Comparison Lemma

Fix genus g and $\beta \in H_2(X;\mathbb{Z})$

Let $\mathcal{M}_n = \overline{\mathcal{M}}_{g,n}(X,\beta)$

$\pi : \mathcal{U}_n = \mathcal{M}_{n+1} \to \mathcal{M}_n$ universal curve

$$\begin{tikzcd}
\mathcal{U}_{n+1} \arrow[r, hook, "c"] \arrow[d, "\tilde{\pi}"] & \mathcal{U}_n \times_{\mathcal{M}_n} \mathcal{U}_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{U}_n \arrow[d, "\pi"] \\
\mathcal{M}_{n+1} \arrow[u, bend left, "\tilde{s}_i"] \arrow[r, phantom, "="] & \mathcal{U}_n \arrow[r, "\pi"] \arrow[u, bend left, "\pi^* s_i"] \arrow[u, bend right, "\Delta"'] & \mathcal{M}_n \arrow[u, bend right, "s_i"']
\end{tikzcd}$$

$\tilde{\mathcal{L}}_i = \hat{s}_i^* \omega_{\tilde{\pi}}$ $\qquad\qquad$ $\mathcal{L}_i = s_i^* \omega_\pi$

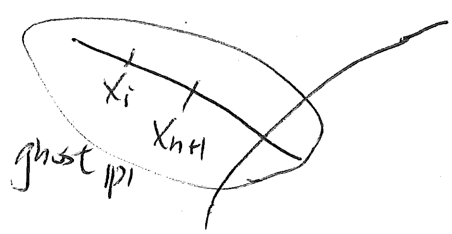
$\hat{\psi}_i = c_1(\tilde{\mathcal{L}}_i) \in A^1(\mathcal{M}_{n+1};\mathbb{Q})$ $\qquad$ $\psi_i = c_1(\mathcal{L}_i) \in A^1(\mathcal{M}_n,\mathbb{Q})$

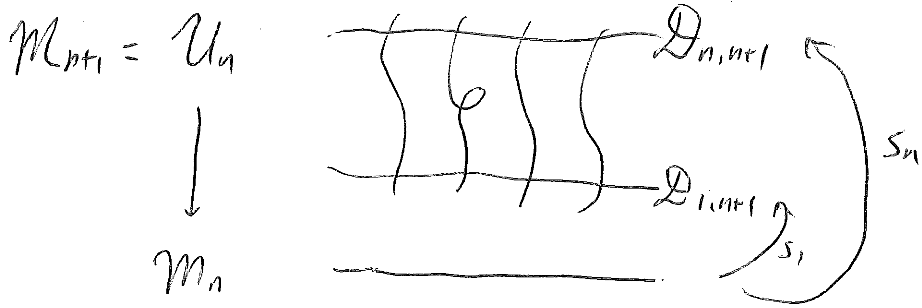
$i = 1, \ldots, n+1$ $\qquad\qquad$ $i = 1, \ldots, n$

$c \circ \hat{s}_i = \pi^* s_i \quad i = 1 \to n$

$c \circ \hat{s}_{n+1} = \Delta$

Let $\mathcal{D}_{i,n+1} \subset \mathcal{M}_{n+1}$ divisor





Lemma (Comparison Lemma)

(a) $i = 1, \dots, n$ $\quad \tilde{\mathcal{L}}_i = \pi^* \mathcal{L}_i \otimes \mathcal{O}_{\mathcal{M}_{n+1}}(\mathcal{D}_{i,n+1})$

(b) $\tilde{\mathcal{L}}_{n+1} = \omega_\pi \otimes \mathcal{O}\left(\sum_{i=1}^{n} \mathcal{D}_{i,n+1}\right)$

Proof (a) $\tilde{\mathcal{L}}_i \big|_{\mathcal{M}_{n+1} - \mathcal{D}_{i,n+1}} \cong \pi^* \mathcal{L}_i \big|_{\mathcal{M}_{n+1} - \mathcal{D}_{i,n+1}}$

$$\tilde{\mathcal{L}}_i = \pi^* \mathcal{L}_i \otimes \mathcal{O}_{\mathcal{M}_{n+1}}(m \mathcal{D}_{i,n+1})$$

