Fact

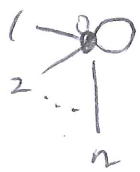
$$\pi_1(\overline{\mathcal{M}}_{g,n}) = 0 \Rightarrow H_1(\overline{\mathcal{M}}_{g,n}; \mathbb{Z}) = 0$$

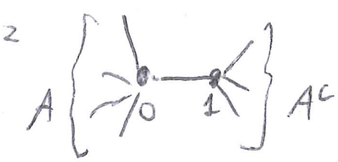
$$\Rightarrow H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Z}) \text{ torsion free}$$

Fact

$$H^2(\overline{\mathcal{M}}_{1,n}; \mathbb{Q}) \left(= H^2(\overline{M}_{1,n}; \mathbb{Q}) \right)$$

$$= \mathbb{Q}\delta_{irr} \oplus \bigoplus_{\substack{A \subset \{1,\dots,n\} \\ |A| \geq 2}} \mathbb{Q}\,\delta_{0,A} \quad \cong \mathbb{Q}^{\oplus n}$$





δ_{irr}, $\{\delta_{0,A} : A \subset \{1,\dots,n\}, |A| \geq 2\}$

is a $\mathbb{Q}$-basis of $H^2(\overline{\mathcal{M}}_{1,n}; \mathbb{Q})$

but not a $\mathbb{Z}$ basis of $H^2(\overline{\mathcal{M}}_{1,n}; \mathbb{Z})$

Example $\underset{\mathbb{Z}\psi_1}{\underset{\shortparallel}{H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Z})}} \supsetneq \mathbb{Z}\delta_{irr} \supsetneq \underset{\mathbb{Z}H}{\underset{\shortparallel}{H^2(\overline{M}_{1,1}; \mathbb{Z})}}$

$H = 2\delta_{irr}$

$\delta_{irr} = 12\psi_1$

$$0 \to \mathbb{Z}\delta_{irr} \to \underset{\underset{\mathbb{Z}\psi_1}{\parallel}}{H^2(\overline{\mathcal{M}}_{1,1};\mathbb{Z})} \to \underset{\underset{\mathbb{Z}/12\mathbb{Z}}{\parallel}}{H^2(\mathcal{M}_{1,1};\mathbb{Z})} \to 0$$

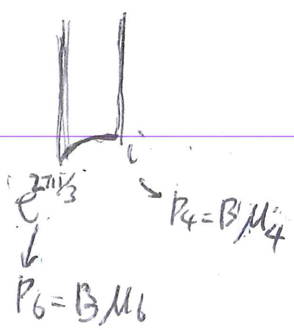
Mumford, "Picard groups of moduli problems" (1965)

<u>Aside</u> G finite group

$$\mathrm{Hom}(G, \mathbb{C}^*) \cong \mathrm{Pic}(BG)$$

$$\begin{array}{ccc} \chi : G \to \mathbb{C}^* & \mapsto & \mathcal{L}_\chi = [\mathbb{C}/G] \\ \text{character} & & \downarrow \\ & & BG = [\{0\}/G] \end{array}$$

$$\mathcal{M}_{1,1} = \left[\mathbb{H}/SL(2,\mathbb{Z})\right]$$



$\mu_m \subset \mathbb{C}^*$
group of m-th roots of 1

$\mathrm{Pic}(B\mu_m) \cong \mathbb{Z}/m\mathbb{Z}$

$$\begin{array}{ccc} \mathcal{L}_1|_{\mathcal{M}_{1,1}} & \ni & T_0(\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{M}_{1,1} & \ni & [\tau] \end{array}$$

$$\mathrm{Pic}(\mathcal{M}_{1,1}) \hookrightarrow \mathrm{Pic}(P_4) \times \mathrm{Pic}(P_6)$$

$$\mathcal{L}_1|_{\mathcal{M}_{1,1}} \mapsto (1, 1)$$

Descendant Integrals

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \in \mathbb{Q}$$

zero unless $3g-3+n = d_1 + \cdots + d_n$

$$\langle \tau_0^3 \rangle_0 = \langle \tau_0 \tau_0 \tau_0 \rangle_0 = \int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$$

$$\langle \tau_1 \rangle_1 = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$$

