}$ with $\mathcal{N} = \widetilde{\mathcal{N}}^1_{F_{w_0}/F_{0,v_0},n}$, the auxiliary Rapoport–Zink space defined in §10.2.

Analogous to Remark 10.2.2, we obtain a cycle $\mathcal{Z}^\flat(T,\varphi_K)$ on $\mathcal{M}_{K\cap K^\sharp}$, which can serve as an integral model of the pullback of the generic fiber of $\mathcal{Z}(T,\varphi_K)$ along π_1. More precisely, first assume that φ_K is a special v_0-admissible Schwartz function as in (13.3.0.10). For a locally noetherian $O_{E,(\nu)}$-scheme S, define $\mathcal{Z}^\flat(T,\varphi_K)(S)$ to be the groupoid of tuples $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\bar\eta^p,\bar\eta_p^{v_0},A^\sharp,\iota^\sharp,\lambda^\sharp,\bar\eta^{\sharp,p},\bar\eta_p^{\sharp,v_0},\alpha,\mathbf{x})$, where

(1) $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\bar\eta^p,\bar\eta_p^{v_0},A^\sharp,\iota^\sharp,\lambda^\sharp,\bar\eta^{\sharp,p},\bar\eta_p^{\sharp,v_0},\alpha) \in \mathcal{M}_{K\cap K^\sharp}(S)$,

(2) $\mathbf{x} = [x_1,\ldots,x_m] \in \mathbb{V}(A_0,A^\sharp)^m$ with fundamental matrix $T(\mathbf{x}) = T$, and each x_i is orthogonal to $\alpha|_{A_0}$ (as elements of the hermitian lattice $\mathbb{V}(A_0,A^\sharp)$ defined in §13.2),

(3) $\eta^p((\alpha^{-1}\mathbf{x})_*) \in (\Omega_i^{(p)}) \subseteq (\mathbb{V}_f^{(p)})^m$. Notice that by (2) we have $\alpha^{-1}\mathbf{x} \in \mathrm{Hom}^\circ_{O_F}(A_0,A)^m$, and $(\alpha^{-1}\mathbf{x})_* \in \mathrm{Hom}_F(\hat V^p(A_0),\hat V^p(A))^m$ is the element induced by $\alpha^{-1}\mathbf{x}$.

(4) $\eta_v((\alpha^{-1}\mathbf{x})_*) \in (\Omega_i^{(p)}) \subseteq (\mathbb{V}_f^{(p)})^m$ for all $\bar\eta_v \in \bar\eta_p^{v_0}$. Here $(\alpha^{-1}\mathbf{x})_* \in \mathrm{Hom}_{O_{F_{\overline{w}}}}(A_0[\overline{w}^\infty],A[\overline{w}^\infty])^m \otimes_{O_{F_{\overline{w}}}} F_{\overline{w}}$ is the element induced by $\alpha^{-1}\mathbf{x}$.

The functor $S \mapsto \mathcal{Z}^\flat(T,\varphi_K)(S)$ is represented by a (possibly empty) Deligne–Mumford stack which is finite and unramified over $\mathcal{M}_{K\cap K^\sharp}$ and thus defines a cycle $\mathcal{Z}^\flat(T,\varphi_K) \in \mathrm{Z}^*(\mathcal{M}_{K\cap K^\sharp})$. For a general v_0-admissible Schwartz function $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$, by extending $\mathbb{C}$-linearly we obtain a cycle $\mathcal{Z}^\flat(T,\varphi_K) \in \mathrm{Z}^*(\mathcal{M}_{K\cap K^\sharp})_{\mathbb{C}}$.

13.5. **The local arithmetic intersection number** $\mathrm{Int}_{T,v_0}(\varphi_K)$**.** Assume $T \in \mathrm{Herm}_n(O_{F,(p)})_{>0}$ is totally positive definite. Let $t_1,\ldots,t_n$ be the diagonal entries of T. Let $\varphi_K \in \mathscr{S}(\mathbb{V}_f^n)$ be a special Schwartz function as in (13.3.0.10). We have a natural decomposition (cf. [KR14, (11.2)]),

$$\mathcal{Z}(t_1,\varphi_1)\cap\cdots\cap\mathcal{Z}(t_n,\varphi_n) = \bigsqcup_{T\in\mathrm{Herm}_n(F)} \mathcal{Z}(T,\varphi_K), \tag{13.5.0.12}$$

here $\cap$ denotes taking fiber product over $\mathcal{M}$, and the indexes T have diagonal entries $t_1,\ldots,t_n$.

