6.6 Examples

Example 1 $\overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n}\times X$ proper smooth DM stack

$$\left[\begin{array}{c} f:(C,x_1,\dots,x_n)\to X \\ f(y)=z \quad \forall y\in C\end{array}\right] \mapsfrom \left([(C,x_1,\dots,x_n)],z\right)$$

virtual dimension $(\dim X-3)(1-g)+n$

actual dim $3g-3+n+\dim X$

$\xi=\left([(C,x_1,\dots,x_n)],z\right)\in \overline{\mathcal{M}}_{g,n}\times X=\overline{\mathcal{M}}_{g,n}(X,0)$

$f:C\to X$ constant map to z $\Rightarrow$ $f^*TX=\mathcal{O}_C\otimes T_zX$

$$0\to \mathrm{Ext}^0\Big(\Omega_C\big(\textstyle\sum_{i=1}^n x_i\big),\mathcal{O}_C\Big)\to H^0(C,\mathcal{O}_C)\otimes T_zX\to T^1_\xi$$

$$\to \mathrm{Ext}^1\Big(\Omega_C\big(\textstyle\sum_{i=1}^n x_i\big),\mathcal{O}_C\Big)\to H^1(C,\mathcal{O}_C)\otimes T_zX\to T^2_\xi\to 0$$

$[(C,x_1,\dots,x_n)]$ stable $\Rightarrow$ $\mathrm{Ext}^0\Big(\Omega_C\big(\sum_{i=1}^n x_i\big),\mathcal{O}_C\Big)=0.$

$$\begin{tikzcd} 0 \arrow[r] & T_zX \arrow[r] & T^1_\xi \arrow[r] & T_{[(C,x_1,\dots,x_n)]}\overline{\mathcal{M}}_{g,n} \arrow[r] \arrow[ll, bend left, leftrightarrow, "T_\xi(\overline{\mathcal{M}}_{g,n}\times X)"] & \mathbb{E}^\vee_{[(C,x_1,\dots,x_n)]}\otimes T_zX \arrow[r] & T^2_\xi \arrow[r] & 0 \end{tikzcd}$$

$\mathbb{E}_{[(C,x_1,\dots,x_n)]}=H^0(C,\omega_C)$

$$\begin{tikzcd} \mathbb{E} = \pi_* \omega_\pi \arrow[d] \\ \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

Hodge bundle

ω_π relative dualizing sheaf of $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ (universal curve)

Over $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n} \times X$

$$\begin{tikzcd} & \overline{\mathcal{M}}_{g,n} \times X \arrow[dl, "p_1"'] \arrow[d, "p_2"] \\ \overline{\mathcal{M}}_{g,n} & X \end{tikzcd}$$

$$0 \to p_2^* TX \to T^1 \to p_1^* T\overline{\mathcal{M}}_{g,n} \to p_1^* \mathbb{E}^\vee \otimes p_2^* TX \to T^2 \to 0$$

$\therefore$ $T^1_{\mathcal{M}} = p_1^* T\overline{\mathcal{M}}_{g,n} \oplus p_2^* TX$ tangent bundle

$T^2_{\mathcal{M}} = p_1^* \mathbb{E}^\vee \otimes p_2^* TX$ obstruction bundle

$\mathrm{rank}(T^2_{\mathcal{M}}) = g \dim X$

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook, "0"] \arrow[d] \arrow[dr, phantom, "\square"] & T^2_{\mathcal{M}} \arrow[d] \\ \underline{C}_{\mathcal{M}} = {[\mathcal{M}/T\mathcal{M}]} \arrow[r, hook] & \underline{E} = {[T^2_{\mathcal{M}}/T\mathcal{M}]} \end{tikzcd}$$

$$\begin{tikzcd} E^\bullet = {[} (T^2_{\mathcal{M}})^\vee \arrow[r, "0"] & \Omega^1_{\mathcal{M}} {]} \arrow[d, "id"] \\ L^\bullet_{\mathcal{M}} = {[} 0 \arrow[r, "0"] & \Omega^1_{\mathcal{M}} {]} \end{tikzcd}$$

$$[\overline{\mathcal{M}}_{g,n}(X,0)]^{vir} = 0^!(0_*[\mathcal{M}])$$
$$= e(T^2_{\mathcal{M}}) \cap [\mathcal{M}]$$
$$= e(p_1^*\mathbb{E}^\vee \otimes p_2^* TX) \cap ([\overline{\mathcal{M}}_{g,n}] \times [X])$$

$$[\overline{\mathcal{M}}_{g,n}] \times [X] \in A_{3g-3+n+\dim X}(\mathcal{M};\mathbb{Q})$$

$$e(p_1^*\mathbb{E}^\vee \otimes p_2^* TX) \in A^{g\dim X}(\mathcal{M};\mathbb{Q})$$

$$[\overline{\mathcal{M}}_{g,n}(X,0)]^{vir} \in A_{(\dim X-3)(1-g)+n}(\mathcal{M};\mathbb{Q})$$

