Math V1202. Calculus IV, Section 004, Spring 2007

Solutions to Practice Final Exam

Problem 1 Consider the integral

$$\int_1^2 \int_x^{x^2} 12x \; dy \; dx + \int_2^4 \int_x^4 12x \; dy \; dx$$

(a) Sketch the region of integration.

Solution: See Figure 1.

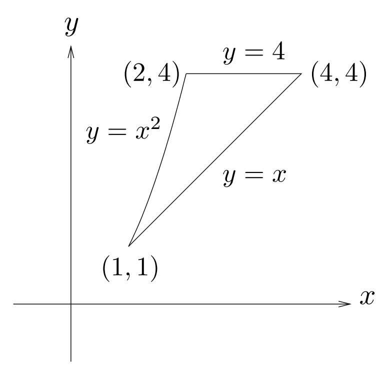

Figure 1: $\{(x, y) \mid 1 \leq y \leq 4, \sqrt{y} \leq x \leq y\}$

(b) Reverse the order of integration and evaluate the integral that you get.
Solution:

$$\int_1^2 \int_x^{x^2} 12x \; dy \; dx + \int_2^4 \int_x^4 12x \; dy \; dx = \int_1^4 \int_{\sqrt{y}}^{y} 12x dx dy$$

$$= \int_1^4 6x^2 \Big|_{x=\sqrt{y}}^{x=y} dy = \int_1^4 (6y^2 - 6y) dy = (2y^3 - 3y^2)\Big|_{y=1}^{y=4} = 81$$

Problem 2 Consider the transformation of $\mathbb{R}^2$ defined by the equations given by $x = u/v$, $y = v$.

(a) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformaion.

Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

(b) Let R be the region in the first quadrant bounded by the lines $y = x$, $y = 2x$ and the hyperbolas $xy = 1$, $xy = 2$. Sketch the region S in the uv-plane corresponding to R.

Solution: The lines $y = x$ and $y = 2x$ in the xy-plane correspond to $v = u/v$, $v = 2u/v$ in the uv-plane, respectively. The part in the first quadrant can be rewritten as $v = \sqrt{u}$ and $v = \sqrt{2u}$, respectively. The hyperbolas $xy = 1$, $xy = 2$ in the xy-plane correspond to the lines $u = 1$, $u = 2$ in the uv-plane, respectively.

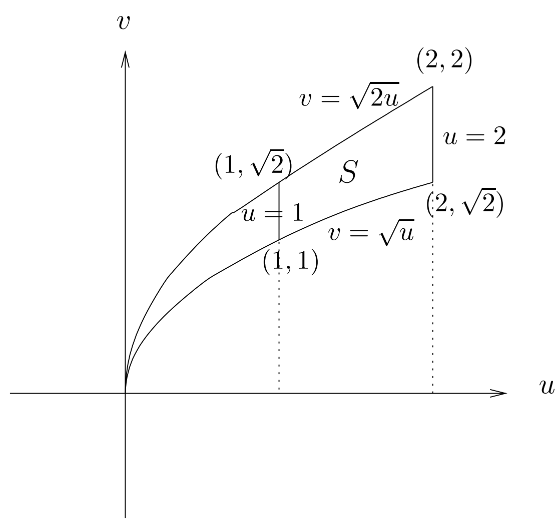

Figure 2: $S = \{(u,v) \in \mathbb{R} \mid 1 \le u \le 2, \sqrt{u} \le v \le \sqrt{2u}\}$

(c) Evaluate $\iint_R y^4 dA$.

Solution:

$$\iint_R y^4 dA = \iint_S v^4 \left|\frac{1}{v}\right| dudv = \int_1^2 \int_{\sqrt{u}}^{\sqrt{2u}} v^3 dvdu = \int_1^2 \frac{v^4}{4}\Big|_{v=\sqrt{u}}^{v=\sqrt{2u}} dv$$

$$= \int_1^2 \frac{3u^2}{4} du = \frac{u^3}{4}\Big|_{u=1}^{u=2} = \frac{7}{4}$$

Problem 3 Let S be the boundary of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$, with outward orientation.

(a) Find the surface area of S. Note that the surface S consists of a portion of the paraboloid $z = x^2 + y^2$ and a portion of the plane $z = 4$.