When Λ_{v_0} is self-dual, define

$$\mathrm{Int}_{T,\nu}(\varphi_K) \coloneqq \chi(\mathcal{Z}(T,\varphi_K),\mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)}\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}(t_n,\varphi_n)})\cdot\log q_\nu, \tag{13.5.0.13}$$

where q_ν denotes the size of the residue field k_ν of E_ν, $\mathcal{O}_{\mathcal{Z}(t_i,\varphi_i)}$ denotes the structure sheaf of the semi-global Kudla–Rapoport divisor $\mathcal{Z}(t_i,\varphi_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{M}$, and χ denotes the Euler–Poincaré characteristic (an alternating sum of lengths of $\mathcal{O}_{E,(\nu)}$-modules). Notice that the derived tensor product $\mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)}\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}(t_n,\varphi_n)}$ has the structure of a complex of $\mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)\cap\cdots\cap\mathcal{Z}(t_n,\phi_n)}$-modules, hence has a natural decomposition by support according to the decomposition (13.5.0.12).

Similarly, when Λ_{v_0} is almost self-dual, define

$$\mathrm{Int}_{T,\nu}(\varphi_K) \coloneqq \frac{1}{\deg \pi_1}\chi(\mathcal{Z}^\flat(T,\varphi_K), \mathcal{O}_{\mathcal{Z}^\flat(t_1,\varphi_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^\flat(t_n,\varphi_n)}) \cdot \log q_\nu, \tag{13.5.0.14}$$

where $\deg \pi_1$ is the generic degree of the generically finite morphism π_1 (§13.4).

Finally, when Λ_{v_0} is self-dual or almost self-dual, define

$$\mathrm{Int}_{T,v_0}(\varphi_K) \coloneqq \frac{1}{[E:F_0]} \cdot \sum_{\nu | v_0} \mathrm{Int}_{T,\nu}(\varphi_K).$$

We extend the definition of $\mathrm{Int}_{T,v_0}(\varphi_K)$ to a general v_0-admissible $\varphi_K \in \mathscr{S}(\mathbb{V}_f^n)$ by extending $\mathbb{C}$-linearly.

13.6. **The semi-global identity.** Recall that we are in the situation of §11.2 (hyperspecial level) or §11.3 (almost self-dual level).

Theorem 13.6.1. *Assume $p>2$. Assume $\varphi_K \in \mathscr{S}(\mathbb{V}_f^n)$ is v_0-admissible (§13.3). Then for any $T \in \mathrm{Herm}_n(O_{F,(p)})_{>0}$,*

$$\mathrm{Int}_{T,v_0}(\varphi_K) q^T = c_K \cdot \partial\mathrm{Eis}_{T,v_0}(\mathsf{z}, \varphi_K),$$

where $c_K = \frac{(-1)^n}{\mathrm{vol}(K)}$ is a nonzero constant independent of T and φ_K, and $\mathrm{vol}(K)$ is the volume of K under a suitable Haar measure on $\widetilde{G}(\mathbb{A}_f)$.

Proof. As explained in [Ter13a, Remark 7.4], this follows routinely from our main Theorem 3.4.1 in the hyperspecial case. We briefly sketch the argument. The support of $\mathcal{Z}(T)$ lies in the supersingular locus $\mathcal{M}_{k_\nu}^{\mathrm{ss}}$ by the same proof of [KR14, Lemma 2.21]. We may then compute the left-hand-side via p-adic uniformization §13.1 as the product of the arithmetic intersection numbers on the Rapoport–Zink space $\mathcal{N}$ and a theta integral of $\varphi_{K,f}^{v_0}$. The arithmetic intersection number is equal to $W'_{T,v_0}(1,0,\varphi_{K,v_0})$ up to a nonzero constant independent of T and φ_K by our main Theorem 3.4.1 and Remark 3.4.2 (as $p>2$). Since T is nonsingular, the theta integral of $\varphi_{K,f}^{v_0}$ evaluates to $\prod_{v \neq v_0, v\nmid\infty} W_{T,v}(1,0,\varphi_{K,v})$ up to a constant independent of T and φ_K ([Ich04, §5–6]). The result then follows from the factorization (12.4.0.8) of the right-hand-side $\partial\mathrm{Eis}_{T,v_0}$.

