6. Virtual Fundamental Classes

- K. Behrend and B. Fantechi "The intrinsic normal cone," Invent. 1997
- K. Behrend, "Gromov-Witten invariants in algebraic geometry," Invent. 1997
- Junliang Shen's master thesis 2014

Another construction:

Jun Li and G. Tian, "Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties," JAMS 1998

6.1 The intrinsic normal cone and sheaf

$\mathcal{M}$ Deligne-Mumford stack, locally of finite type over k

↳ field, $k = \bar{k}$, $\mathrm{char}(k) = 0$

$\mathcal{M}$ has an étale open cover by affine schemes of finite type over k.

Consider a "chart"

$$i : U \to \mathcal{M}$$

where U is an affine scheme of finite type over $\mathrm{Spec}(k)$, and i is an étale morphism.

$U = \mathrm{Spec}\, A$

where A is a finitely generated k-algebra

$\exists\ \phi : R = k[x_1, \dots, x_n] \to A$

surjective ring homomorphism

$I = \mathrm{Ker}\,\phi \qquad A = R/I$

$f : U = \mathrm{Spec}\, A \to W = \mathrm{Spec}\, R$

closed immersion

W smooth affine scheme of finite type over k.

$A = R/I =$ degree zero part

$$\widehat{\bigoplus_{n=0}} \mathrm{Sym}^n(I/I^2) \longrightarrow \widehat{\bigoplus_{n=0}} I^n/I^{n+1}$$

: surjective morphism of graded A-algebras

normal cone of U in W

$$C_{U/W} := \mathrm{Spec}\left(\widehat{\bigoplus_{n=0}} I^n/I^{n+1}\right)$$

normal sheaf of U in W

abuse of terminology

$$N_{U/W} := \mathrm{Spec}\left(\widehat{\bigoplus_{n=0}} \mathrm{Sym}^n(I/I^2)\right)$$

(cf. Fulton "Intersection theory")

$C_{U/W} \hookrightarrow N_{U/W}$ closed subscheme

$C_{U/W}$ " " pure dimension $= \dim W$

$C_{U/W}$ cone over U

$N_{U/W}$ abelian cone over U

abelian hull of $C_{U/W}$

If I/J^2 is a free A-module of rank r then $C_{U/W} = N_{U/W}$ is a vector bundle of rank r : normal bundle of U in W

$L_U^{\cdot} = [\, I/J^2 \to \Omega^1_{W}|_U \,]$ cotangent complex of U

$$\begin{array}{cc} \shortparallel & \shortparallel \\ L^{-1} & L^{0} \end{array}$$

In general, $C_{U/W} \subsetneqq N_{U/W}$

$[\, I/I^2 \to \Omega^1_W|_U \,] = L_U^{\geq -1}$

truncated cotangent complex of U

If $f' : U \to W'$ is another closed immersion to a smooth affine scheme of finite type $/\mathbb{k}$ then we have isomorphism of cone stacks over U:

$$[C_{U/W} / TW|_U] \cong [C_{U/W'} / TW'|_U]$$

isomorphism of abelian cone stacks over U

$$[N_{U/W}/TW|_U] \cong [N_{U/W'}/TW'|_U]$$

$$N_{U/W} = \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(I/I^2)\right)$$
$$TW|_U = \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(\Omega^1_{W}|_U)\right)$$

Define the <u>intrinsic normal cone of U</u>

$$\underline{C}_U := [C_{U/W}/TW|_U]$$

$U \hookrightarrow W$ any closed immersion to a smooth affine scheme W of finite type / $\Bbbk$.

pure dim $= \dim W - \dim W = 0$

<u>intrinsic normal sheaf of U</u>

abuse of terminology

$$\underline{N}_U := [N_{U/W}/TW|_U]$$

$\underline{C}_U \subset \underline{N}_U$ closed subcone stack over U

$\underline{N}_U$ is the abelian hull of $\underline{C}_U$

$\underline{C}_U$, $\underline{N}_U$: glued to

intrinsic normal cone $\underline{C}_{\mathcal{M}}$

and intrinsic normal sheaf $\underline{N}_{\mathcal{M}}$

over the DM stack $\mathcal{M}$

$\underline{C}_{\mathcal{M}} \subset \underline{N}_{\mathcal{M}}$: closed subcone stack over $\mathcal{M}$