Solution: Let S_1 be the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 4$, and let S_2 be the disk $x^2 + y^2 \leq 4$, $z = 4$. Then S is the union of S_1 and S_2, and

$$\text{Area}(S) = \text{Area}(S_1) + \text{Area}(S_2)$$

where $\text{Area}(S_2) = 4\pi$ since S_2 is a disk of radius 2. To find $\text{Area}(S_1)$, consider a vector equation of S_1 given by

$$\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle, \quad (x, y) \in D,$$

where $g(x, y) = x^2 + y^2$ and $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$. We have

$$\begin{aligned} \mathbf{r}_x \times \mathbf{r}_y &= \langle -g_x, -g_y, 1\rangle = \langle -2x, -2y, 1\rangle \\ |\mathbf{r}_x \times \mathbf{r}_y| &= \sqrt{4x^2 + 4y^2 + 1} \end{aligned}$$

$$\text{Area}(S_1) = \iint_D |\mathbf{r}_x \times \mathbf{r}_y| dxdy = \iint_D \sqrt{4x^2 + 4y^2 + 1} dxdy$$

We use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$.

$$\iint_D \sqrt{4x^2 + 4y^2 + 1} dxdy = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} rdrd\theta$$

Let $u = 4r^2 + 1$, $du = 8rdr$. Then

$$\int_0^2 \sqrt{4r^2 + 1} rdr = \int_1^{\sqrt{17}} u^{1/2} \frac{du}{8} = \frac{u^{3/2}}{12}\Big|_{u=1}^{u=\sqrt{17}} = \frac{17\sqrt{17} - 1}{12}$$

So

$$\int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} rdrd\theta = \int_0^{2\pi} \frac{17\sqrt{17} - 1}{12} d\theta = \frac{\pi}{6}(17\sqrt{17} - 1)$$

$$\text{Area}(S) = \text{Area}(S_1) + \text{Area}(S_2) = \frac{\pi}{6}(17\sqrt{17} - 1) + 4\pi = \frac{\pi}{6}(17\sqrt{17} + 23)$$

(b) Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = (x + y^2z^2)\mathbf{i} + (y + z^2x^2)\mathbf{j} + (z + x^2y^2)\mathbf{k}$.

Solution: $S = \partial E$, where E is the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. By the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F} dV$$

where

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x + y^2z^2) + \frac{\partial}{\partial y}(y + z^2x^2) + \frac{\partial}{\partial z}(z + x^2y^2) = 1 + 1 + 1 = 3.$$

We use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dV = rdzdrd\theta$.

$$\iint_E \operatorname{div}\mathbf{F} dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 3rdzdrd\theta = \int_0^{2\pi}\int_0^2 3r(4 - r^2)drd\theta$$
$$= \int_0^{2\pi}(6r^2 - \frac{3}{4}r^4)\Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi} 12d\theta = 24\pi$$

So $\iint_S \mathbf{F}\cdot d\mathbf{S} = 24\pi$.

Problem 4 Let

$$\mathbf{F} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}.$$

Note that $\mathbf{F}$ is defined on $\{(x, y) \in \mathbb{R} \mid (x, y) \neq (0, 0)\}$.

(a) Evaluate $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$, where C_1 is the circle $x^2 + y^2 = 1$, oriented counterclockwise.

Solution: A vector equation of C_1 is given by

$$\mathbf{r}(t) = \langle \cos t, \sin t\rangle, \quad 0 \le t \le 2\pi$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= \mathbf{F}(\cos t, \sin t) = \langle -\sin t, \cos t\rangle \\ \mathbf{r}'(t) &= \langle -\sin t, \cos t\rangle\end{aligned}$$

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^{2\pi} \langle -\cos t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle dt \\ &= \int_0^{2\pi} 1dt = 2\pi\end{aligned}$$

(b) Compute curl $\mathbf{F}$.

Solution:

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y/(x^2+y^2) & x/(x^2+y^2) & 0 \end{vmatrix} \\
&= \mathbf{k}\left(\frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) + \frac{\partial}{\partial y}(\frac{y}{x^2+y^2})\right) \\
&= \mathbf{k}\left(\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2} + \frac{(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}\right) \\
&= \mathbf{0}.
\end{aligned}$$

(c) Use Green's Theorem to evaluate $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, where C_2 is the circle $(x-2)^2+(y-2)^2=1$, oriented counterclockwise.

Solution: $C_2 = \partial D$, where D is the disk $(x-2)^2+(y-2)^2 \leq 1$. Note that D does not contain the origin $(0,0)$, and the components $-x/(x^2+y^2)$, $y/(x^2+y^2)$ of $\mathbf{F}$ are defined and has continuous partial derivatives on D. By the vector form of Green's theorem,

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \iint_D \operatorname{curl}\mathbf{F}\cdot\mathbf{k}dA = \iint_D 0dA = 0.$$

(d) Is $\mathbf{F}$ conservative?

Solution: $\mathbf{F}$ is not conservative because the line integral of $\mathbf{F}$ along the simple closed curve C_1 is $2\pi \neq 0$.