The identity follows in a similar way from our main Theorem 10.3.1 and Remark 10.3.2 in the almost self-dual case. In fact, by the same proof of [San17, Theorem 4.13], it remains to check that for $\Lambda = \langle 1\rangle^{n-1} \oplus \langle \varpi \rangle$ an almost self-dual lattice and $L \subseteq \mathbb{V}$ any O_F-lattice of full rank n, we have the following identity

$$\frac{\mathrm{Den}(\Lambda,\Lambda)}{\mathrm{Den}(\langle 1\rangle^{n-1}, \langle 1\rangle^{n-1})} = \frac{\partial\mathrm{Den}_\Lambda(L)}{\mathrm{Int}(L)}. \tag{13.6.1.1}$$

By Theorem 9.1.2, the left-hand-side of (13.6.1.1) is equal to $\mathrm{Den}(\Lambda^\sharp)$. By (3.6.1.1), $\mathrm{Den}(\Lambda^\sharp)$ is equal to the number of self-dual lattices containing $\Lambda^\sharp$. Since $\Lambda^\sharp$ is a vertex lattice of type 2, the latter is equal to the number of isotropic lines in a 2-dimensional (non-degenerate) k_F/k-hermitian space, which is $q+1$ (cf. Remark 10.2.1). By Theorem 10.3.1, the right-hand-side of (13.6.1.1) is also equal to $q+1$, and thus the desired identity (13.6.1.1) is proved. $\square$

14.1. **Global integral models at minimal levels.** In this subsection we will define a global integral model over O_E of the Shimura variety Sh_K introduced in §11.1. We will be slightly more general than [RSZ20, §5], allowing F/F_0 to be unramified at all finite places.

We consider an O_F-lattice $\Lambda \subseteq V$ and let

$$K_G^\circ = \{g \in G(\mathbb{A}_f) : g(\Lambda \otimes_{O_F} \widehat{O}_F) = \Lambda \otimes_{O_F} \widehat{O}_F\}.$$

Assume that for any finite place v of F_0 (write p its residue characteristic),

(G0) if $p = 2$, then v is unramified in F.

(G1) if v is inert in F and V_v is split, then $\Lambda_v \subseteq V_v$ is self-dual. If v is further ramified over p, then (H6) is satisfied.

(G2) if v is inert in F and V_v is nonsplit, then v is unramified over p and $\Lambda_v \subseteq V_v$ is almost self-dual.

(G3) if v is split in F, then $\Lambda_v \subseteq V_v$ is self-dual.

(G4) if v is ramified in F, then v is unramified over p and $\Lambda_v \subseteq V_v$ is self-dual.

We take $K^\circ = K_{Z^{\mathbb{Q}}} \times K_G^\circ$, where $K_{Z^{\mathbb{Q}}}$ is the unique maximal open compact subgroup of $Z^{\mathbb{Q}}(\mathbb{A}_f)$ as in §11.2.

Notice the assumptions (G0)—(G4) ensure that each finite place v_0 and the level K_{G,v_0} belongs one of the four cases considered in §11.2, §11.3, §11.4, §11.6. Define an integral $\mathcal{M}_{K^\circ}$ of Sh_{K° over O_E as follows. For a locally noetherian O_E-scheme S, we consider $\mathcal{M}_{K^\circ}(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$, where

(1) A_0 (resp. A) is an abelian scheme over S.

(2) ι_0 (resp. ι) is an action of O_F on A_0 (resp. A) satisfying the Kottwitz condition of signature $\{(1,0)_{\phi\in\Phi}\}$ (resp. signature $\{(r_\phi, r_{\bar{\phi}})_{\phi\in\Phi}\}$).

(3) λ_0 (resp. λ) is a polarization of A_0 (resp. A) whose Rosati involution induces the automorphism given by the nontrivial Galois automorphism of F/F_0 via ι_0 (resp. ι).

We require that the triple $(A_0, \iota_0, \lambda_0)$ satisfies (M6), and for any finite place ν of E (write p its residue characteristic), the triple (A, ι, λ) over $S_{O_{E,(\nu)}}$ satisfies the conditions (M7), (M9), and moreover (MS) when v_0 is split in F and (MR) when v_0 is ramified in F. We may and do choose the axillary ideal $\mathfrak{a} \subseteq O_{F_0}$ in (M6) to be divisible only by primes split in F.

Then the functor $S \mapsto \mathcal{M}_{K^\circ}(S)$ is represented by a Deligne–Mumford stack $\mathcal{M}_{K^\circ} = \mathcal{M}_{K^\circ}$ flat over $\operatorname{Spec} O_E$. It has isolated singularities only in ramified characteristics, and we may further obtain a regular model by blowing up (the *Krämer model*) which we still denote by $\mathcal{M}_K$. For each finite place ν of E, the base change $\mathcal{M}_{K^\circ, O_{E,(\nu)}}$ is canonically isomorphic to the semi-global integral models defined in §11.2, §11.3, §11.4, §11.6.

