Evaluation Maps

$$ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X \qquad i=1,\dots,n$$

Set theoretic definition

$$[f:(C,x_1,\dots,x_n)\to X] \mapsto f(x_i)$$

Natural transformation of functors

$$ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to h_X$$

$B \in Sch$

$$\overline{\mathcal{M}}_{g,n}(X,\beta)(B) \xrightarrow{ev_i(B)} h_X(B) = Mor_{Sch}(B,X)$$

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \\ B \arrow[u, bend left, "s_i"] \end{tikzcd} \quad \mapsto \quad f\circ s_i : B \to X$$

$i=1,\dots,n$

$$B_1 \xrightarrow{\phi} B_2$$

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n}(X,\beta)(B_2) \arrow[r] \arrow[d, "ev_i(B_2)"'] & \overline{\mathcal{M}}_{g,n}(X,\beta)(B_1) \arrow[d, "ev_i(B_1)"] \\
Mor_{Sch}(B_2, X) \arrow[r, "h_X(\phi)"] & Mor_{Sch}(B_1, X)
\end{tikzcd}$$

$$[u: B_2 \to X] \longmapsto [u \circ \phi: B_1 \to X]$$

$$\begin{tikzcd}
 & X \\
B_1 \times_{B_2} C \arrow[ur, "\phi^* f"] \arrow[r] \arrow[d, "\phi^*\pi"] & C \arrow[u, "f"'] \arrow[d, "\pi"] \\
B_1 \arrow[u, bend left, "\phi^* s_i"] \arrow[r, "\phi"] & B_2 \arrow[u, bend left, "s_i"]
\end{tikzcd}$$

$$\phi^* f \circ \phi^* s_i = f \circ s_i \circ \phi$$

Forget the map

$\overline{\mathcal{M}}^{pre}_{g,n}$ fine moduli stack of n-pointed genus g prestable curves

Smooth Artin stack of dim $3g-3+n$

not Deligne-Mumford

highly non-separated

$\overline{\mathcal{M}}_{g,n}(X,\beta)$ fine moduli stack of n-pointed genus g degree β stable maps to X

proper DM stack

(usually singular)

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}^{pre}_{g,n}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & X \\ B \arrow[u, bend left, "s_i"] \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] \\ B \arrow[u, bend left, "s_i"] \end{tikzcd}$$

$B \leftarrow$ scheme

5.4 Deformation of the domain $(C, x_1, \ldots, x_n)$

$(C, x_1, \ldots, x_n)$ n-pointed genus g prestable curve

$D = x_1 + \cdots + x_n$

infinitesimal automorphisms of $(C, x_1, \ldots, x_n)$

$\mathrm{Aut}(C, x_1, \ldots, x_n) = \mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C)$

infinitesimal deformation of $(C, x_1, \ldots, x_n)$

$\mathrm{Def}(C, x_1, \ldots, x_n) = \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C)$

$(C, x_1, \ldots, x_n)$ is stable $\Leftrightarrow$ $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C) = 0$

If C is smooth then
$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C) &= \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C, T_C(-D)) \\ &= H^1(C, T_C(-D)) \end{aligned}$$

$H^0(C, T_C(-D))$ space of hol. vector fields on C vanishing at $x_1, \ldots, x_n$

In general, we may relate $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C))$ where C nodal to $\mathrm{Ext}^1_{\mathcal{O}_{\tilde{C}}}(\Omega_{\tilde{C}}(\tilde{D}), \mathcal{O}_{\tilde{C}}) = H^1(\tilde{C}, T_{\tilde{C}}(-\tilde{D}))$