Example 2

$i: X \hookrightarrow \mathbb{P}^m$ smooth hypersurface of degree ℓ

$X = s^{-1}(0)$ $s \in H^0(\mathbb{P}^m, \mathcal{O}(\ell))$

$m>3$ $i_*: H_2(X;\mathbb{Z}) \xrightarrow{\cong} H_2(\mathbb{P}^m;\mathbb{Z}) = \mathbb{Z}[\mathbb{P}^1]$

$$\overline{\mathcal{M}}_{g,n}(X,d) \hookrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^m,d)$$

$$[f:(C,x_1,\dots,x_n) \to X] \mapsto [i\circ f:(C,x_1,\dots,x_n) \to \mathbb{P}^m]$$

$g=0$
$$\begin{tikzcd}\overline{\mathcal{M}}_{0,n+1}(\mathbb{P}^m,d) \arrow[r, "ev_{n+1}"] \arrow[d, "\pi"'] & \mathbb{P}^m \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d) & \end{tikzcd}$$

$$\begin{tikzcd} E_d := \pi_* ev_{n+1}^* \mathcal{O}_{\mathbb{P}^m}(\ell) \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d) \end{tikzcd}$$

$\to$ vector bundle of rank $d\ell+1$
$(R^1\pi_* ev_{n+1}^* \mathcal{O}_{\mathbb{P}^m}(\ell)=0)$

$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)$ smooth DM stack of dim
$(m+1)d+(m-3)+n$

$$\begin{tikzcd} E_d \arrow[d] & \supset H^0(C, f^*\mathcal{O}(\ell)) \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d) & \ni [f:(C,x_1,\dots,x_n)\to\mathbb{P}^m] \end{tikzcd}$$

$\hat{s}: \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d) \to E_d$ section

$\xi = [f:(C,x_1,\dots,x_n)\to \mathbb{P}^m]$

$\hat{s}(\xi) = f^*s \in H^0(C, f^*\mathcal{O}_{\mathbb{P}^m}(\ell)) = (E_d)_\xi$

$\hat{s}^{-1}(0) = \overline{\mathcal{M}}_{0,n}(X,d)$

p.104

$\mathcal{M} = \overline{\mathcal{M}}_{0,n}(X,d)$ virtual dim

$$(m+1-\ell)d + \underbrace{\dim X - 3}_{m-4} + n$$

$W = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ smooth

$$\xi := [f : (C, x_1 - x_n) \to X] \in \mathcal{M} \subset W$$

$$0 \to \mathrm{Ext}^0(\Omega_C(x_1 + \dots + x_n), \mathcal{O}_C) \to H^0(C, f^*T\mathbb{P}^m) \to T^1_W$$
$$\to \mathrm{Ext}^1(\Omega_C(x_1 + \dots + x_n), \mathcal{O}_C) \to \underset{0}{\cancel{H^1(C, f^*T\mathbb{P}^m)}} \to T^2_W \to 0$$

$$T^2_W = 0 \Rightarrow [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)]^{vir} = [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)]$$

$$0 \to \mathrm{Ext}^0(\Omega_C(x_1 + \dots + x_n), \mathcal{O}_C) \to H^0(C, f^*T\mathbb{P}^m) \to T^1_W$$
$$\to \mathrm{Ext}^1(\Omega_C(x_1 + \dots + x_n), \mathcal{O}_C) \to 0$$

$$0 \to \mathrm{Ext}^0(\Omega_C(x_1 + \dots + x_n), \mathcal{O}_C) \to H^0(C, f^*TX) \to T^1_{\mathcal{M}}$$
$$\to \mathrm{Ext}^1(\Omega_C(x_1 + \dots + x_n), \mathcal{O}_C) \to H^1(C, f^*TX) \to T^2_{\mathcal{M}} \to 0$$