$\underline{N}_{\mathcal{M}}$ abelian hull of $\underline{C}_{\mathcal{M}}$

For any étale morphism $i : U \to \mathcal{M}$

U affine scheme of finite type / $\Bbbk$

$$\begin{tikzcd} \underline{C}_U \cong i^*\underline{C}_{\mathcal{M}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{C}_{\mathcal{M}} \arrow[d] \\ U \arrow[r, "i"] & \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} \underline{N}_U \cong i^*\underline{N}_{\mathcal{M}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{N}_{\mathcal{M}} \arrow[d] \\ U \arrow[r, "i"] & \mathcal{M} \end{tikzcd}$$

pure dim = dim W

$$\begin{tikzcd} C_{U/W} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & N_{U/W} \arrow[d] \\ \underline{C}_{\mathcal{M}} \arrow[r, hook] & \underline{N}_{\mathcal{M}} \end{tikzcd}$$

pure dim 0

<u>Proposition</u>

$\mathcal{M}$ is a local complete intersection

$\Leftrightarrow \underline{C}_{\mathcal{M}}$ is a vector bundle stack over $\mathcal{M}$

$\Leftrightarrow \underline{C}_{\mathcal{M}} = \underline{N}_{\mathcal{M}}$

<u>Example</u>

$\mathcal{M}$ smooth DM stack locally of finite type over k

$T_{\mathcal{M}}$ tangent bundle, vector bundle over $\mathcal{M}$

$\underline{C}_{\mathcal{M}} = \underline{N}_{\mathcal{M}} = [\mathcal{M} / T_{\mathcal{M}}] = B\,T_{\mathcal{M}}$

6.2 Obstruction theory

We say an object

$L^\bullet = [\cdots L^{-1} \to L^0 \to L^1 \to \cdots]$

in $D(\mathcal{M})$ = derived category of sheaves of $\mathcal{O}_{X_{\acute{e}t}}$-modules

satisfies (*) if

$$\begin{cases} h^i(L^\bullet) = 0 & \text{for } i > 0 \\ h^i(L^\bullet) = 0 & \text{for } i = 0, -1 \end{cases}$$

If $L^\bullet \in ob(D(\mathcal{M}))$ satisfies (*)

then we may construct

$h^1/h^0((L^\bullet[1])^\vee)$ abelian cone stack over $\mathcal{M}$

Example The cotangent complex $L^\bullet_{\mathcal{M}} \in ob(D(\mathcal{M}))$

satisfies (*)

$h^1/h^0((L^\bullet_{\mathcal{M}}[1])^\vee) = \underline{N}_{\mathcal{M}}$ intrinsic normal sheaf

$$L^{\cdot} \to h^1/h^0((L^{\cdot}_{\mathrm{fl}})^\vee)$$

is a contravariant functor

$E^{\cdot}, L^{\cdot} \in Ob(D(M))$ satisfying $(*)$

$\phi : E^{\cdot} \to L^{\cdot} \in Mor_{D(M)}(E^{\cdot}, L^{\cdot})$

$\rightsquigarrow \phi^\vee : h^1/h^0((L^{\cdot}_{\mathrm{fl}})^\vee) \to h^1/h^0((E^{\cdot}_{\mathrm{fl}})^\vee)$

morphism of abelian cones over $\mathcal{M}$

Proposition

(1) $\phi^\vee$ is representable $\Leftrightarrow$ $h^0(\phi)$ is surjective

(2) $\phi^\vee$ is a closed immersion

$\Leftrightarrow$ $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective

(3) $\phi^\vee$ is an isomorphism

$\Leftrightarrow$ $h^0(\phi)$ and $h^{-1}(\phi)$ are isomorphisms.