$\nu : \tilde{C} \to C$ normalization

smooth, possibly disconnected

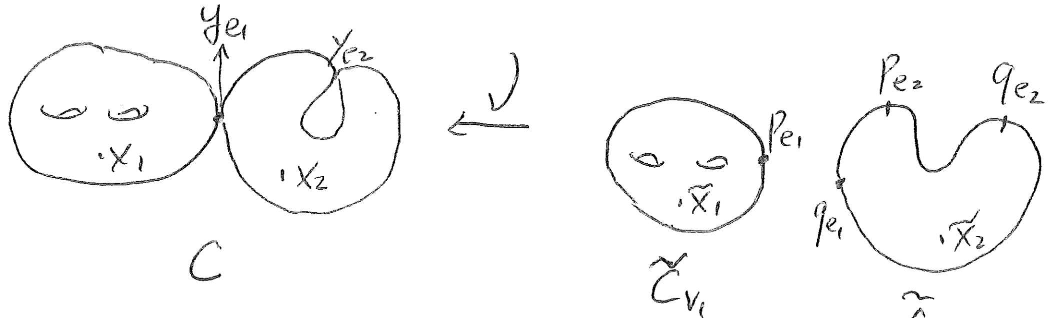
Γ dual graph of $(C, x_1, \ldots, x_n)$

$C_{sing} = \{ y_e : e \in E(\Gamma) \}$

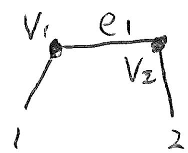
$\nu^{-1}(y_e) = \{ p_e, q_e \} \subset \tilde{C}$

$\nu^{-1}(x_i) = \{ \tilde{x}_i \}$

$\hat{D} = \sum_{i=1}^{n} \tilde{x}_i + \sum_{e \in E(\Gamma)} (p_e + q_e)$



Γ



$\tilde{D}_{v_1} = \tilde{x}_1 + p_{e_1}$ $\qquad$ $\tilde{D}_{v_2} = \tilde{x}_2 + q_{e_1} + p_{e_2} + q_{e_2}$

$g(v_1) = 2, \; g(v_2) = 0$

$(\tilde{C}, \tilde{D}) = \bigsqcup_{v \in V(\Gamma)} (\tilde{C}_v, \tilde{D}_v)$

disjoint union of connected components

$$\begin{tikzcd} \tilde{U} = \nu^{-1}(U) \arrow[r, phantom, "\overset{\text{open}}{\subset}"] \arrow[d, phantom, "\wr\|"] & \tilde{C} \arrow[d, "\nu"] \\ U := C - C_{\text{sing}} \arrow[r, phantom, "\overset{\text{open}}{\subset}"] & C \end{tikzcd}$$

$$\mathcal{H}om_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C)|_U = \mathcal{H}om_{\mathcal{O}_C}(\Omega^1_U(D), \mathcal{O}_U) = T_U(-x_1 - \cdots x_n)$$

$$(\nu|_{\tilde{U}})^*(\mathcal{H}om_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C)|_U) = T_{\tilde{U}}(-\tilde{x}_1 \cdots \tilde{x}_n)$$

Near a node $y_e \in C$

Local model $p = (0,0) \in \hat{C} = \operatorname{Spec}\left(\frac{\mathbb{C}[x,y]}{\langle xy \rangle}\right) \subset \mathbb{C}^2$

Consider the conormal sequence, which is a free resolution of $\Omega^1_{\hat{C}}$:

$$0 \to \underset{\substack{\| \\ N^*_{\hat{C}/\mathbb{C}^2} \\ \text{conormal line}}}{\mathcal{O}_{\mathbb{C}^2}(-\hat{C}) \otimes \mathcal{O}_{\hat{C}}} \to \Omega^1_{\mathbb{C}^2} \otimes \mathcal{O}_{\hat{C}} \to \Omega^1_{\hat{C}} \to 0$$

$$0 \to Hom_{\mathcal{O}_{\hat{C}}}(\Omega^1_{\hat{C}}, \mathcal{O}_{\hat{C}}) \to Hom_{\mathcal{O}_{\hat{C}}}(\Omega^1_{\mathbb{C}^2} \otimes \mathcal{O}_{\hat{C}}, \mathcal{O}_{\hat{C}})$$

$$\to Hom_{\mathcal{O}_{\hat{C}}}(\mathcal{O}_{\mathbb{C}^2}(-\hat{C}) \otimes \mathcal{O}_{\hat{C}}, \mathcal{O}_{\hat{C}}) \to Ext^1_{\mathcal{O}_{\hat{C}}}(\Omega^1_{\hat{C}}, \mathcal{O}_{\hat{C}}) \to 0$$