String Equation (to be proved later)

$2g-2+n>0$

$$\langle \tau_{d_1} \cdots \tau_{d_n} \tau_0 \rangle_g = \sum_{j=1}^{n} \langle \tau_{d_1} \cdots \tau_{d_{j-1}} \tau_{d_j - 1} \tau_{d_{j+1}} \cdots \tau_n \rangle_g$$

$\overline{\mathcal{M}}_{g,n+1}$ $\qquad\qquad$ $\overline{\mathcal{M}}_{g,n}$

Exercise string equation + $\langle \tau_0^3 \rangle_0 = 1$

$$\Rightarrow \langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \begin{cases} \binom{n-3}{d_1 \cdots d_n} = \dfrac{(n-3)!}{\prod_{i=1}^{n} d_i!} & \text{if } d_1 + \cdots + d_n = n-3 \\ 0 & \text{otherwise} \end{cases}$$

Dilaton Equation (to be proved later)

$$\langle \tau_{d_1} \cdots \tau_{d_n} \tau_1 \rangle_g = (2g-2+n) \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$$

$\overline{\mathcal{M}}_{g,n+1}$ $\qquad\qquad$ $\overline{\mathcal{M}}_{g,n}$

Witten Conjecture

$\{ \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \}$ satisfy KdV

KdV + string equation + $\langle \tau_0^3 \rangle = 1$

$\Rightarrow$ evaluation of all descendant integrals $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$

First proof: Kontsevich

Other proofs: Okounkov-Pandharipande, Mirzakhani, Kim-Liu, Kazarian-Lando, Chen-Li-Liu, Mulase-Zhang, ...

Keteng Liu

5. Moduli of Stable Maps

X nonsingular projective variety $/\mathbb{C}$

5.1 Prestable maps and stable maps

Definition $g, n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X;\mathbb{Z})$

An n-pointed genus g deg β prestable map to X is a morphism

$$f : (C, x_1, \ldots, x_n) \to X$$

where $(C, x_1, \ldots, x_n)$ is an n-pointed genus g prestable curve and $f_*[C] = \beta$

isomorphism

$$\begin{tikzcd} (C, x_1 \ldots x_n) \arrow[d, "\psi"] \arrow[dr, "f"] & \\ (C', x_1' \ldots x_n') \arrow[r, "f'"] & X \end{tikzcd}$$

$\psi : C \to C'$ isomorphism

$\psi(x_i) = x_i'$

$f' \circ \psi = f$

<u>Definition</u>

$f : (C, x_1, \ldots, x_n) \to X$ prestable map

An irreducible component $C' \subset C$ is a <u>ghost component</u> if $f(C')$ is a point.

<u>Definition</u>

A prestable map $\xi = [f : (C, x_1 \ldots x_n) \to X]$ is <u>stable</u> if $\mathrm{Aut}(\xi)$ is finite

<u>Definition</u> $\xi = [f : (C, x_1, \ldots, x_n) \to X]$ prestable map to X. TFAE:

(1) $\mathrm{Aut}(\xi)$ is finite

(2) If $C' \subset C$ is a ghost component then $(C', \underbrace{(\{x_1, \ldots, x_n\} \sqcup C_{sing}) \cap C'}_{(y_1, \ldots, y_k)})$ is a (pointed) <u>stable</u> curve.

5.2 Moduli functors and moduli spaces

Definition $g, n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X;\mathbb{Z})$

A family of n-pointed genus g, deg β stable map to X over a scheme B (over $\mathbb{C}$) is

$$\begin{array}{c} C \xrightarrow{f} X \\ s_i \left(\big\downarrow \pi \right. \\ B \end{array}$$

$i = 1, \dots, n$

where $\begin{array}{c} C \\ s_i \left(\big\downarrow \pi \right. \\ B \end{array}$ is a family of n-pointed,

genus g prestable curves. (in particular, π is flat and projective) such that $\forall b \in B$

$f|_{C_b} : (C_b, s_1(b), \dots, s_n(b)) \to X$ is an n-pointed genus g deg β stable map to X.