Definition

$E^{\cdot} \in \mathrm{ob}(D(\mathcal{M}))$ satisfies $(*)$

$\phi: E^{\cdot} \to L^{\cdot}_{\mathcal{M}} \in \mathrm{Mor}_{D(\mathcal{M})}(E^{\cdot}, L^{\cdot}_{\mathcal{M}})$

cotangent complex

is an <u>obstruction theory</u> if
$h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective

Remark

By part (2) of proposition,

$$\phi^{\vee}: \mathfrak{N}_{\mathcal{M}} = h^1/h^0(((L^{\cdot}_{\mathcal{M}})_{\geq -1})^{\vee}) \to \mathfrak{E} := h^1/h^0((E^{\cdot}_{\geq -1})^{\vee})$$

intrinsic normal cone

is a closed immersion of abelian cone stacks

<u>Definition</u> An obstruction theory $\phi: E^{\cdot} \to L^{\cdot}_{\mathcal{M}}$
is <u>perfect</u> if $E^{\cdot}$ is of perfect amplitude
contained in $[-1, 0]$: locally

$E^{\cdot} = [E^{-1} \to E^0]$ $\qquad$ $\mathrm{rank}\, E^{\cdot} = \mathrm{rank}\, E^0 - \mathrm{rank}\, E^{-1}$

vector bundles $\qquad$ <u>virtual dimension</u>

<u>Remark</u> If $\phi^{\cdot}: E^{\cdot} \to L_{\mathcal{M}}$ is a <u>perfect</u> obstruction theory then

$\underline{E} = h^1/h^0((E^{\cdot}{}_{[1]})^{\vee})$ locally $[(E^{-1})^{\vee}/(E^0)^{\vee}]$

is a vector bundle stack of rank $-d$

$d = \operatorname{rank}(E^{\cdot}) =$ virtual dimension

6.3 Virtual fundamental classes

$$\begin{tikzcd} E \arrow[d] \\ M \arrow[u, bend left, "0_E"] \end{tikzcd}$$

E vector bundle of rank r

M scheme

0_E zero section

Gysin homomorphism

$$0_E^! : A_*(E) \to A_{*-r}(E)$$

A. Kresch "Cycles groups for Artin stacks"
Invent. 1999

We now have

$$\begin{tikzcd} \underline{E} \arrow[d] \\ \mathcal{M} \arrow[u, bend left, "0_{\underline{E}}"] \end{tikzcd}$$

$\underline{E}$ vector bundle stack of rank $-d$ (Artin stack)

$\mathcal{M}$: DM stack

$0_{\underline{E}}$ zero section

$$0_{\underline{E}}^! : A_*(\underline{E}) \to A_{*+d}(\mathcal{M})$$

The virtual fundamental class of $(\mathcal{M}, E^\cdot)$ is defined to be

$$[\mathcal{M}, E^\cdot]^{vir} := 0_{\underline{E}}^! \underbrace{[\underline{C}_{\mathcal{M}}]}_{\substack{\cap \\ A_0(\underline{E})}} \in A_d(\mathcal{M})$$

Original construction of Behrend-Fantechi requires global resolution $E^\cdot = [F^{-1} \to F^0]$ where F^{-1}, F^0 are global vector bundles on $\mathcal{M}$

Then $\underline{E} = [F_1 / F_0]$ where $F_1 = (\mathcal{F}^{-1})^\vee$

$F_0 = (\mathcal{F}^0)^\vee$

$\operatorname{rank} F_0 - \operatorname{rank} F_1 = d$

$$\begin{tikzcd} C(F^\cdot) := \underline{C}_M \times_{\underline{E}} F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F_1 \arrow[d, "\text{relative dim} = \operatorname{rank} F_0"] \\ \underline{C}_M \arrow[r] & \underline{E} \end{tikzcd}$$

$C(F^\cdot) \hookrightarrow F_1$ closed immersion

pure dim $= \operatorname{rank} F_0$

$0^!_{F_1} : A_*(F_1) \to A_{*-\operatorname{rank} F_1}(\mathcal{M})$

F_1, $\mathcal{M}$ DM stacks

$$[\mathcal{M}, E^\cdot]^{vir} := 0^!_{F_1} \underbrace{[C(F^\cdot)]}_{\substack{\cap \\ A_{\operatorname{rank} F_0}(F_1)}} \in A_d(\mathcal{M})$$

independent of choice of the global resolution

6.4 Relative version

$$\begin{array}{ll} \mathcal{M} & \text{DM stack} \\ \downarrow & \\ \mathcal{D} & \text{smooth DM stack of dim } m \end{array}$$

Definition

$E^{\cdot} \in Ob(D(\mathcal{M}))$ satisfies $(*)$

$\phi : E^{\cdot} \to L^{\cdot}_{\mathcal{M}/\mathcal{D}}$

↳ relative cotangent complex

is a relative obstruction theory if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective

It is perfect if locally

$E^{\cdot} = [E^{-1} \to E^{0}]$

↑ ↗

vector bundles

If $\phi : E^{\cdot} \to L^{\cdot}_{\mathcal{M}/\mathcal{D}}$ is a perfect relative obstruction theory then

$$\underset{\substack{\text{relative}\\ \text{intrinsic}\\ \text{normal cone}}}{\mathfrak{C}_{\mathcal{M}/\mathcal{D}}} \hookrightarrow \underset{\substack{\downarrow \\ \text{relative intrinsic}\\ \text{normal sheaf}}}{\mathfrak{N}_{\mathcal{M}/\mathcal{D}}} = h^1/h^0(\underbrace{((L^{\cdot}_{\mathcal{M}/\mathcal{D}})_{\geq -1})^{\vee}}_{T_{\mathcal{M}/\mathcal{D}}}) \xrightarrow{\phi^{\vee}} \mathfrak{E} = h^1/h^0((E^{\cdot}_{\geq -1})^{\vee})$$

where $\underline{C}_{\mathfrak{M}/\mathfrak{D}} \hookrightarrow \underline{N}_{\mathfrak{M}/\mathfrak{D}} \overset{\phi^\vee}{\hookrightarrow} \underline{E}$

closed immersions

$\underline{C}_{\mathfrak{M}/\mathfrak{D}}$ cone stack over $\mathfrak{M}$, pure dim m

$\underline{N}_{\mathfrak{M}/\mathfrak{D}}$ abelian cone stack over $\mathfrak{M}$

$\underline{E}$ vector bundle stack over $\mathfrak{M}$

rank $\underline{E} = -d$.

$d =$ rank$(E^\bullet) =$ relative virtual dim

$$[\mathfrak{M}, E^\bullet]^{vir} = 0^!_{\underline{E}} \underbrace{[\underline{C}_{\mathfrak{M}/\mathfrak{D}}]}_{\overset{\cap}{A_m(\underline{E})}} \in A_{m+d}(\mathfrak{M})$$

Application to GW theory

$\mathfrak{M} = \overline{\mathcal{M}}_{g,n}(X, \beta)$

$\mathfrak{D} = \mathfrak{M}^{pre}_{g,n}$ dim $3g-3+n$

$(E^\bullet)^\vee = R^\bullet\tilde{\pi}_* \tilde{f}^* TX$ relative virtual dim $\langle c_1(TX), \beta \rangle + \dim X (1-g)$

6.5 Virtual Fundamental Classes in Symplectic Gromov-Witten Theory

Many versions

- Fukaya-Ono : Kuranishi structures
- Li-Tian : Fredholm V-bundle over smoothly stratified orbispace
- Ruan : Virtual neighborhoods
  → Chen-Tian : virtual manifolds / orbifolds
  → Chen-Li-Wang
- Siebert
- Hofer-Wysoski-Zehnder : polyfolds
- McDuff-Wehrheim : Kuranishi atlases
  → Castellano (genus 0)
- Pardon : implicit atlases

(and more!)

(X, ω, J)

X compact manifold

ω symplectic form

J almost complex stucture compatible with ω, i.e.

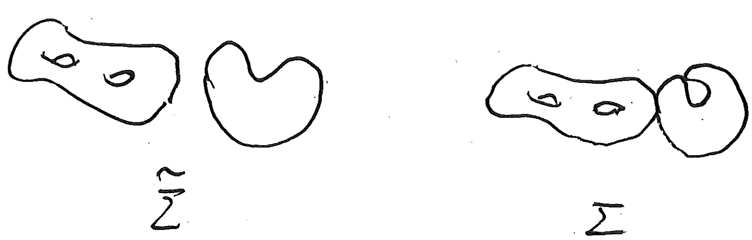
$g(u,v) = \omega(u, Jv)$ Riemannian metric