Problem 5 Let E be a solid in the first octant bounded by the cone $z^2 = x^2+y^2$ and the plane $z=1$. Evaluate $\iiint_E xyz^2dV$.

Solution: We use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$dV = dxdydz = rdrd\theta dz.$

$$\iiint_W xyz^2 dV = \int_0^{\pi/2}\int_0^1\int_r^1 r\cos\theta r\sin\theta z^2 r dz dr d\theta$$

$$= \int_0^{\pi/2}\int_0^1 \frac{\sin(2\theta)}{6} r^3 z^3\Big|_{z=r}^{z=1} drd\theta = \int_0^{\pi/2}\int_0^1 \frac{\sin(2\theta)}{6}(r^3 - r^6)drd\theta$$

$$= \int_0^{\pi/2} \frac{\sin(2\theta)}{6}\left(\frac{r^4}{4} - \frac{r^7}{7}\right)\Big|_{r=0}^{r=1} d\theta = \int_0^{\pi/2}\frac{\sin(2\theta)}{56}d\theta$$

$$= \frac{-\cos(2\theta)}{112}\Big|_{\theta=0}^{\theta=\pi/2} = \frac{1}{56}$$

Problem 6 Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F} = e^{y^2}\mathbf{i} + (y + \sin(z^2))\mathbf{j} + (z-1)\mathbf{k},$$

and S is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented upward. Note that the surface S does NOT include the bottom of the hemisphere.

Solution : Consider the solid $E = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1, z \geq 0\}$. Its boundary ∂E is the union of S and the disk

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, z = 0\},$$

where S_1 is oriented downward. By the Divergence Theorem

$$\iint_S \mathbf{F}\cdot d\mathbf{S} + \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iiint_W \operatorname{div}\mathbf{F} dV$$

where

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(e^{y^2}) + \frac{\partial}{\partial y}(y + \sin(z^2)) + \frac{\partial}{\partial z}(z-1) = 2$$

$$\iiint_E \operatorname{div}\mathbf{F} dV = \iiint_E 2dV = 2\text{volume}(E) = \text{volume}(B) = \frac{4\pi}{3}$$

where B is the unit ball $x^2 + y^2 + z^2 \leq 1$.

The downward unit normal of S_1 is the constant vector $-\mathbf{k} = \langle 0, 0, -1\rangle$, so

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot(-\mathbf{k})dS = \iint_{S_1}(-z+1)dS = \iint_{S_1} 1dS = \text{Area}(S_1) = \pi$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}dV - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$$

Problem 7 Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where

$$\mathbf{F} = x^2y\mathbf{i} - xy^2\mathbf{j} + z^3\mathbf{k},$$

and C is the curve of intersection of the plane $3x + 2y + z = 6$ and the cylinder $x^2 + y^2 = 4$, oriented clockwise when viewed from above.

Solution: Let S be the part of the plane $3x + 2y + z = 6$ that lies inside the cylinder $x^2 + y^2 = 1$, oriented downward. Then $C = \partial S$. By Stokes' Theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

where

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -xy^2 & z^3 \end{vmatrix} = (-y^2 - x^2)\mathbf{k}.$$

A vector equation of S is given by

$$\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle, \quad (x,y)\in D$$

where $g(x,y) = 6 - 3x - 2y$ and $D = \{(x,y)\in\mathbb{R}^2 \mid x^2 + y^2 \le 4\}$. We have

$$\begin{aligned} \operatorname{curl}\mathbf{F}(\mathbf{r}(x,y)) &= \langle 0, 0, -x^2 - y^2\rangle \\ \mathbf{r}_x \times \mathbf{r}_y &= \langle -g_x, -g_y, 1\rangle = \langle 3, 2, 1\rangle \end{aligned}$$

$\mathbf{r}_x \times \mathbf{r}_y$ is upward, so

$$\iint_S \operatorname{curl}\mathbf{F}\cdot dS = \iint_D \operatorname{curl}\mathbf{F}(\mathbf{r}(x,y))\cdot(-\mathbf{r}_x\times\mathbf{r}_y)dxdy$$
$$= \iint_D \langle 0,0,-x^2-y^2\rangle\cdot\langle -3,-2,-1\rangle dxdy = \iint_D (x^2+y^2)dxdy$$

We use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$.

$$\iint_D (x^2+y^2)dxdy = \int_0^{2\pi}\int_0^2 r^3 drd\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi} 4d\theta = 8\pi.$$