Claim: $m = 1$. We consider

$$\begin{tikzcd} C \coloneqq \pi^{-1}(\xi) \arrow[r, "\Phi"] \arrow[d] \arrow[dr, "\phi"] & \mathcal{U}_n = \mathcal{M}_{n+1} \arrow[d, "\pi"] \\ \xi & \mathcal{M}_n \end{tikzcd}$$

$\xi = [f : (C, x_1, \dots, x_n) \to X]$

$\phi \coloneqq \pi \circ \Phi : C \to \mathcal{M}_n$

constant map to ξ

$\Phi^* \pi^* \mathcal{L}_i = \phi^* \mathcal{L}_i = \mathcal{O}_C$

$\Phi^* \mathcal{D}_{i,n+1} = x_i$

C is smooth

It remains to show

$\deg \phi^* \mathcal{L}_i = 1$

$$Bl_{\{(x_1,x_1),\dots,(x_n,x_n)\}}(C\times C)$$
$$\parallel$$

$$\begin{tikzcd}
S := \phi^*\mathcal{U}_{n+1} \arrow[rr, "\phi^* c"] \arrow[dr, "\widetilde{\phi^*\pi}"] & & \phi^*\mathcal{U}_n = C\times C \arrow[dl, "\phi^*\pi"'] \\
& C \arrow[ul, bend left, "\widetilde{\phi^* s_i}"] \arrow[ur, bend right, "\phi^* s_i"'] &
\end{tikzcd}$$

$i=1,\dots,n+1$ $\qquad\qquad$ $i=1,\dots,n$

$\phi^*\pi : C\times C \to C$
$(y_1,y_2) \mapsto y_1$

$\phi^* s_i : C \to C\times C$
$y \mapsto (y, x_i)$

$i=1,\dots,n$

$D_i = C\times\{x_i\} \subset C\times C \qquad D_i \cdot D_i = 0$ (in $C\times C$)

$E_i = (\phi^* c)^{-1}(x_i,x_i) \subset S$ exceptional divisor

$E_i \cong \mathbb{P}^1 \qquad E_i\cdot E_i = -1$ (in S)

$\hat{D}_i \subset S$ proper transform of D_i

$\hat{D}_{n+1} \subset S$ proper transform of $\Delta \subset C\times C$ (Δ: diagonal)

$i=1,\dots,n$

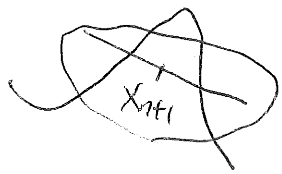
$$\tilde{\phi}^*\hat{\mathcal{L}}_i = (\widetilde{\phi^* s_i})^*\omega_{\tilde{\pi}} = (\widetilde{\phi^* s_i})^*\mathcal{O}_S(-\hat{D}_i)$$

$$\deg \tilde{\phi}^*\hat{\mathcal{L}}_i = -\hat{D}_i\cdot\tilde{D}_i = -(\pi^*D_i - E_i)\cdot(\pi^*D_i - E_i)$$

where $\pi^*D_i\cdot\pi^*D_i = \pi^*D_i\cdot E_i = 0, \quad E_i\cdot E_i = -1$

So $\deg \tilde{\Phi}^* \tilde{\mathcal{L}}_i = -1$

(b) $\delta \subset \mathcal{M}_{n+1}$ codim 2



ghost $\mathbb{P}^1$

$$\Sigma := \bigsqcup_{i=1}^{n} \mathcal{D}_{i,n+1} \sqcup \delta$$

$$\mathcal{L}_{n+1}\big|_{\mathcal{M}_{n+1}\setminus\Sigma} \cong \omega_\pi\big|_{\mathcal{M}_{n+1}\setminus\Sigma}$$

$$\mathcal{L}_{n+1} \cong \omega_\pi \otimes \mathcal{O}_{\mathcal{M}_{n+1}}\Big(\sum_{i=1}^{n} \mathcal{D}_{i,n+1}\Big)$$

$$\tilde{\Phi}^* \omega_\pi = \omega_C$$

$$\tilde{\Phi}^* \mathcal{O}_{\mathcal{M}_{n+1}}(\mathcal{D}_{i,n+1}) = \mathcal{O}_C(x_i)$$

$$\tilde{\Phi}^* \tilde{\mathcal{L}}_{n+1} = \tilde{\Phi}^* \tilde{s}_i^* \mathcal{O}_{\tilde{S}}(-\tilde{D}_{n+1})$$