$\omega(u,v) = \omega(Ju, Jv)$

(Σ, j) (possibly nodal) Riemann surface

$\nu : (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ normalization

smooth, possibly disconnected



Let $f : \Sigma \to X$ be a continuous map

We say f is C^∞ if $\hat{f} := f \circ \nu : \hat{\Sigma} \to X$ is C^∞

We say f is J-holomorphic if it is C^∞

and $J \circ d\hat{f} = d\hat{f} \circ j : T\tilde{\Sigma} \to TX$

$$\Updownarrow$$

$$\bar{\partial}_J f \; \tfrac{1}{2}(df + J \circ df \circ j) = 0$$

We have

$$\begin{array}{c} \mathcal{E} \\ \pi \big\downarrow \big\uparrow s \\ \mathcal{M} = s^{-1}(0) \subset \mathcal{W} \end{array}$$

$\mathcal{W}$ = moduli of n-pointed, genus g, deg β stable C^∞ maps to X

(More precisely, we should consider $W^{1,p}$ maps where $p > 2$.)

$\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X, \beta)$ = moduli of n-pointed genus g, deg β, stable J-hol. maps to X

Suppose that $\xi = [f : (\Sigma, j, x_1, \ldots, x_n) \to (X, J)] \in \mathcal{W}$

and Σ is smooth

$$S(\xi) = \bar{\partial}_J f = \frac{1}{2}(df + J \circ df \circ j) \in \Omega^{0,1}((\Sigma, j), f^*TX) = \mathcal{E}_\xi$$

If moreover, $\mathrm{Aut}(\Sigma, j, x_1, \ldots, x_n)$ is trivial

then $\mathcal{W}$ is an ∞ dim'l manifold near ξ

$$T_\xi \mathcal{W} = T_{(\Sigma, j, x_1, \ldots, x_n)} \mathcal{M}_{g,n} \oplus T_f C^\infty(\Sigma, X)$$

$$= H^1(\Sigma, T_\Sigma(-x_1 \ldots -x_n)) \oplus C^\infty(\Sigma, f^*TX)$$

Let $dS_\xi : T_\xi \mathcal{W} \to T_{S(\xi)} \mathcal{E}$ denote the differential of $S : \mathcal{W} \to \mathcal{E}$ at ξ

If $\xi \in S^{-1}(0) = \mathcal{M}$

$$T_\xi \mathcal{W} \xrightarrow{dS_\xi} \underset{\substack{\| \\ (\xi, 0)}}{T_{S(\xi)}} \mathcal{E} = T_\xi \mathcal{W} \oplus \mathcal{E}_\xi$$

$$dS_\xi = (v, DS(\xi)(v))$$

$DS(\xi) : T_\xi \mathcal{W} \to \mathcal{E}_\xi$ linear

$i : \mathcal{W} \to \mathcal{E}$ inclusion of the zero section

$$T_\xi \mathcal{W} \xrightarrow{di_\xi} T_{(\xi,0)}\mathcal{E} = T_\xi \mathcal{W} \oplus \mathcal{E}_\xi$$

$$di_\xi(v) = (v, 0)$$

s intersects the zero section transversally at ξ

$\Leftrightarrow$ $Ds(\xi) : T_\xi \mathcal{W} \to \mathcal{E}_\xi$ is surjective

Restrict $Ds(\xi)$ to $C^\infty(\Sigma, f^*TX) \subset T_\xi \mathcal{W}$

$$D_\xi : C^\infty(\Sigma, f^*TX) \to \Omega^{0,1}(\Sigma, f^*TX)$$

If J is integrable then

$\mathrm{Ker}(D_\xi) = H^0(\Sigma, f^*TX)$

$\mathrm{Coker}(D_\xi) = H^1(\Sigma, f^*TX)$

If $H^1(\Sigma, f^*TX) = 0$ then $Ds(\xi)$ is surjective

$\mathcal{M}$ is smooth near ξ

$$0 \to T_\xi \mathcal{M} \to T_\xi \mathcal{W} \xrightarrow{Ds(\xi)} \mathcal{E}_\xi \to 0$$

$$\dim_{\mathbb{R}} T_\xi \mathcal{M} = 2(3g-3+n+h^0(\Sigma, f^*TX) - h^1(\Sigma, f^*TX))$$
$$= 2(\langle c_1(TX), \beta\rangle + (\dim_{\mathbb{C}} X - 3)(1-g) + n)$$

Kuranishi structure

$p = [f:(\Sigma, j, x_1 \cdots x_n) \to (X, J)] \in \mathcal{M} \subset \mathcal{W}$