14.2. **Global integral models at Drinfeld levels.** With the same set-up as §14.1, but now we allow Drinfeld levels at split primes. Let $\mathbf{m} = (m_v)$ be a collection of integers $m_v \geq 0$ indexed by finite places v of F_0. Further assume

(G5) if $m_v > 0$, then v satisfies (S), and each place ν of E above v satisfies (D).

We take $K_G^{\mathbf{m}} \subseteq K_G^\circ$ such that $(K_G^{\mathbf{m}})_v = (K_G^\circ)_v$ if $m_v = 0$ and $(K_G^{\mathbf{m}})_v = (K_G^\circ)_v^{m_v}$ to be the principal congruence subgroup mod $\varpi_v^{m_v}$ if $m_v > 0$. Write $K^{\mathbf{m}} = K_{Z^{\mathbb{Q}}} \times K_G^{\mathbf{m}}$. Define $\mathcal{M}_{K^{\mathbf{m}}}$ to be the normalization of $\mathcal{M}_{K^\circ}$ in $\mathrm{Sh}_{K^{\mathbf{m}}}(\widetilde{G}, h_{\widetilde{G}})$.

Then $\mathcal{M}_{K^{\mathbf{m}}}$ is a Deligne–Mumford stack finite flat over $\mathcal{M}_{K^\circ}$. Moreover for each finite place ν of E, the base change $\mathcal{M}_{K^{\mathbf{m}}, O_{E,(\nu)}}$ is canonically isomorphic to the semi-global integral models defined in §11.2, §11.3, §11.4, §11.5, §11.6. Thus $\mathcal{M}_{K^{\mathbf{m}}}$ is smooth at places over v_0 in (G1), (G3), semi-stable at places over v_0 in (G2) when ν is unramified over p, and regular at places over v_0 in (G4), (G5). In particular, assume all places ν over v_0 in (G2) are unramified over p, then $\mathcal{M}_{K^{\mathbf{m}}}$ is regular. When $\mathbf{m}$ is sufficiently large, $\mathcal{M}_{K^{\mathbf{m}}}$ is relatively representable over $\mathcal{M}_0^{\mathfrak{a},\xi}$.

14.3. **Global Kudla–Rapoport cycles $\mathcal{Z}(T, \varphi_K)$.** We continue with the same set-up as §14.2. From now on write $K = K^{\mathbf{m}}$ and $\mathcal{M} = \mathcal{M}_{K^{\mathbf{m}}}$ for short. Let $\varphi_K = (\varphi_i) \in \mathscr{S}(\mathbb{V}_f^m)$ be K-invariant. Let $t_1, \ldots, t_m \in F$. Let $Z(t_i, \varphi_i)$ be the (possibly empty) Kudla–Rapoport cycle on the generic fiber of $\mathcal{M}$ (defined similarly as in §13.3) and let $\mathcal{Z}(t_i, \varphi_i)$ be its Zariski closure in the global integral model $\mathcal{M}$. Then we have a decomposition into the global *Kudla–Rapoport cycles* $\mathcal{Z}(T, \varphi_K)$ over $\mathcal{M}$ (cf. [KR14, (11.2)]),

$$\mathcal{Z}(t_1, \varphi_1) \cap \cdots \cap \mathcal{Z}(t_m, \varphi_m) = \bigsqcup_{T \in \mathrm{Herm}_m(F)} \mathcal{Z}(T, \varphi_K),$$

here $\cap$ denotes taking fiber product over $\mathcal{M}$, and the indexes T have diagonal entries $t_1, \ldots, t_m$.

14.4. **The arithmetic intersection number $\mathrm{Int}_T(\varphi_K)$.** For nonsingular $T \in \mathrm{Herm}_n(F)$, define

$$\mathrm{Int}_T(\varphi_K) \coloneqq \sum_v \mathrm{Int}_{T,v}(\varphi_K)$$

to be the sum over all finite places v of F of local arithmetic intersection numbers defined as in §13.5. By the same proof of [KR14, Lemma 2.21], this sum is nonzero only if $\mathrm{Diff}(T, \mathbb{V}) = \{v\}$ is a singleton, and in this case v is necessarily nonsplit in F. Hence

$$\mathrm{Int}_T(\varphi_K) \neq 0 \Longrightarrow \mathrm{Diff}(T, \mathbb{V}) = \{v\} \text{ and } \mathrm{Int}_T(\varphi_K) = \mathrm{Int}_{T,v}(\varphi_K). \tag{14.4.0.2}$$