$$\deg \tilde{\Phi}^* \tilde{\mathcal{L}}_{n+1} = -\tilde{D}_{n+1} \cdot \tilde{D}_{n+1} = -\Big(\pi^*\Delta - \sum_{i=1}^{n} E_i\Big)\cdot\Big(\pi^*\Delta - \sum_{i=1}^{n} E_i\Big)$$

$$\Delta\cdot\Delta = 2-2g \qquad \pi^*\Delta\cdot E_i = 0 \qquad E_i\cdot E_j = -\delta_{ij}$$

$$\deg \tilde{\Phi}^* \tilde{\mathcal{L}}_{n+1} = 2g-2+n$$

Corollary (of Comparison Lemma (a))

$$\prod_{i=1}^{n} \tilde{\psi}_i^{a_i} = \pi^*\Big(\prod_{i=1}^{n} \psi_i^{a_i}\Big) + \sum_{i=1}^{n} D_{i,n+1}\, \pi^*\big(\psi_1^{a_1}\cdots \psi_{i-1}^{a_{i-1}}\, \psi_i^{a_i-1}\, \psi_{i+1}^{a_{i+1}} \cdots \psi_n^{a_n}\big)$$

where $\tilde{\psi}_i = c_1(\tilde{\mathcal{L}}_i) \in A^1(M_{n+1};\mathbb{Q})$

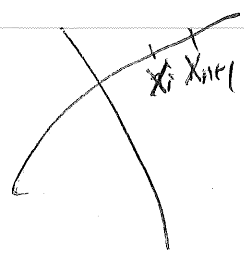
$\psi_i = c_1(\mathcal{L}_i) \in A^1(M_n;\mathbb{Q})$

$D_{i,n+1} = c_1(\mathcal{O}_{M_{n+1}}(\mathcal{D}_{i,n+1})) \in A^1(M_{n+1};\mathbb{Q})$

Proof $\tilde{\psi}_i^{a_i} - \pi^*\psi_i^{a_i} = (\tilde{\psi}_i - \pi^*\psi_i)\Big(\sum_{k=0}^{a_i-1} \tilde{\psi}_i^{k}\, (\pi^*\psi_i)^{a_i-1-k}\Big)$

where $\tilde{\psi}_i - \pi^*\psi_i = D_{i,n+1}$ by Comparison Lemma (a)

$\tilde{\mathcal{L}}_i\big|_{\mathcal{D}_{i,n+1}} \cong \mathcal{O}_{\mathcal{D}_{i,n+1}} \Rightarrow D_{i,n+1}\cdot \tilde{\psi}_i = 0$



$$\tilde{\psi}_i^{a_i} - \pi^*\psi_i^{a_i} = D_{i,n+1}\cdot \pi^*(\psi_i^{a_i-1})$$

$$\tilde{\psi}_i^{a_i} = \pi^*(\psi_i^{a_i-1})\,(\pi^*\psi_i + D_{i,n+1})$$

$$\prod_{i=1}^{n} \tilde{\psi}_i^{a_i} = \prod_{i=1}^{n}(\pi^*\psi_i^{a_i-1}) \cdot \prod_{i=1}^{n}(\pi^*\psi_i - D_{i,n+1})$$

$D_{i,n+1}\cdot D_{j,n+1} = 0$ for $i \neq j$

$$\prod_{i=1}^{n} \tilde{\psi}_i^{a_i} = \pi^*\Big(\prod_{i=1}^{n}\psi_i^{a_i-1}\Big)\cdot\Big(\pi^*(\psi_1\cdots\psi_n) + \sum_{i=1}^{n} D_{i,n+1}\,\pi^*(\psi_1\cdots\psi_{i-1}\psi_{i+1}\cdots\psi_n)\Big)$$

$\square$

<u>String Equation</u> $\gamma_1, \dots, \gamma_n \in H^*(X;\mathbb{Q})$

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n)\, \tau_0(1) \rangle^X_{g,n+1,\beta}$$

$$= \sum_{i=1}^{n} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_i-1}(\gamma_i) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta}$$

<u>Proof</u>

$$\begin{tikzcd} \mathcal{M}_{n+1} \arrow[dr, "\widetilde{ev}_i"] \arrow[d, "\pi"'] & \\ \mathcal{M}_n \arrow[r, "ev_i"'] & X \end{tikzcd}$$