For simplicity assume that $\mathrm{Aut}(p)$ is trivial

$E_p \subset \mathcal{E}_p$ finite dimensional vector space

$\mathrm{Im}(Ds(p)) + E_p = \mathcal{E}_p$

Then $\exists$ open neighborhood $\mathcal{U}_p$ of p in $\mathcal{W}$

$$\begin{tikzcd}
 & \mathcal{E}|_{\mathcal{U}_p} \cong \mathcal{U}_p \times \mathcal{E}_p \arrow[r, "pr_2"] \arrow[d] & \mathcal{E}_p \arrow[d] \\
 & \mathcal{U}_p \arrow[u, bend left, "s|_{\mathcal{U}_p}"] \arrow[u, bend right, "(id, s_p)"'] \arrow[ur, "s_p"] \arrow[r, "\tilde{s}_p"'] & \mathcal{E}_p/E_p
\end{tikzcd}$$

$p \in s^{-1}(0) \cap \mathcal{M} \subset \tilde{s}_p^{-1}(0) \subset \mathcal{U}_p$

$s^{-1}(0) \cap \mathcal{M} = \mathcal{M}_p$, $\quad s_p^{-1}(E_p) = \tilde{s}_p^{-1}(0) = U_p$ ← smooth manifold

$\tilde{s}_p$ ← submersion

$(d\tilde{s}_p)_\xi : \mathcal{U}_p \to \mathcal{E}_p/E_p$

surjective $\forall \xi \in \mathcal{U}_p$

$$s_p : U_p \to E_p$$

$$\mathcal{M}_p = s_p^{-1}(0)$$

$$\dim_{\mathbb{R}} U_p - \mathrm{rank}_{\mathbb{R}} E_p = 2\,\mathrm{virdim}$$

$$\mathrm{virdim} = \langle c_1(TX), \beta \rangle + (\dim_{\mathbb{C}} X - 3)(1-g) + n$$

$$\underbrace{s_p + \nu_p}_{\text{perturbation}} : U_p \to E_p \quad \text{submersion}$$

$$\mathcal{M}_p^{\nu} := (s_p + \nu_p)^{-1}(0) \subset U_p$$

smooth submfd, $\dim_{\mathbb{R}} \mathcal{M}_p^{\nu} = 2\,\mathrm{virdim}$

More generally $\Gamma_p := \mathrm{Aut}(p)$ finite group

$$\begin{array}{l} \mathcal{E}|_{\mathcal{U}_p} \cong (\mathcal{V}_p \times \mathcal{E}_p)/\Gamma_p \\ \downarrow \\ \mathcal{U}_p = \mathcal{V}_p/\Gamma_p \end{array}$$

$s_p : \mathcal{V}_p \to \mathcal{E}_p$ Γ_p-equivariant

$E_p \subset \mathcal{E}_p$ Γ_p-invariant

$$\tilde{s}_p : \mathcal{V}_p \to \mathcal{E}_p / E_p$$

$$V_p = \tilde{s}_p^{-1}(0) = s_p^{-1}(E_p)$$

$s_p : V_p \to E_p$ $\quad\Gamma_p$-equivariant

$$\mathcal{M}_p = s_p^{-1}(0)/\Gamma_p$$

$s_p + \nu_p : V_p \to E_p$ submersion

ν_p: perturbation, not Γ_p-invariant in general

$\mathcal{M}_p^\nu = (s_p + \nu_p)^{-1}(0) \subset V_p$ smooth submfd $\dim_{\mathbb{R}} \mathcal{M}_p^\nu = 2$ virdim

$\pi_p : V_p \to V_p/\Gamma_p \subset \mathcal{V}_p/\Gamma_p = \mathcal{U}_p$

$\frac{1}{|\Gamma_p|} \pi_p(\mathcal{M}_p^\nu)$ $\quad$ 2virdim-chain in V_p/Γ_p

Technical issues

① When $\mathrm{Aut}(\Sigma, j, x_1, \ldots x_n)$ has positive dim

e.g. $(\Sigma, j) = \mathbb{P}^1$, $n \le 2$

action of the group of reparametrization

not smooth in the usual sense

(2) lower strata : gluing profile

(3) How to patch the $\mathbb{Q}$-chains to obtain a $\mathbb{Q}$-cycle whose class is independent of choices of the Kuranishi structure and perturbation

(1) and (2) analytic

(3) topological