$$0 \to Hom_{\mathcal{O}_{\hat{C}}}(\Omega^1_{\hat{C}}, \mathcal{O}_{\hat{C}}) \to H^0(\hat{C}, \underbrace{T_{\mathbb{C}^2} \otimes \mathcal{O}_{\hat{C}}}_{\substack{\text{rank } 2 \\ \text{vector bundle}}}) \to H^0(\hat{C}, \underbrace{\mathcal{O}_{\mathbb{C}^2}(\hat{C}) \otimes \mathcal{O}_{\hat{C}}}_{\substack{= N_{\hat{C}/\mathbb{C}^2} \\ \text{conormal line}}})$$

$$\to Ext^1_{\mathcal{O}_{\hat{C}}}(\Omega^1_{\hat{C}}, \mathcal{O}_{\hat{C}}) \to 0$$

$$A = \mathbb{C}[x,y]/\langle xy \rangle$$

$$0 \to Hom_{\mathcal{O}_{\hat{C}}}(\Omega^1_{\hat{C}}, \mathcal{O}_{\hat{C}}) \to A\frac{\partial}{\partial x} \oplus A\frac{\partial}{\partial y} \to A \to Ext^1_{\mathcal{O}_{\hat{C}}}(\Omega^1_{\hat{C}}, \mathcal{O}_{\hat{C}}) \to 0$$

$$Hom_{\mathcal{O}_{\hat{C}}}(\Omega^1_{\hat{C}}, \mathcal{O}_{\hat{C}}) = Ax\frac{\partial}{\partial x} \oplus Ay\frac{\partial}{\partial y} = \mathbb{C}[x]x\frac{\partial}{\partial x} \oplus \mathbb{C}[y]y\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial x} \longmapsto y$$

$$\frac{\partial}{\partial y} \longmapsto x$$

Globally,

$$Hom_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C) = \nu_* T_{\tilde{C}}(-\tilde{D})$$

$$Ext^1_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C) = \bigoplus_{e \in E(\Gamma)} T_{p_e}\tilde{C} \otimes T_{q_e}\tilde{C} \otimes \mathcal{O}_{y_e}$$

By the "local-to-global" spectral sequence for Ext's

$$\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C) = H^0(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C))$$

$$= H^0(C, \nu_* T_{\tilde{C}}(-\tilde{D})) = H^0(\tilde{C}, \nu_* T_{\tilde{C}}(-\tilde{D}))$$

$$= \bigoplus_{v \in V(\Gamma)} H^0(\tilde{C}_v, T_{\tilde{C}_v}(-\tilde{D}_v))$$

We have a short exact sequence of vector spaces over $\mathbb{C}$:

$$0 \to H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C)) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C)$$
$$\to H^0(C, \mathcal{E}xt_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C)) \to 0$$

where $H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C))$

$$= H^1(C, \nu_* T_{\tilde{C}}(-\tilde{D})) = \bigoplus_{v \in V(\Gamma)} H^1(\hat{C}_v, T_{\tilde{C}_v}(-\tilde{D}_v))$$

$$H^0(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega^1_C(D), \mathcal{O}_C)) = H^0(C, \bigoplus_{e \in E(\Gamma)} (T_{p_e}\hat{C} \otimes T_{q_e}\tilde{C}) \otimes \mathcal{O}_{y_e})$$

$$= \bigoplus_{e \in E(\Gamma)} \underbrace{T_{p_e}\hat{C} \otimes T_{q_e}\tilde{C}}_{\substack{\text{smoothing of the node } y_e \\ xy = t}}$$

Reference : Arbarello, Cornalba, Griffiths
Geometry of Algebraic Curves II
Chapter XI Section 3. Deformation of nodal curves