$i = 1, \dots, n$

$$\widetilde{ev}_i^* \gamma_i = \pi^* ev_i^* \gamma_i$$

$\pi^* : A_k(\mathcal{M}_n) \to A_{k+1}(\mathcal{M}_{n+1})$ flat pullback

$$\pi^*[\mathcal{M}_n]^{vir} = [\mathcal{M}_{n+1}]^{vir}$$

It suffices to prove

$$\boxed{\pi_*(\tilde\psi_1^{a_1} \cdots \tilde\psi_n^{a_n}) = \sum_{i=1}^{n} \psi_1^{a_1} \cdots \psi_{i-1}^{a_{i-1}} \psi_i^{a_i-1} \psi_{i+1}^{a_{i+1}} \cdots \psi_n^{a_n}}$$

where $\pi_* : A^\bullet(\mathcal{M}_{n+1}) \to A^{\bullet-1}(\mathcal{M}_n)$

$$\pi_*((\pi^*\alpha)\beta) = \alpha\, \pi_*\beta$$

for all $\alpha \in A^*(\mathcal{M}_n)$, $\beta \in A^*(\mathcal{M}_{n+1})$

By Corollary on P.117

$$\prod_{i=1}^{n} \tilde\psi_i^{a_i} = \pi^*\left(\prod_{i=1}^{n} \psi_i^{a_i}\right) + \sum_{i=1}^{n} D_{i,n+1}\, \pi^*\left(\psi_1^{a_1} \cdots \psi_{i-1}^{a_{i-1}} \psi_i^{a_i-1} \psi_{i+1}^{a_{i+1}} \cdots \psi_n^{a_n}\right)$$

$$\pi_*\left(\prod_{i=1}^n \hat{\psi}_i^{a_i}\right) = \prod_{i=1}^n \psi_i^{a_i}\, \pi_* 1 + \sum_{i=1}^n \psi_1^{a_1} \cdots \psi_{i-1}^{a_{i-1}} \psi_i^{a_i - 1} \psi_{i+1}^{a_{i+1}} \cdots \psi_n^{a_n}\, \pi_* D_{i,n+1}$$

where $\pi_* 1 = 0$, $\pi_* D_{i,n+1} = 1$

Dilaton Equation $\gamma_1 \ldots \gamma_n \in H^*(X;\mathbb{Q})$

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n)\, \tau_1(1) \rangle^X_{g,n+1,\beta}$$

$$= (2g-2+n) \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta}$$

Proof It suffices to prove

$$\boxed{\pi_*\left(\hat{\psi}_1^{a_1} \cdots \hat{\psi}_n^{a_n} \hat{\psi}_{n+1}\right) = (2g-2+n)\, \psi_1^{a_1} \cdots \psi_n^{a_n}}$$

By Corollary on p. 117

$$\prod_{i=1}^n \hat{\psi}_i^{a_i} \cdot \tilde{\psi}_{n+1} = \pi^*\left(\prod_{i=1}^n \psi_i^{a_i}\right) \cdot \tilde{\psi}_{n+1} \quad \text{since } \underbrace{\tilde{\psi}_{n+1} \cdot D_{i,n+1} = 0}_{\Uparrow}$$

$$\tilde{\mathcal{L}}_{n+1}\big|_{D_{i,n+1}} \cong \mathcal{O}_{D_{i,n+1}}$$

$$\pi_*\left(\prod_{i=1}^n \hat{\psi}_i^{a_i} \cdot \hat{\psi}_{n+1}\right) = \prod_{i=1}^n \psi_i^{a_i} \left(\pi_* \hat{\psi}_{n+1}\right)$$

$$\text{where } \pi_* \hat{\psi}_{n+1} \overset{\text{Comparison Lemma (b)}}{=} \pi_*\left(K_\pi + \sum_{i=1}^n D_{i,n+1}\right) = 2g-2+n$$

$$K_\pi = c_1(\omega_\pi)$$

□

Divisor Equation

$\gamma_1, \ldots \gamma_n \in H^*(X;\mathbb{Q})$, $\gamma \in H^2(X;\mathbb{Q})$