isomorphism

$$\begin{tikzcd} & X & \\ \mathcal{C} \arrow[ur, "f"] \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & \mathcal{C}' \arrow[ul, "f'"'] \arrow[dl, "\pi'"] \\ & B \arrow[ul, bend left, "s_i"] \arrow[ur, bend right, "s_i'"'] & \end{tikzcd}$$

$\psi : \mathcal{C} \to \mathcal{C}'$ isomorphism

$\pi' \circ \psi = \pi$

$s_i' = \psi \circ s_i$

$f' \circ \psi = f$

Define two contravariant functors

(1) $\overline{F}_{g,n}(X,\beta) : Sch \longrightarrow Set$

$$B \longmapsto \left\{ \begin{array}{c} \mathcal{C} \xrightarrow{f} X \\ s_i \uparrow \downarrow \pi \\ B \\ i=1,\dots,n \end{array} \quad \begin{array}{c} \text{family of} \\ \text{n-pointed genus } g \\ \deg \beta \text{ stable maps} \\ \text{to } X \text{ over } B \end{array} \right\} \Big/ \cong$$

set of isomorphism classes of familie of n-pointed genus g deg β stable maps to X over B

(2) $\overline{\mathcal{F}}_{g,n}(X,\beta)$: Sch $\to$ Groupoid

$$B \mapsto \overline{\mathcal{F}}_{g,n}(X,\beta)(B)$$

<u>objects</u>: families of n-pointed, genus g, deg β stable maps to X over B

<u>morphisms</u> isomorphisms (*) on p.52.

<u>Fact</u> (1) $\overline{\mathcal{F}}_{g,n}(X,\beta)$ is represented by a proper Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$

(2) There exists a projective scheme $\overline{M}_{g,n}(X,\beta)$ which is the coarse moduli space of $\overline{\mathcal{F}}_{g,n}(X,\beta)$

K. Behrend, Y. Manin, "Stacks of stable maps and Gromov-Witten invariants" Duke 1996

W. Fulton and R. Pandharipande, "Notes on stable maps and quantum cohomology." Proc. Sympos. Pure Math, 1997.

$i=1,\dots,n$

$$ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$$
$$[f:(C,x_1,\dots,x_n)\to X] \mapsto f(x_i)$$

evaluation at the i-th marked pt

5.3 Universal family

$\overline{\mathcal{M}}_{g,n}(X,\beta)$ is the fine moduli, so there is a universal family of n-pointed genus g, deg β stable maps to X over $\overline{\mathcal{M}}_{g,n}(X,\beta)$

$$\begin{tikzcd} \mathcal{U}_{g,n}(X,\beta) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}"] & X \\ \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[u, bend left, "\tilde{s}_i"] & \end{tikzcd}$$
$(i=1,\dots,n)$

such that if
$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & X \\ B \arrow[u, bend left, "s_i"] & \end{tikzcd}$$
$(i=1,\dots,n)$ is a family

of n-pointed, genus g, deg β stable maps to X over a scheme B, then there exists a morphism

$\phi : B \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ B \arrow[u, bend left, "s_i"] & \end{tikzcd}$$

is isomorphic to

$$\begin{tikzcd} B \times_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \mathcal{U}_{g,n}(X,\beta) \arrow[r, "\tilde{\phi}"] \arrow[d, "\phi^*\tilde{\pi}"] \arrow[rr, bend left, "\bar{f}\circ\tilde{\phi}"] \arrow[dr, phantom, "\square"] & \mathcal{U}_{g,n}(X,\beta) \arrow[r, "\bar{f}"] \arrow[d, "\tilde{\pi}"] & X \\ B \arrow[r, "\phi"] \arrow[u, bend left, "\phi^*\tilde{s}_i"] & \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[u, bend left, "\tilde{s}_i"] & \end{tikzcd}$$