$$0 \to TX \to T\mathbb{P}^m|_X \to \mathcal{O}_{\mathbb{P}^m}(l)|_X \to 0$$

$$0 \to H^0(C, f^*TX) \to H^0(C, f^*T\mathbb{P}^m) \to H^0(C, f^*\mathcal{O}_{\mathbb{P}^m}(l)) \to H^1(C, f^*TX) \to 0$$

$$E^\bullet = [\, E_d^\vee \to \Omega^1_{W/\mathfrak{m}} \,]$$

$$L_M^\bullet = [\to I/I^2 \to \Omega^1_{W/\mathfrak{m}}]$$

$$\begin{tikzcd}
& C_{M/W} \arrow[r, hook] \arrow[d] & E_d \arrow[d] \\
{[C_{M/W}/T_{W/\mathfrak{m}}]} = & \underline{C}_M \arrow[r, hook] & \underline{E}_M = [E_d/T_{W/\mathfrak{m}}]
\end{tikzcd}$$

Fulton intersection theory

$$\begin{tikzcd}
& E \arrow[d] \arrow[r] & \text{vector bundle} \\
M = \tilde{s}^{-1}(0) \arrow[r, hook, "i"] & W \arrow[u, bend left, "\tilde{s}"] \arrow[r, squiggly] & \text{smooth scheme}
\end{tikzcd}$$

Localized top Chern class of $\bar{E}$ w.r.t. the section $\tilde{s}$:

$$e_{loc}(E,\tilde{s}) = 0_E^!([C_{M/W}]) \in A_{\dim W - \mathrm{rank}\, E}(M)$$

$$[C_{M/W}] \in A_{\dim W}(E)$$

$$i_*: A_*(M) \to A_*(W)$$

$$i_*(e_{loc}(E,\tilde{s})) = c_{\mathrm{rank}\, E}(E) \cap [W]$$

$$[\overline{\mathcal{M}}_{0,n}(X,d)]^{vir} = 0^![C_{m/w}]$$

$$= e_{loc}(E_d,\tilde{s})$$

$$\in A_{\dim W - \mathrm{rank}\, E_d}(\overline{\mathcal{M}}_{0,n}(X,d);\mathbb{Q})$$

where $\dim W = (m+1)d + (m-3) + n$

$\mathrm{rank}\, E_d = ld + 1$

$\dim W - \mathrm{rank}\, E_d = (m+1-l)d + (m-4) + n$

$= \text{virtual dim } \overline{\mathcal{M}}_{0,n}(X,d)$

$$i_*: A_*(\overline{\mathcal{M}}_{0,n}(X,d)) \to A_*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d))$$

$$i_*[\overline{\mathcal{M}}_{0,n}(X,d)]^{vir} = e(E_d) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)]$$

<u>Special case</u> $m=4$, $l=5$

X is a degree 5 hypersurface in $\mathbb{P}^4$

$\mathcal{O}_X(K_X) \cong \mathcal{O}_X \Rightarrow c_1(TX)=0$

X is a quintic CY 3-fold

$$[\overline{M}_{0,0}(X,d)]^{vir} \in \underset{H_0}{A_0}(\overline{M}_{0,0}(X,d);\mathbb{Q})$$

The genus 0, deg $d>0$ GW invariant of X

$$N_{0,d} := \langle \underset{H^0}{\underset{\cap}{1}}, \underset{H_0}{\underset{\cap}{[\overline{M}_{0,0}(X,d)]^{vir}}} \rangle$$

$i: \overline{M}_{0,0}(X,d) \hookrightarrow \overline{M}_{0,0}(\mathbb{P}^4,d)$

$$= \langle i^*1, [\overline{M}_{0,0}(X,d)]^{vir} \rangle$$

$$= \langle 1, i_*[\overline{M}_{0,0}(X,d)]^{vir} \rangle$$

$$= \langle 1, e(E_d) \cap [\overline{M}_{0,0}(\mathbb{P}^4,d)] \rangle$$

Kontsevich:

$$N_{0,d} = \int_{[\overline{M}_{0,0}(\mathbb{P}^4, d)]} e(E_d)$$

can be computed by localization

$(\mathbb{C}^*)^5 \curvearrowright \mathbb{P}^4$

induces $(\mathbb{C}^*)^5 \curvearrowright \overline{M}_{0,0}(\mathbb{P}^4, d)$

E_d $(\mathbb{C}^*)^5$-equivariant vector bundle

7 Gromov-Witten invariants

X smooth projective variety

7.1 Hodge classes and ψ-classes

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[r, "ev_{n+1}"] \arrow[d, "\pi"] & X \\
\overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[u, bend left, "s_i"] &
\end{tikzcd}$$