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_0(\gamma) \rangle_{g,n+1,\beta}$$
$$= \int_\beta \gamma \, \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,n,\beta}$$
$$+ \sum_{i=1}^{n} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_{i-1}}(\gamma_{i-1}) \tau_{a_i - 1}(\gamma_i \cup \gamma) \tau_{a_{i+1}}(\gamma_{i+1}) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,n,\beta}$$

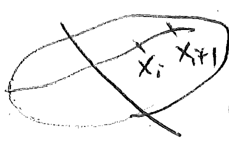
<u>Proof</u> It suffices to prove

$$\pi_*\left(\hat{\psi}_1^{a_1} \cdots \hat{\psi}_n^{a_n} \widetilde{ev}_{n+1}^* \gamma\right) = \int_\beta \gamma \cdot \psi_1^{a_1} \cdots \psi_n^{a_n}$$
$$+ \sum_{i=1}^{n} \psi_1^{a_1} \cdots \psi_{i-1}^{a_{i-1}} \psi_i^{a_i - 1} \psi_{i+1}^{a_{i+1}} \cdots \psi_n^{a_n} ev_i^* \gamma$$

By Corollary on p.117

$$\hat{\psi}_1^{a_1} \cdots \hat{\psi}_n^{a_n} \widetilde{ev}_{n+1}^* \gamma$$
$$= \pi^*(\psi_1^{a_1} \cdots \psi_n^{a_n}) \widetilde{ev}_{n+1}^* \gamma + \sum_{i=1}^{n} \pi^*(\psi_1^{a_1} \cdots \psi_{i-1}^{a_{i-1}} \psi_i^{a_i - 1} \psi_{i+1}^{a_{i+1}} \cdots \psi_n^{a_n}) \cdot D_{i,n+1} \cdot \widetilde{ev}_{n+1}^* \gamma$$

$$\widetilde{ev}_i \big|_{D_{i,n+1}} = \widetilde{ev}_{n+1} \big|_{D_{i,n+1}} \Rightarrow D_{i,n+1} \cdot \widetilde{ev}_{n+1}^* \gamma = D_{i,n+1} \, \widetilde{ev}_i^* \gamma = D_{i,n+1} \pi^* ev_i^* \gamma$$



ghost $\mathbb{P}^1$

$$\hat{\psi}_1^{a_1} \cdots \hat{\psi}_n^{a_n} \widetilde{ev}_{n+1}^* \gamma$$
$$= \pi^*(\psi_1^{a_1} \cdots \psi_n^{a_n}) \widetilde{ev}_{n+1}^* \gamma + \sum_{i=1}^{n} \pi^*(\psi_1^{a_1} \cdots \psi_{i-1}^{a_{i-1}} \psi_i^{a_i - 1} \psi_{i+1}^{a_{i+1}} \cdots \psi_n^{a_n} ev_i^* \gamma) D_{i,n+1}$$

$$\pi_*(\hat\psi_1^{a_1}\cdots\hat\psi_n^{a_n}\hat{ev}_{n+1}^*\gamma)$$
$$= \psi_1^{a_1}\cdots\psi_n^{a_n}\pi_*(\hat{ev}_{n+1}^*\gamma) + \sum_{i=1}^n \psi_1^{a_1}\cdots\psi_{i-1}^{a_{i-1}}\psi_i^{a_i-1}\psi_{i+1}^{a_{i+1}}\cdots\psi_n^{a_n} ev_i^*\gamma(\pi_* D_{i,n+1})$$

where $\pi_*(\hat{ev}_{n+1}^*\gamma) = \int_\beta \gamma$, $\pi_*(D_{i,n+1}) = 1$ $\square$

7.3 Genus zero Gromov-Witten potential

Choose homogeneous $\mathbb{Q}$-basis

$T_0, T_1, \ldots, T_m$ of $H^*(X;\mathbb{Q})$

$T_0 = 1$, $T_1, \ldots, T_p \in H^2(X;\mathbb{Q})$, $T_{p+1}, \ldots, T_m \in H^{>2}(X;\mathbb{Q})$

(For simplicity, we assume $H^*(X;\mathbb{Q}) = H^{even}(X;\mathbb{Q})$)