14.5. **The global Kudla–Rapoport conjecture for nonsingular Fourier coefficients.** Assume that we are in the situation of §14.2. We say $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$ is *inert-admissible* if it is v-admissible at all v inert in F (§13.3). When φ_K is inert-admissible, the base change of the global Kudla–Rapoport cycle $\mathcal{Z}(T, \varphi_K)$ to $\mathrm{Spec}\, O_{E,(\nu)}$ above an inert prime agrees with the semi-global Kudla–Rapoport cycle defined in §13.3. We say a nonsingular $T \in \mathrm{Herm}_n(F)$ is *inert* if $\mathrm{Diff}(T, \mathbb{V}) = \{v\}$ where v is inert in F and not above 2.

Theorem 14.5.1. *Assume $\varphi_K \in \mathscr{S}(\mathbb{V}_f^n)$ is inert-admissible. Let $T \in \mathrm{Herm}_n(F)$ be inert. Then*

$$\mathrm{Int}_T(\varphi_K) q^T = c_K \cdot \partial\mathrm{Eis}_T(\mathsf{z}, \varphi_K),$$

where $c_K = \frac{(-1)^n}{\mathrm{vol}(K)}$ as in Theorem 13.6.1.

Proof. Since T is inert, we know that $T > 0$, and moreover by (14.4.0.2) and (12.4.0.7) both sides are contributed non-trivially only by the term at $\mathrm{Diff}(T, \mathbb{V}) = \{v\}$. Since φ_K is inert-admissible, both sides are zero unless $T \in \mathrm{Herm}_n(O_{F,(p)})$ (p the residue characteristic of v). So we can apply Theorem 13.6.1 to obtain $\mathrm{Int}_{T,v}(\varphi_K)q^T = c_K \cdot \partial\mathrm{Eis}_{T,v}(\mathsf{z}, \varphi_K)$. $\square$

Corollary 14.5.2. *Kudla–Rapoport's global conjecture* [KR14, Conjecture 11.10] *holds.*

Proof. We take $F_0 = \mathbb{Q}$ and $K = K^\circ$. We also take the axillary ideal $\mathfrak{a}$ to be trivial (see (M6)). Then the global integral model $\mathcal{M}_{K^\circ}$ agrees with the moduli stack $\mathcal{M}^V$ in [KR14, Proposition 2.12]. The test function φ in [KR14] satisfies $\varphi_K = \mathbf{1}_{(\hat{\Lambda})^n}$ and φ_∞ is the Gaussian function, so φ_K is inert-admissible. The assumption $\mathrm{Diff}_0(T) = \{p\}$ with $p > 2$ in [KR14, Conjecture 11.10] ensures that T is inert. The result then follows from Theorem 14.5.1. $\square$

15. The arithmetic Siegel–Weil formula

15.1. **Complex uniformization.** Assume we are in the situation of §11.1. Under the decomposition (11.2.0.1), we may identify the the $\widetilde{G}(\mathbb{R})$-conjugacy class $\{h_{\widetilde{G}}\}$ as the product $\{h_{Z^{\mathbb{Q}}}\} \times \prod_{\phi\in\Phi}\{h_{G,\phi}\}$. Notice $\{h_{Z^{\mathbb{Q}}}\}$ is a singleton as $Z^{\mathbb{Q}}$ is a torus, and $\{h_{G,\phi}\}$ is also a singleton for $\phi \neq \phi_0$ as $h_{G,\phi}$ is the trivial cocharacter. For $\phi = \phi_0$ the cocharacter is given by $h_{G,\phi_0}(z) = \mathrm{diag}\{1_{n-1}, \bar{z}/z\}$, and $\{h_{G,\phi_0}\}$ is the hermitian symmetric domain

$$\mathcal{D}_{n-1} \cong \mathrm{U}(n-1,1)/(\mathrm{U}(n-1)\times \mathrm{U}(1)).$$