If $\xi = [(C, x_1, \ldots x_n)]$ is stable then

$$\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C) = 0$$

$$T_\xi \overline{\mathcal{M}}_{g,n} = \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C)$$

$$T_\xi \mathcal{M}_\Gamma = \bigoplus_{v \in V(\Gamma)} H^1(\tilde{C}_v, T_{\tilde{C}_v}(-\hat{D}_v))$$

$$\mathcal{M}_\Gamma = \left[\left(\prod_{v \in V(\Gamma)} \mathcal{M}_{g(v), n_v} \right) / \mathrm{Aut}(\Gamma) \right]$$

$$n_v = \deg(\hat{D}_v)$$

$$\left(N_{\mathcal{M}_\Gamma / \overline{\mathcal{M}}_{g,n}} \right)_\xi = \bigoplus_{e \in E(\Gamma)} T_{p_e} \tilde{C} \otimes T_{q_e} \tilde{C}$$

5.5 Deformation of the map f

$\mathcal{T} = \mathcal{T}^1 \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ tangent sheaf

$Ob = \mathcal{T}^2 \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ obstruction sheaf

Given $\xi = [f : (C, x_1 \ldots x_n) \to X] \in \overline{\mathcal{M}}_{g,n}(X,\beta)$, we have the following exact sequence of vector spaces over $\mathbb{C}$

$$0 \to \mathrm{Aut}(C, x_1, \ldots, x_n) \to \mathrm{Def}(f) \to \mathcal{T}^1_\xi$$
$$\to \mathrm{Def}(C, x_1 \ldots x_n) \to \mathrm{Obs}(f) \to \mathcal{T}^2_\xi \to 0$$

$\mathrm{Def}(f) = H^0(C, f^*TX)$ infinitesimal deformation of f (for fixed domain C)

$\mathrm{Obs}(f) = H^1(C, f^*TX)$ obstruction to deforming f

$\mathcal{T}^1_\xi = \mathcal{T}_\xi$ tangent space at ξ

$\mathcal{T}^2_\xi = Ob_\xi$ obstruction space at ξ

$$h^0(C, f^*TX) - h^1(C, f^*TX)$$
$$= \deg(f^*TX) + \mathrm{rank}(f^*TX)(1-g)$$
$$= \langle c_1(TX), \beta \rangle + \dim X (1-g)$$

virtual / expected dimension

$= \dim T^1_\xi - \dim T^2_\xi = 3g-3+n+\langle c_1(TX),\beta\rangle + \dim X(1-g)$

$= \langle c_1(TX),\beta\rangle + (\dim X - 3)(1-g)+n$

Suppose that $(C, x_1 \dots x_n)$ is stable

($\Leftrightarrow \mathrm{Aut}(C, x_1 \dots x_n) = 0$) and $H^1(C, f^*TX) = 0$.

Then $T^2_\xi = 0$, and we have

$$0 \to \underset{\mathrm{Def}(f)}{\underset{\shortparallel}{H^1(C, f^*TX)}} \to T^1_\xi \to \underset{\mathrm{Def}(C, x_1 \dots x_n)}{\underset{\shortparallel}{\mathrm{Ext}^1(\Omega_C(x_1+\dots+x_n), \mathcal{O}_C)}} \to 0$$

Write $\mathcal{M}_X = \overline{\mathcal{M}}_{g,n}(X,\beta)$, $\mathcal{U}_X = \mathcal{U}_{g,n}(X,\beta)$

$\mathcal{M} = \overline{\mathcal{M}}^{\,pre}_{g,n}$

We have $T_{\mathcal{M}_X/\mathcal{M}} \to \mathcal{M}_X$ relative tangent sheaf

$Ob_{\mathcal{M}_X/\mathcal{M}} \to \mathcal{M}_X$ relative obstruction sheaf

$$\begin{array}{ccc} \mathcal{U}_X & \xrightarrow{\hat{f}} & X \\ \downarrow \hat{\pi} & & \\ \mathcal{M}_X & & \end{array} \qquad \begin{array}{l} T_{\mathcal{M}_X/\mathcal{M}} = \hat{\pi}_* \hat{f}^* TX \\ Ob_{\mathcal{M}_X/\mathcal{M}} = R^1\hat{\pi}_* \hat{f}^* TX \end{array}$$

$$\begin{tikzcd}
\mathcal{U}_X = \mathcal{U}_{g,n}(X,\beta) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}"'] & X \\
\mathcal{M}_X = \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[d] & \\
\mathcal{M} = \overline{\mathcal{M}}^{pre}_{g,n} &
\end{tikzcd}$$