The genus zero GW potential of X is defined to be

$$\Phi^X(t_0,\ldots,t_m) = \sum_{\substack{\beta\in H_2(X,\mathbb{Z})\\ n\geq 0}} \frac{\langle \gamma^n\rangle^X_{0,n,\beta}}{n!} \quad \text{where } \gamma = \sum_{i=1}^m t_i T_i \in H^*(X;\mathbb{C})$$

$$\in \mathbb{C}[\![t_0,t_1,\ldots,t_m]\!]$$

$$\gamma = \gamma_0 + \gamma_2 + \gamma'$$
$$\gamma_0 \in H^0,\quad \gamma_2 \in H^2,\quad \gamma' \in H^{>2}$$

$\gamma_0 = t_0 1$

$\gamma_2 = \sum_{i=1}^p t_i T_i$

$\gamma' = \sum_{i=p+1}^m t_i T_i$

$\boxed{\beta=0}$ $\overline{\mathcal{M}}_{0,n}(X,0)=\overline{\mathcal{M}}_{0,n}\times X$ (empty if $n\leq 2$)

$$[\overline{\mathcal{M}}_{0,n}(X,0)]^{vir}=[\overline{\mathcal{M}}_{0,n}]\times[X]$$

$ev_i:\overline{\mathcal{M}}_{0,n}(X,0)=\overline{\mathcal{M}}_{0,n}\times X\to X$

projection to the 2nd factor

$$\langle\gamma^n\rangle_{g=0,n,\beta=0}=\int_{[\overline{\mathcal{M}}_{0,n}]}1\cdot\int_{[X]}\gamma^n=\begin{cases}\int_X\gamma^3 & n=3\\ 0 & \text{otherwise}\end{cases}$$

$$\Phi^X(t_0,\dots,t_m)=\frac{1}{6}\int_X\gamma^3+\sum_{\substack{\beta>0\\ n\geq 0}}\frac{1}{n!}\langle\gamma^n\rangle^X_{0,n,\beta}$$

$\boxed{\beta>0}$

$$\sum_{n\geq 0}\frac{1}{n!}\langle\gamma^n\rangle^X_{0,n,\beta}=\sum_{n_0,n_1,n_2\geq 0}\frac{t_0^{n_0}}{n_0!\,n_1!\,n_2!}\langle 1^{n_0},\gamma_2^{n_1},(\gamma')^{n_2}\rangle^X_{0,n_0+n_1+n_2,\beta}$$

string equation

$$=\sum_{n_1,n_2\geq 0}\frac{1}{n_1!\,n_2!}\langle\gamma_2^{n_1},(\gamma')^{n_2}\rangle^X_{0,n_1+n_2,\beta}$$

divisor equation

$$=\sum_{n_1,n_2\geq 0}\frac{1}{n_1!\,n_2!}\left(\int_\beta\gamma_2\right)^{n_1}\langle(\gamma')^{n_2}\rangle^X_{0,n_2,\beta}$$

$$=e^{\int_\beta\gamma_2}\sum_{n_2\geq 0}\frac{1}{n_2!}\langle(\gamma')^{n_2}\rangle^X_{0,n_2,\beta}$$

$$\Phi^X(t_0,\dots,t_m)=\frac{1}{6}\int_X\gamma^3+\sum_{\substack{\beta>0\\ n\geq 0}}e^{\int_\beta\gamma_2}\frac{\langle(\gamma')^n\rangle^X_{0,n,\beta}}{n!}$$

<u>Example</u> $X = \mathbb{P}^2$, $\gamma = t_0 \cdot 1 + t_1 H + t_2 H^2$

$$\Phi^{\mathbb{P}^2}(t_0, t_1, t_2) = \frac{1}{6}\int_{\mathbb{P}^2} (t_0 + t_1 H + t_2 H^2)^3 + \sum_{d=1}^{\infty} \sum_{n \geq 0} e^{dt_1} \frac{t_2^n \langle (H^2)^n \rangle_{0,n,d}^{\mathbb{P}^2}}{n!}$$

Where $\langle H^2, \ldots, H^2 \rangle_{0,n,d}^{\mathbb{P}^2} = 0$ unless

$$2n = 3d - 1 + n \iff n = 3d - 1$$

Let $N_d := \langle H^2, \ldots, H^2 \rangle_{0,n,d}^{\mathbb{P}^2}$

= number of deg d rational curves in $\mathbb{P}^2$ passing through $3d-1$ points

$$\Phi^{\mathbb{P}^2}(t_0, t_1, t_2) = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum_{d=1}^{\infty} e^{dt_1} \frac{N_d t_2^{3d-1}}{(3d-1)!}$$

