Math V1202. Calculus IV, Section 004, Spring 2007

Solutions to Midterm 1

Problem 1. Evaluate $\iint_D (x+y)dA$, where D is the triangular region with vertices $(0,0)$, $(-1,1)$, $(2,1)$.

Solution:

$$\begin{aligned}\iint_D (x+y)dA &= \int_0^1 \int_{-y}^{2y} (x+y)dxdy = \int_0^1 \left(\frac{x^2}{2}+xy\right)\Big|_{x=-y}^{x=2y} \\ &= \int_0^1 \frac{9y^2}{2}dy = \frac{3y^3}{2}\Big|_{y=0}^{y=1} = \frac{3}{2}.\end{aligned}$$

Problem 2. Evaluate the iterated integral

$$\int_0^2 \int_{x^2}^4 x\sin(y^2)\, dy\, dx$$

by reversing the order of integration.

Solution:

$$\int_0^2 \int_{x^2}^4 x\sin(y^2)dydx = \int_0^4 \int_0^{\sqrt{y}} x\sin(y^2)dxdy = \int_0^4 \frac{x^2}{2}\sin(y^2)\Big|_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^4 \frac{y}{2}\sin(y^2)dy = \frac{-1}{4}\cos(y^2)\Big|_{y=0}^{y=4} = \frac{1}{4}(1-\cos 16)$$

Problem 3. Evaluate the integral

$$\iint_R e^{4x^2+9y^2} dA,$$

where R is the region bounded by the ellipse $4x^2+9y^2=1$.

Solution: We use the transformation $u=2x$, $v=3y$. Then

$$x=\frac{u}{2},\quad y=\frac{v}{3},\quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 0 \\ 0 & 1/3 \end{vmatrix} = \frac{1}{6},$$

so $dA = dxdy = \frac{1}{6}dudv$. The region R is transformed to S bounded by the circle $u^2+v^2=1$. Then we use polar coodinates $u=r\cos\theta$, $v=r\sin\theta$, $dudv = rdrd\theta$.

$$\begin{aligned}\iint_R e^{4x^2+9y^2}dA &= \iint_S e^{u^2+v^2}\frac{1}{6}dudv = \int_0^{2\pi}\int_0^1 e^{r^2}\frac{1}{6}rdrd\theta \\ &= \int_0^{2\pi} \frac{e^{r^2}}{12}\Big|_{r=0}^{r=1} d\theta = \int_0^{2\pi}\frac{e-1}{12} = \frac{\pi}{6}(e-1)\end{aligned}$$

Problem 4. Evaluate $\iiint_E z\, dV$, where E is the solid that lies above the paraboloid $z = x^2 + y^2$ and below the half cone $z = \sqrt{x^2 + y^2}$.

Solution: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dV = rdrd\theta dz$.

$$\iiint_E z\, dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} zrdzdrd\theta = \int_0^{2\pi}\int_0^1 \left.\frac{z^2 r}{2}\right|_{z=r^2}^{z=r} drd\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{r^3}{2} - \frac{r^5}{2}\right) drd\theta = \int_0^{2\pi}\left.\left(\frac{r^4}{8} - \frac{r^6}{12}\right)\right|_{r=0}^{r=1} d\theta = \int_0^{2\pi}\frac{1}{24}d\theta = \frac{\pi}{12}$$

Problem 5. Evaluate $\iiint_E y\, dV$, where E is enclosed by the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Solution:

$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi, \quad dV = \rho^2\sin\varphi d\rho d\theta d\varphi$$

$$\begin{aligned}
\iiint_E y\, dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho\sin\varphi\sin\theta\rho^2\sin\varphi d\rho d\theta d\varphi \\
&= \int_0^{\pi/2}\sin^2\varphi d\varphi \int_0^{\pi/2}\sin\theta \int_0^1 \rho^3 d\rho \\
&= \int_0^{\pi/2}\frac{1-\cos(2\varphi)}{2}d\varphi \cdot \left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi/2}\right)\cdot\left(\frac{\rho^4}{4}\Big|_{\rho=0}^{\rho=1}\right) \\
&= \left(\frac{\varphi}{2} - \frac{\sin(2\varphi)}{4}\Big|_{\varphi=0}^{\varphi=\pi/2}\right)\cdot 1\cdot\frac{1}{4} = \frac{\pi}{16}
\end{aligned}$$

Problem 6. Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the cylinder $x^2 + y^2 = 1$.

Solution: Let E be the solid described above.

$$x = r\cos\theta, \quad y = r\cos\theta, \quad z = z, \quad dV = rdzdrd\theta$$

$$\begin{aligned}
\text{volume}(E) &= \iiint_E 1dV = \int_0^{2\pi}\int_1^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} rdzdrd\theta \\
&= \int_0^{2pi}\int_1^e 2r\sqrt{9-r^2}drd\theta = \int_0^{2\pi}\frac{-2}{3}(9-r^2)^{3/2}\Big|_{r=1}^{r=3}d\theta \\
&= \int_0^{2\pi}\frac{32\sqrt{2}}{3}d\theta = \frac{64\pi\sqrt{2}}{3}
\end{aligned}$$

Problem 7. Let E be the solid that lies inside both cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.

(a) (6%) Express the volume of E as an iterated integral in the order $dx\,dy\,dz$.

Solution:

$$\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 1dxdydz$$

(b) (6%) Evaluate the iterated integral in (a).

Solution:

$$\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 1dxdydz = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 2\sqrt{1-z^2}dydz$$

$$= \int_{-1}^{1} 4(1-z^2)dz = (4z - \frac{4z^3}{3})\Big|_{z=-1}^{z=1} = \frac{16}{3}$$