<u>Proposition</u> There exists a two term complex of vector bundles

$$\mathbb{F}^{\cdot} = [\,0 \to F^0 \xrightarrow{\psi} F^1 \to 0\,] \quad \text{over } \mathcal{M}_X$$

such that

$$h^0(\mathbb{F}^{\cdot}) = \ker(\psi) = R^0\tilde{\pi}^*\tilde{f}^*T_X$$
$$h^1(\mathbb{F}^{\cdot}) = \operatorname{coker}(\psi) = R^1\tilde{\pi}^*\tilde{f}^*T_X$$

i.e. we have the following exact sequence

$$0 \to R^0\hat{\pi}^*\hat{f}^*T_X \to F^0 \xrightarrow{\psi} F^1 \to R^1\hat{\pi}^*\hat{f}^*T_X \to 0$$

Outline of proof

Let M be an ample line bundle on the target X.

If $f:(C, x_1, \dots, x_n) \to X$ is a stable map then $L := \omega_C(x_1 + \dots + x_n) \otimes (f^*M)^{\otimes 3}$ is ample

(Check: $C' \subset C$ irreducible component $\Rightarrow \deg(L|_{C'}) > 0$)

$g, n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X; \mathbb{Z})$ fixed

$\exists$ a positive integer $N \gg 0$ such that for any n-pointed, genus g, deg β stable map $f:(C, x_1, \dots, x_n) \to X$

① $H^0(C, f^*T_X \otimes L^{\otimes N}) \otimes \mathcal{O}_C \to f^*TX \otimes L^{\otimes N}$ is surjective i.e. $f^*T_X \otimes L^{\otimes N}$ is generated by global sections

② $H^1(C, f^*TX \otimes L^{\otimes N}) = 0$

③ $H^0(C, L^{-N}) = 0$

By ①, we have the following short exact sequence of vector bundles over C

$$0 \to H \to \underbrace{H^0(C, f^*TX \otimes L^{\otimes N}) \otimes L^{-N}}_{F} \to f^*TX \to 0$$

where $\operatorname{rank} F = \dim H^0(C, f^*TX \otimes L^N)$

$$\overset{②}{=} \deg(f^*TX \otimes L^N) + \dim X(1-g)$$

$$= \langle c_1(TX), \beta\rangle + N\left(2g-2+n+3\langle c_1(M), \beta\rangle\right) + \dim X(1-g)$$

Globally, we have the following short exact sequence of vector bundles over $\mathcal{U}_X$

$$0 \to \hat{H} \to \tilde{F} \to \hat{f}^*T_X \to 0$$

where $\hat{F} = \tilde{\pi}_*\left(\hat{f}^*T_X \otimes (\omega_{\tilde{\pi}}(\tilde{D}) \otimes \tilde{f}^*M^{\otimes 3})^N\right)$

$$\begin{array}{ccc} \tilde{D} \subset \mathcal{U}_X & \xrightarrow{\tilde{f}} & X \\ \tilde{\pi}\downarrow & & \\ \mathcal{M}_X & & \end{array}$$

$\tilde{D}$ universal divisor of marked pts

$$\begin{array}{c} H^0(C, L^{-N})^{\oplus \operatorname{rank} F} \overset{(3)}{=} 0 \\ \| \end{array}$$

$$0 \to H^0(C,H) \to H^0(C,F) \to H^0(C, f^*TX)$$
$$\to H^1(C,H) \to H^1(C,F) \to H^1(C, f^*TX) \to 0$$

Globally, we have the following sequence of $\mathcal{O}_{\mathcal{M}_X}$-modules on $\mathcal{M}_X$:

$$0 \to \tilde{\pi}_* \hat{f}^* TX \to \underbrace{R^1 \tilde{\pi}_* \tilde{H}}_{F^0} \to \underbrace{R^1 \hat{\pi}_* \tilde{F}}_{F^1} \to R^1 \hat{\pi}_* f^* TX \to 0$$

vector bundles since
$$\hat{\pi}_* \tilde{H} = \hat{\pi}_* \hat{F} = 0$$