7.4 WDVV Equation

Witten–Dijkgraaf–Verlinde–Verlinde

$g_{ab} = \int_X T_a \cup T_b$, g^{ab} inverse matrix

$$\Phi^X_{ijk} := \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k} = \int_X T_i \cup T_j \cup T_k + \sum_{\substack{\beta > 0 \\ n \geq 0}} \frac{1}{n!} \langle \gamma^n, T_i, T_j, T_k \rangle^X_{0, n+3, \beta}$$

$$\Phi^X_{0ij} = g_{ij}$$

Theorem (WDVV Equation)

$$\sum_{a,b} \Phi_{ija}\, g^{ab}\, \Phi_{bkl} = \sum_{a,b} \Phi_{jka}\, g^{ab}\, \Phi_{bil}$$

Example $\mathbb{P}^2$

$T_0 = 1,\ T_1 = H,\ T_2 = H^2$

$$(g_{ab}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (g^{ab})$$

$$\therefore\ \Phi = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum_{d>0} e^{dt_1} \frac{N_d}{(3d-1)!} t_2^{3d-1}$$

$(i, j, k, l) = (1, 1, 2, 2)$

$$\Phi_{112} \underset{0}{\cancel{\Phi_{220}}} + \underbrace{\Phi_{110}}_{=1} \Phi_{222} + \Phi_{111} \Phi_{221} = \underset{0}{\cancel{\Phi_{120}}} \Phi_{122} + \Phi_{122} \underset{0}{\cancel{\Phi_{120}}} + \Phi_{121} \Phi_{121}$$

$$\Phi_{222} = \Phi_{121}^2 - \Phi_{111}\Phi_{221}$$

where $\Phi_{222} = \sum_{d\geq 1} e^{dt_1} \frac{N_d}{(3d-4)!} t_2^{3d-4}$

$$\Phi_{121} = \sum_{d\geq 0} e^{dt_1} \frac{d^2 N_d}{(3d-2)!} t_2^{3d-2}$$

$$\Phi_{111} = \sum_{d\geq 0} e^{dt_1} \frac{d^3 N_d}{(3d-1)!} t_2^{3d-1}$$

$$\Phi_{122} = \sum_{d\geq 0} e^{dt_1} \frac{d N_d}{(3d-3)!} t_2^{3d-3}$$

$$N_1 = \langle H^2, H^2 \rangle^{\mathbb{P}^2} = \boxed{1}$$

$d \geq 2$

$$\frac{N_d}{(3d-4)!} = \sum_{\substack{d_1, d_2 > 0 \\ d_1 + d_2 = d}} \left(\frac{d_1^2 d_2^2}{(3d_1-2)!(3d_2-2)!} N_{d_1} N_{d_2} - \frac{d_1^3 d_2}{(3d_1-1)!(3d_2-3)!} N_{d_1} N_{d_2} \right)$$

$$N_d = \sum_{\substack{d_1, d_2 > 0 \\ d_1 + d_2 = d}} \left(\binom{3d-4}{3d_1-2} d_1^2 d_2^2 - \binom{3d-4}{3d_1-1} d_1^3 d_2 \right) N_{d_1} N_{d_2}$$

$$N_2 = \left(\binom{2}{1} - \binom{2}{2} \right) N_1 \cdot N_1 = \boxed{1}$$

P.126

$$N_3 = N_1 N_2 \left[\binom{5}{1} 2^2 - \binom{5}{2} \cdot 2 \right] + N_2 N_1 \left[\binom{5}{4} \cdot 2^2 - \binom{5}{5} 2^3 \right]$$

$$= (20-20) + (20-8) = \boxed{12}$$

$$N_4 = N_1 N_3 \left[\binom{8}{1} 3^2 - \binom{8}{2} \cdot 3 \right] + N_2 N_2 \left[\binom{8}{4} 2^4 - \binom{8}{5} 2^4 \right]$$

$$+ N_3 N_1 \left[\binom{8}{7} 3^2 - \binom{8}{8} \cdot 3^3 \right]$$

$$= 12 \cdot (72-84) + (70-56) \cdot 16 + 12 \cdot (72-27)$$

$$= -144 + 224 + 540 = \boxed{620}$$

Kontsevich-Manin   del Pezzo surfaces.