$F : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)$

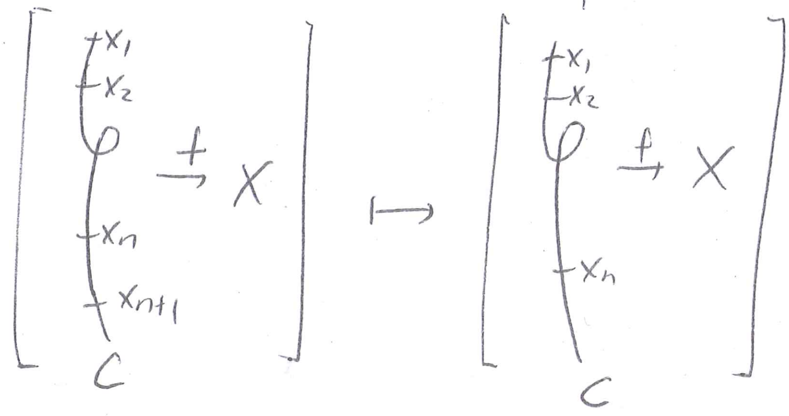
forget the $(n+1)$-th marked point

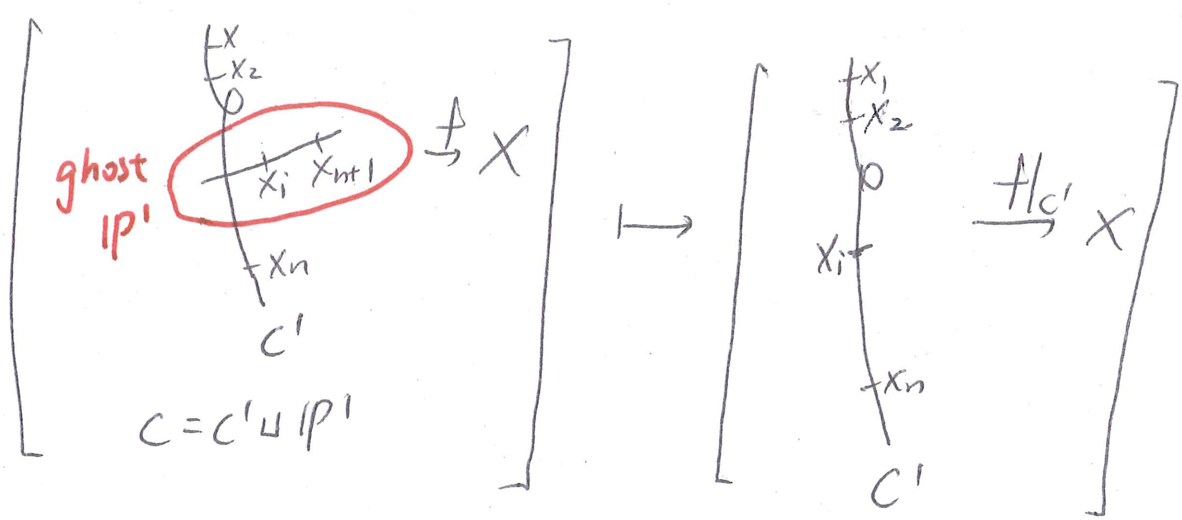
(and contracting the unstable ghost component)