$\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ universal curve

$ev_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to X$ universal map

ω_π relative dualizing sheaf on $\overline{\mathcal{M}}_{g,n+1}(X,\beta)$

Hodge bundle

$$\begin{tikzcd}
\mathbb{E} := \pi_* \omega_\pi \arrow[d] & \supset H^0(C,\omega_C) \arrow[d] \\
\overline{\mathcal{M}}_{g,n}(X,\beta) & \ni [f:(C,x_1,..,x_n) \to X]
\end{tikzcd}$$

rank $\mathbb{E} = g$

$\lambda_j := c_j(\mathbb{E}) \in A^j(\overline{M}_{g,n}(X,\beta))$ $\quad j=1,\dots,g$

$\qquad\qquad H^{2j}(\overline{M}_{g,n}(X,\beta))$

Hodge classes

$$\begin{tikzcd}
\mathbb{L}_i := s_i^*\omega_\pi \arrow[d] & \supset & T^*_{x_i}C \arrow[d] \\
\overline{M}_{g,n}(X,\beta) & \ni & {[f:(C,x_1,\dots,x_n)\to X]}
\end{tikzcd}$$

$\psi_i := c_1(\mathbb{L}_i) \in A^1(\overline{M}_{g,n}(X,\beta))$ $\quad i=1,\dots,n$

$\qquad\qquad H^2(\overline{M}_{g,n}(X,\beta))$

ψ-classes

Remark
$$\begin{tikzcd}
\overline{M}_{g,n}(X,\beta) \arrow[r] \arrow[rr, bend right, "p"'] & M^{pre}_{g,n} \arrow[r, "c"] & \overline{M}_{g,n}
\end{tikzcd}$$

$\psi_1',\dots,\psi_n'$ ψ-classes on $\overline{M}_{g,n}$

$p^*\psi_i' \neq \psi_i$

7.2 Descendant Gromov-Witten Invariants

$\gamma_1, \dots, \gamma_n \in H^*(X; \mathbb{Q})$ homogeneous element

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n \left(ev_i^*(\gamma_i) \psi_i^{a_i} \right)$$

descendant Gromov-Witten invariants of X

$$\prod_{i=1}^n \left(ev_i^*(\gamma_i) \psi_i^{a_i} \right) \in H^{\sum_{i=1}^n (\deg \gamma_i + 2a_i)}(\overline{M}_{g,n}(X,\beta))$$

$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta} = 0$ unless

$$\sum_{i=1}^n (\deg \gamma_i + 2a_i) = 2 \left(\langle c_1(TX), \beta \rangle + (\dim X - 3)(1-g) + n \right)$$

$a_1 = \cdots = a_n = 0$

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,n,\beta} := \langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_n) \rangle^X_{g,n,\beta}$$

primary GW invariants of X

Theorem $(g,n,\beta) \neq (0,0,0), (0,1,0), (0,2,0), (1,0,0)$

(1) (string equation)

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_0(1) \rangle^X_{g,n+1,\beta}$$

$$= \sum_{i=1}^{n} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_i-1}(\gamma_i) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta}$$

(2) (dilaton equation)

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_1(1) \rangle^X_{g,n+1,\beta}$$

$$= (2g-2+n) \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,n,\beta}$$

(3) (divisor equation) $\gamma \in H^2(X;\mathbb{Q})$

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_0(\gamma) \rangle^X_{g,n+1,\beta}$$

$$= \left(\int_\beta \gamma\right) \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta}$$

$$+ \sum_{i=1}^{n} \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_i-1}(\gamma_i \cup \gamma) \cdots \tau_{a_n}(\gamma_n) \rangle^X_{g,n,\beta}$$

Corollary

(1)' $\langle \gamma_1 \cdots \gamma_n, 1 \rangle^X_{g,n+1,\beta} = 0$

(3)' $\gamma \in H^2(X;\mathbb{Q}) \Rightarrow \langle \gamma_1 \cdots \gamma_n, \gamma \rangle^X_{g,n+1,\beta} = \left(\int_\beta \gamma\right) \langle \gamma_1 \cdots \gamma_n \rangle^X_{g,n,\beta}$