We may identify $\mathcal{D}_{n-1} \subseteq \mathbb{P}(V_{\phi_0})(\mathbb{C})$ as the open subset of negative $\mathbb{C}$-lines in V_{ϕ_0}, and $\widetilde{G}(\mathbb{R})$ acts on $\mathcal{D}_{n-1}$ via its quotient $\mathrm{PU}(V_{\phi_0})(\mathbb{R})$. We may also identity it with the open $(n-1)$-ball

$$\mathcal{D}_{n-1} \xrightarrow{\sim} \{z \in \mathbb{C}^{n-1} : |z| < 1\}, \quad [z_1, \dots, z_n] \longmapsto (z_1/z_n, \dots, z_{n-1}/z_n),$$

under the standard basis of V_{ϕ_0}. In this way we obtain a complex uniformization (via ϕ_0),

$$\mathrm{Sh}_K(\mathbb{C}) = \widetilde{G}(\mathbb{Q})\backslash[\mathcal{D}_{n-1} \times \widetilde{G}(\mathbb{A}_f)/K]. \tag{15.1.0.1}$$

15.2. **Green currents.** Write $\mathcal{D} = \mathcal{D}_{n-1}$ for short. Let $x \in V_{\phi_0}$ be a nonzero vector. For any $z \in \mathcal{D}$, we let $x = x_z + x_{z^\perp}$ be the orthogonal decomposition with respect to z (i.e., $x_z \in z$ and $x_{z^\perp} \perp z$). Let $R(x,z) = -(x_z, x_z)$. Define

$$\mathcal{D}(x) = \{z \in \mathcal{D} : z \perp x\} = \{z \in \mathcal{D} : R(x,z) = 0\}.$$

Then $\mathcal{D}(x)$ is nonempty if and only if $(x,x) > 0$, in which case $\mathcal{D}(x)$ is an analytic divisor on $\mathcal{D}$. Define *Kudla's Green function* to be

$$g(x,z) = -\mathrm{Ei}(-2\pi R(x,z)),$$

where $\mathrm{Ei}(u) = -\int_1^\infty \frac{e^{ut}}{t}dt$ is the exponential integral. Then $g(x,-)$ is a smooth function on $\mathcal{D}\backslash\mathcal{D}(x)$ with a logarithmic singularity along $\mathcal{D}(x)$. By [Liu11a, Proposition 4.9], it satisfies the $(1,1)$-current equation for $\mathcal{D}(x)$,

$$\mathrm{dd}^c[g(x)] + \delta_{\mathcal{D}(x)} = [\omega(x)],$$

where $\omega(x,-) = e^{2\pi(x,x)}\varphi_{\mathrm{KM}}(x,-)$, and $\varphi_{\mathrm{KM}}(-,-) \in (\mathscr{S}(V_{\phi_0}) \otimes A^{1,1}(\mathcal{D}))^{\mathrm{U}(V_{\phi_0})(\mathbb{R})}$ is the *Kudla–Millson Schwartz form* ([KM86]). Here we recall $\mathrm{d} = \partial + \bar\partial$, $\mathrm{d}^c = \frac{1}{4\pi i}(\partial - \bar\partial)$ and $\mathrm{dd}^c = -\frac{1}{2\pi i}\partial\bar\partial$.

More generally, let $\mathbf{x} = (x_1, \ldots, x_m) \in V_{\phi_0}^m$ such that its fundamental matrix $T(\mathbf{x}) = ((x_i, x_j))_{1 \leq i,j \leq m}$ is nonsingular. Define

$$\mathcal{D}(\mathbf{x}) = \mathcal{D}(x_1) \cap \cdots \cap \mathcal{D}(x_m),$$

which is nonempty if and only if $T(\mathbf{x}) > 0$. Define Kudla's Green current by taking star product

$$g(\mathbf{x}) \coloneqq [g(x_1)] * \cdots * [g(x_m)].$$

It satisfies the (m, m)-current equation for $\mathcal{D}(\mathbf{x})$,

$$\mathrm{dd}^c(g(\mathbf{x})) + \delta_{\mathcal{D}(\mathbf{x})} = [\omega(x_1) \wedge \cdots \wedge \omega(x_m)].$$

Here we recall that

$$[g(x)] * [g(y)] \coloneqq [g(x)] \wedge \delta_{D(y)} + [\omega(x)] \wedge [g(y)].$$

15.3. **The local arithmetic Siegel–Weil formula at archimedean places.** Let $T \in \mathrm{Herm}_m(F)$ be nonsingular. Let $\varphi_K \in \mathscr{S}(\mathbb{V}_f^m)$ be K-invariant. Let $Z(T, \varphi_K)$ be the (possibly empty) Kudla–Rapoport cycle on the generic fiber Sh_K. Then

$$Z(T, \varphi_K)(\mathbb{C}) = \sum_{\substack{\mathbf{x} \in \widetilde{G}(\mathbb{Q}) \backslash V^m(F) \\ T(\mathbf{x}) = T}} \sum_{\tilde{g} \in \widetilde{G}_{\mathbf{x}}(\mathbb{A}_f) \backslash \widetilde{G}(\mathbb{A}_f)/K} \varphi_K(\tilde{g}^{-1}\mathbf{x}) \cdot Z(\mathbf{x}, \tilde{g})_K,$$