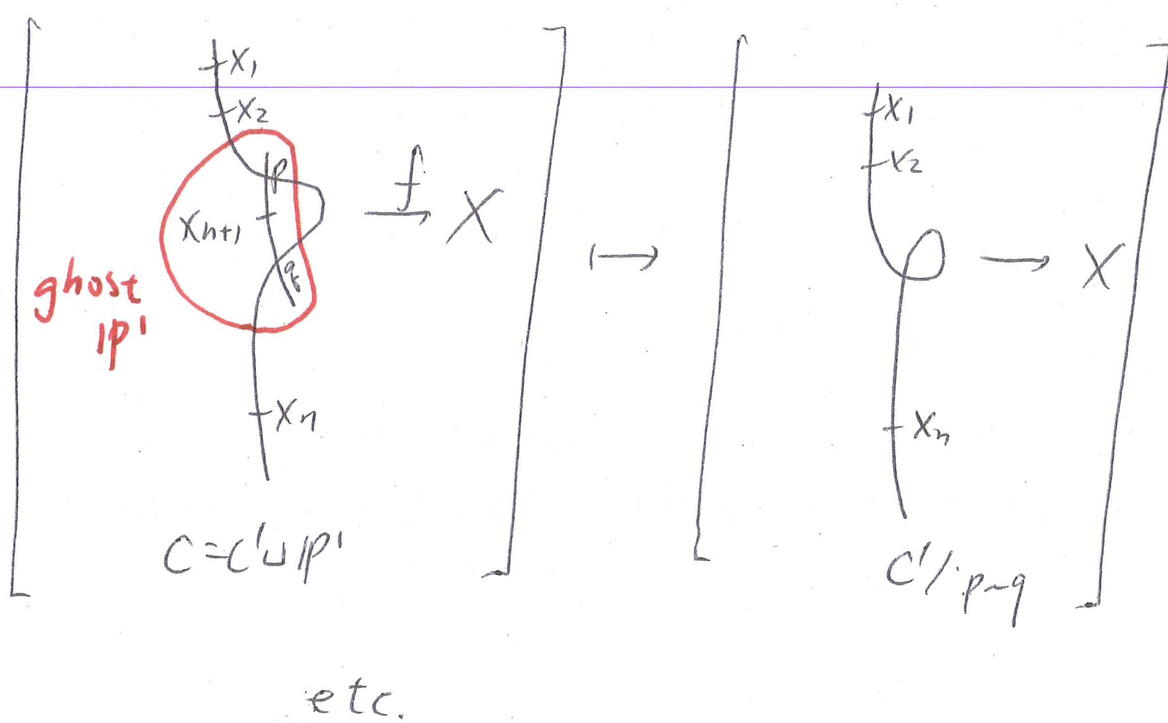
can be identify with $\tilde{\pi} : \mathcal{U}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$

Set-Theoretic Description of

$F: \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$







etc.

Let $\psi : \mathcal{U}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n+1}(X,\beta)$

be the isomophism. Then

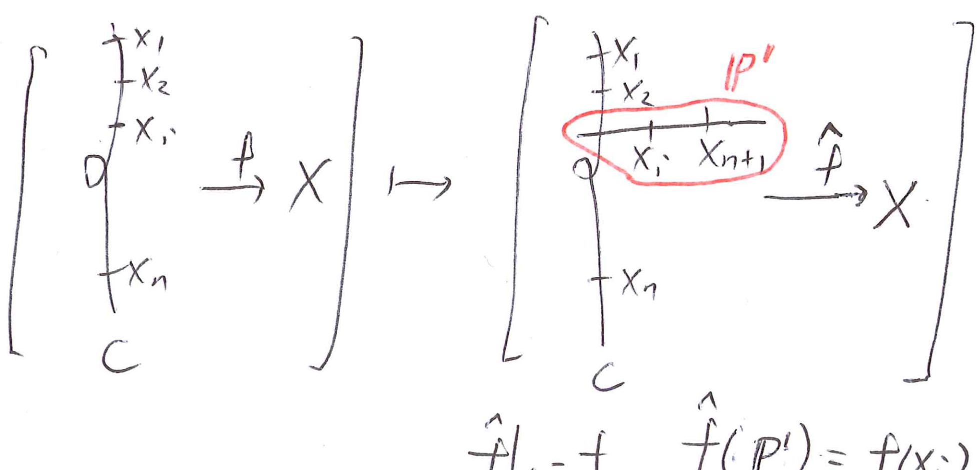
$$\begin{tikzcd}
 & X & \\
\mathcal{U}_{g,n}(X,\beta) \arrow[ur, "\tilde{f}"] \arrow[rr, "\psi"] \arrow[dr, "\pi"] & & \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[ul, "ev_{n+1}"'] \arrow[dl, "F"] \\
 & \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[ul, bend left, "\tilde{s}_i"] &
\end{tikzcd}$$

$ev_{n+1} \circ \psi = \tilde{f}$

$\pi = F \circ \psi$

$\psi \circ \hat{s}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n+1}(X,\beta)$

is given by



$\hat{f}|_C = f, \quad \hat{f}(\mathbb{P}^1) = f(x_i)$