Solutions to Practice Midterm 1

Problem 1. Evaluate $\iint_D y \, dA$, where $D$ is the region in the first quadrant that is bounded by the lines $y = x$, $y = 2$ and the hyperbola $xy = 1$.

Solution:

$$\iint_D y \, dA = \int_1^2 \int_{1/y}^y y \, dx \, dy = \int_1^2 (y - 1/y) \, dy = \int_1^2 (y^2 - 1)$$

$$= \left[ \frac{y^3}{3} - y \right]_{y=1}^{y=2} = \frac{4}{3}$$

Problem 2. Evaluate the integral $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3} \, dx \, dy$.

Solution:

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3} \, dx \, dy = \int_0^2 \int_0^{x^2} e^{x^3} \, dy \, dx = \int_0^2 x^2 e^{x^3} \, dx = \left[ \frac{e^{x^3}}{3} \right]_{x=0}^{x=2} = \frac{e^8 - 1}{3}$$

Problem 3. Evaluate the integral $\iint_R \cos \left( \frac{y-x}{y+x} \right) \, dA$, where $R$ is the triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, 0)$.

Solution: We use the transformation $u = y - x$, $v = y + x$. Then

$$x = \frac{v-u}{2}, \quad y = \frac{u+v}{2}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right| = -\frac{1}{2},$$

so

$$dA = dx \, dy = \left| -\frac{1}{2} \right| \, dudv = \frac{1}{2} dudv.$$

$$\iint_R \cos \left( \frac{y-x}{y+x} \right) \, dA = \int_0^2 \int_{-v}^v \cos \frac{u}{v} \, dv \, du = \int_0^2 \frac{v}{2} \sin \frac{u}{v} \bigg|_{u=-v}^{u=v}$$

$$= \int_0^2 v \sin 1 \, dv = \frac{v^2}{2} \sin 1 \bigg|_{v=0}^{v=2} = 2 \sin 1$$
Problem 4. Evaluate
\[ \iiint_{E} \frac{1}{x^2 + y^2 + z^2} \, dV, \]
where \( E \) is the region bounded by the spheres with center the origin and radii 1 and 3.

Solution: \( x = \rho \sin \varphi \cos \theta, \quad y = \rho \sin \varphi \sin \theta, \quad z = \rho \cos \varphi \)
\[ dV = dx dy dz = \rho^2 \sin \varphi d\rho d\varphi d\theta \]
\[ \iiint_{E} \frac{1}{x^2 + y^2 + z^2} \, dV = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{1}^{3} \sin \varphi \rho^2 d\rho d\varphi d\theta = \int_{0}^{2\pi} \int_{0}^{\pi} 2 \sin \varphi d\varphi d\theta = \int_{0}^{2\pi} (-2 \cos \varphi) \bigg|_{\varphi=0}^{\varphi=\pi} d\theta = 8\pi \]

Problem 5. Evaluate \( \iiint_{E} xz \, dV \), where \( E \) is bounded by the cylinder \( x^2 + y^2 = 4 \) and the planes \( x = 0, \, y = 0, \, z = 0, \) and \( y = z \) in the first octant.

Solution: \( x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \)
\[ dV = dx dy dz = rdz dr d\theta \]
\[ \iiint_{E} xz \, dV = \int_{0}^{\pi/2} \int_{0}^{2} \int_{0}^{r \sin \theta} r^2 \cos \theta \cdot z \, dz \, dr \, d\theta = \int_{0}^{\pi/2} \int_{0}^{2} \left( \frac{r^2 \cos \theta}{2} \right) z \bigg|_{z=0}^{z=r \sin \theta} dr d\theta = \int_{0}^{\pi/2} \int_{0}^{2} \frac{1}{2} r^4 \sin^2 \theta \cos \theta dr d\theta \]
\[ = \int_{0}^{\pi/2} \left( \frac{1}{10} r^5 \sin^2 \theta \cos \theta \right) \bigg|_{r=0}^{r=2} d\theta = \int_{0}^{\pi/2} \frac{16}{5} \sin^2 \theta \cos \theta d\theta = \left( \frac{16}{15} \sin^3 \theta \right) \bigg|_{0}^{\pi/2} = \frac{16}{15} \]

Problem 6. Find the volume of the solid \( E \) that lies above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 4. \)
**Solution:**  
\[ x = \rho \sin \varphi \cos \theta, \quad y = \rho \sin \varphi \sin \theta, \quad z = \rho \cos \varphi, \]

\[ dV = dxdydz = \rho^2 \sin \varphi d\rho d\varphi d\theta \]

\[
\text{volume}(E) = \iiint_W 1dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin \varphi d\rho d\varphi d\theta \\
= \int_0^{2\pi} 1d\theta \int_0^{\pi/4} \sin \varphi d\varphi \int_0^2 \rho^2 d\rho \\
= 2\pi \cdot (\cos \varphi)|_{\varphi=\pi/4}^{\varphi=0} \cdot \left. \frac{\rho^3}{3} \right|_{\rho=0}^{\rho=2} = \frac{16\pi}{3} \left(1 - \frac{\sqrt{2}}{2}\right).
\]

**Problem 7.** Find the area of the region inside the circle \( r = 2 \cos \theta \) and outside the circle \( r = \sqrt{2} \).

**Solution:** Note that the polar curves \( r = 2 \cos \theta \) and \( r = \sqrt{2} \) intersect when \( 2 \cos \theta = \sqrt{2} \), or equivalently, \( \theta = \pm \pi/4 \). The area is given by

\[
\int_{-\pi/4}^{\pi/4} \int_{\sqrt{2}}^{2 \cos \theta} r dr d\theta = \int_{-\pi/4}^{\pi/4} \frac{r^2}{2} \bigg|_{r=\sqrt{2}}^{r=2 \cos \theta} d\theta = \int_{-\pi/4}^{\pi/4} (2 \cos^2 \theta - 1) d\theta \\
= \int_{-\pi/4}^{\pi/4} \cos(2\theta) d\theta = \frac{1}{2} \sin(2\theta) \bigg|_{-\pi/4}^{\pi/4} = 1.
\]