where we define the cycle on $\mathrm{Sh}_K(\mathbb{C})$ via the complex uniformization (15.1.0.1),

$$Z(\mathbf{x}, \tilde{g})_K = \widetilde{G}_{\mathbf{x}}(\mathbb{Q}) \backslash [\mathcal{D}(\mathbf{x}) \times \widetilde{G}_{\mathbf{x}}(\mathbb{A}_f) \tilde{g} K/K],$$

and $\widetilde{G}_{\mathbf{x}} \subseteq \widetilde{G}$ is the stabilizer of $\mathbf{x}$. Define a Green current for $Z(T, \varphi_K)(\mathbb{C})$ by

$$g(\mathsf{y}_{\phi_0}, T, \varphi_K, z, \tilde{g}) \coloneqq \sum_{\substack{\mathbf{x} \in V^m(F) \\ T(\mathbf{x}) = T}} \varphi_K(\tilde{g}^{-1}\mathbf{x}) \cdot g(\mathbf{x} \cdot a, z), \quad (z, \tilde{g}) \in \mathcal{D} \times \widetilde{G}(\mathbb{A}_f),$$

where $a \in \mathrm{GL}(V_{\phi_0}) \cong \mathrm{GL}_n(\mathbb{C})$ and $\mathsf{y}_{\phi_0} = a^t \bar{a}$. Define the *archimedean arithmetic intersection number* (depending on the parameter y_{ϕ_0}) to be

$$\mathrm{Int}_{T,\phi_0}(\mathsf{y}_{\phi_0}, \varphi_K) \coloneqq \frac{1}{2} \int_{\mathrm{Sh}_K(\mathbb{C})} g(\mathsf{y}_{\phi_0}, T, \varphi_K).$$

Replacing the choice of ϕ_0 by another $\phi \in \Phi$ (§11.1) gives rise to a Shimura variety Sh_K^ϕ conjugate to Sh_K, associated to a hermitian space V^ϕ whose signature at ϕ_0, ϕ are swapped compared to V. Thus we can define in the same way the archimedean intersection number for any $\phi \in \Phi$,

$$\mathrm{Int}_{T,\phi}(\mathsf{y}_\phi, \varphi_K) \coloneqq \frac{1}{2} \int_{\mathrm{Sh}_K^\phi(\mathbb{C})} g(\mathsf{y}_\phi, T, \varphi_K). \tag{15.3.0.2}$$

Theorem 15.3.1. *Assume $\varphi_K \in \mathscr{S}(\mathbb{V}_f^n)$ is K-invariant. Let $T \in \mathrm{Herm}_n(F)$ be nonsingular and $\phi \in \Phi$. Then*

$$\mathrm{Int}_{T,\phi}(\mathsf{y}, \varphi_K) q^T = c_K \cdot \partial \mathrm{Eis}_{T,\phi}(\mathsf{z}, \varphi_K),$$

where $c_K = \frac{(-1)^n}{\mathrm{vol}(K)}$ as in Theorem 13.6.1.

Proof. By the main archimedean result of [Liu11a, Proposition 4.5, Theorem 4.17] (the archimedean analogue of our main Theorem 3.4.1) and the standard unfolding argument, we can express the integral (15.3.0.2) as a product involving the derivative $W'_{T,\phi}(g_{\mathsf{z}}, 0, \varphi_K)q^T$ and the product of values $\prod_{v\neq\phi} W_{T,v}(g_{\mathsf{z}}, 0, \varphi_K)$ from the Siegel–Weil formula, up to a nonzero constant independent of T. The result then follows from the factorization (12.2.0.3) of Fourier coefficients and comparing the constant with that of Theorem 13.6.1. See the proof of [Liu11a, Theorem 4.20] and the proof in the orthogonal case [BY21, Theorem 7.1] for details. When V is anisotropic (e.g., when $F_0 \neq \mathbb{Q}$), the result also follows from [GS19, (1.19)] for $r = p + 1 = n$ in the notation there. $\square$