Problem 8 Use Stokes' theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = (\sin(y+z) - yx^2 - \frac{y^3}{3})\,\mathbf{i} + x\cos(y+z)\,\mathbf{j} + \cos(2y)\,\mathbf{k},$$

and S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

Solution: Let S_1 be the bottom of the cube, oriented by the upward unit normal $\mathbf{k}$, and let C be the boundary of S_1 (with the positive orientation). Then $\partial S = C = \partial S_1$. By Stokes's theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{k} dS$$

$$\begin{aligned}
\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(y+z) - yx^2 - \frac{y^3}{3} & x\cos(y+z) & \cos(2y) \end{vmatrix} \\
\operatorname{curl} \mathbf{F} \cdot \mathbf{k} &= \frac{\partial}{\partial x}(x\cos(y+z)) - \frac{\partial}{\partial y}(\sin(y+z) - yx^2 - \frac{y^3}{3}) \\
&= \cos(y+z) - \big(\cos(y+z) - x^2 - y^2\big) = x^2 + y^2
\end{aligned}$$

$$\begin{aligned}
&\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{k} dS = \iint_{S_1} (x^2+y^2) dS = \int_{-1}^{1}\int_{-1}^{1} (x^2+y^2) dx dy \\
&= \int_{-1}^{1} (\frac{x^3}{3} + xy^2)\Big|_{x=-1}^{x=1} dy = \int_{-1}^{1} (\frac{2}{3} + 2y^2) dy = \left(\frac{2y}{3} + \frac{2y^3}{3}\right)\Big|_{y=-1}^{y=1} = \frac{8}{3}
\end{aligned}$$

$$\text{So } \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \frac{8}{3}.$$

Problem 9 Write in the form of $a + bi$:

(a) Find all the fourth roots of -4.

Solution: $-4 = 4(\cos \pi + i \sin \pi)$, so the fourth roots of -4 are

$$4^{1/4}\left(\cos(\frac{\pi + 2k\pi}{4}) + i\sin(\frac{\pi + 2k\pi}{4})\right), \quad k = 0, 1, 2, 3$$

$$\begin{aligned}
k=0: &\quad \sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}) = 1+i \\
k=1: &\quad \sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}) = -1+i \\
k=2: &\quad \sqrt{2}(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}) = -1-i \\
k=3: &\quad \sqrt{2}(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}) = 1-i
\end{aligned}$$

The fourth roots of -4 are $1+i$, $-1+i$, $-1-i$, $1-i$.

(b) Evaluate $(1-i)^{10}$.

Solution:

$$\begin{aligned}
1-i &= \sqrt{2}(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i) = \sqrt{2}(\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})) \\
(1-i)^{10} &= \sqrt{2}^{10}(\cos(-\frac{10\pi}{4})+i\sin(-\frac{10\pi}{4}) = 2^5(-i) = -32i
\end{aligned}$$

(c) Find all the possible values of $(-2)^i$.

Solution: $-2 = 2e^{\pi i}$, so

$$\ln(-2) = \ln 2 + i(\pi + 2k\pi)$$

where k is any integer.

$$\begin{aligned}
(-2)^i &= e^{i\ln(-2)} = e^{i\ln 2 - \pi(2k+1)} = e^{-\pi(2k+1)}(\cos(\ln 2)+i\sin(\ln 2)) \\
&= e^{-\pi(2k+1)}\cos(\ln 2) + e^{-\pi(2k+1)}\sin(\ln 2)i
\end{aligned}$$

where k is any integer.

Problem 10 Let $f(z) = e^{iz}$.

(a) Write $f(z)$ in the form $u+iv$.

Solution:

$$\begin{aligned}
f(z) &= e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}(\cos x + i\sin x) \\
&= e^{-y}\cos x + ie^{-y}\sin x
\end{aligned}$$

(b) Is $f(z)$ analytic?

Solution: $f(z) = u(x, y) + iv(x, y)$, where

$$\begin{aligned} &u(x,y) = e^{-y}\cos x, \quad v(x,y) = e^{-y}\sin x, \\ &\frac{\partial u}{\partial x} = -e^{-y}\sin x, \quad \frac{\partial u}{\partial y} = -e^{-y}\cos x, \\ &\frac{\partial v}{\partial x} = e^{-y}\cos x, \quad \frac{\partial v}{\partial y} = -e^{-y}\sin x. \end{aligned}$$

u, v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

so $f(z)$ is analytic.

Problem 11 Let $f(z)$ be an analytic function which only takes real values, i.e., $\mathrm{Im} f(z) = 0$. Show that $f(z)$ is a constant function. (Hint: Use Cauchy-Riemann equations.)

Solution: Let $u = \mathrm{Re} f$, so that $f(x + iy) = u(x, y)$. It suffices to show that $u(x, y)$ is a constant function. $f(z)$ is analytic, so u and $v = 0$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0.$$

So $u(x, y)$ is a constant function.