15.4. **Arithmetic degrees of Kudla–Rapoport cycles.** Let us come back to the situation of §14.2. Let $T \in \mathrm{Herm}_n(F)$ be nonsingular. Let $\varphi_K = (\varphi_i) \in \mathscr{S}(\mathbb{V}_f^n)$ be K-invariant. Define the *arithmetic degree* (depending on the parameter $\mathsf{y} = (\mathsf{y}_\phi)_{\phi\in\Phi}$)

$$\widehat{\deg}_T(\mathsf{y}, \varphi_K) \coloneqq \mathrm{Int}_T(\varphi_K) + \sum_{\phi\in\Phi} \mathrm{Int}_{T,\phi}(\mathsf{y}_\phi, \varphi_K) \tag{15.4.0.1}$$

to be the sum of all nonarchimedean and archimedean intersection numbers. Define the *generating series of arithmetic degrees* of Kudla–Rapoport cycles to be

$$\widehat{\deg}(\mathsf{z}, \varphi_K) \coloneqq \sum_{\substack{T\in\mathrm{Herm}_n(F)\\ \det T\neq 0}} \widehat{\deg}_T(\mathsf{y}, \varphi_K)q^T.$$

It is related to the usual arithmetic degree on arithmetic Chow groups as we now explain. For nonzero $t_1, \ldots, t_n \in F$, we have classes in the Gillet–Soulé arithmetic Chow group (with $\mathbb{C}$-coefficients) of the regular Deligne–Mumford stack $\mathcal{M}_K$ ([GS90, Gil09]),

$$\widehat{\mathcal{Z}}(\mathsf{y}, t_i, \varphi_i) \coloneqq (\mathcal{Z}(t_i, \varphi_i), g(\mathsf{y}, t_i, \varphi_i)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{M}_K).$$

We have an arithmetic intersection product on n copies of $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{M}_K)$,

$$\langle\ ,\cdots,\ \rangle_{\mathrm{GS}} : \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{M}_K) \times \cdots \times \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{M}_K) \to \widehat{\mathrm{Ch}}^n_{\mathbb{C}}(\mathcal{M}_K),$$

and when $\mathcal{M}_K$ is proper over O_E, a degree map on the arithmetic Chow group of 0-cycles,

$$\widehat{\deg} : \widehat{\mathrm{Ch}}^n_{\mathbb{C}}(\mathcal{M}_K) \to \mathbb{C}.$$

We may compose these two maps and obtain a decomposition

$$\widehat{\deg}\langle \widehat{\mathcal{Z}}(\mathsf{y}, t_1, \varphi_1), \cdots, \widehat{\mathcal{Z}}(\mathsf{y}, t_n, \varphi_n)\rangle_{\mathrm{GS}} = \sum_T \widehat{\deg}_T(\mathsf{y}, \varphi_K),$$

where the matrices T have diagonal entries $t_1, \ldots, t_n$. The terms corresponding to nonsingular T agree with (15.4.0.1), at least in the hyperspecial case at inert primes.

15.5. **The arithmetic Siegel–Weil formula when F/F_0 is unramified.** Assume that we are in the situation of §14.2.

Theorem 15.5.1 (Arithmetic Siegel–Weil formula)**.** *Assume that F/F_0 is unramified at all finite places and split at all places above 2. Assume that $\varphi_K \in \mathscr{S}(\mathbb{V}_f^n)$ is inert-admissible (§14.5) and nonsingular (§12.3) at two places split in F. Then*

$$\widehat{\deg}(\mathsf{z}, \varphi_K) = c_K \cdot \partial\mathrm{Eis}(\mathsf{z}, \varphi_K),$$

where $c_K = \frac{(-1)^n}{\mathrm{vol}(K)}$ *as in Theorem 13.6.1.*

Remark 15.5.2. The assumption that F/F_0 is unramified at all finite places implies that $F_0 \neq \mathbb{Q}$ and hence the Shimura variety Sh_K is projective and the global integral model $\mathcal{M}_K$ is proper over O_E. Moreover, this assumption forces that the hermitian space V to be nonsplit at some inert place, and thus it is necessary to allow almost self-dual level at some inert place (as we did in (G2)).

Remark 15.5.3. The Schwartz function φ_K satisfying the assumptions in Theorem 15.5.1 exists for a suitable choice of K since we allow arbitrary Drinfeld levels at split places.

Proof. Since φ_K is nonsingular at two places, by (12.3.0.5) we know that only nonsingular T contributes non-trivially to the sum (12.4.0.6). For a nonsingular T, by (12.4.0.7) we know that $\mathrm{Diff}(T, \mathbb{V}) = \{v\}$ for v nonsplit in F. By the assumption on F/F_0, we know that either T is inert or v is archimedean. The result then follows from Theorem 14.5.1 and Theorem 15.3.1 depending on T is inert or v is archimedean. $\square$

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