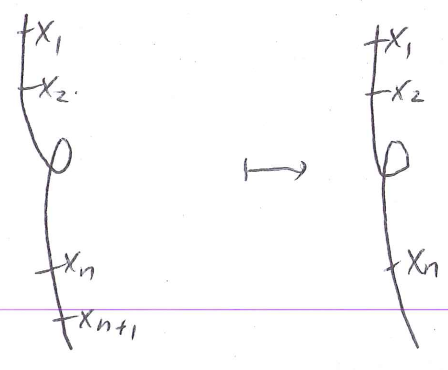
Reference: Arbarello, Cornalba, Griffiths
Geometry of Algebraic Curves Volume II
(Columbia library E-Resource)

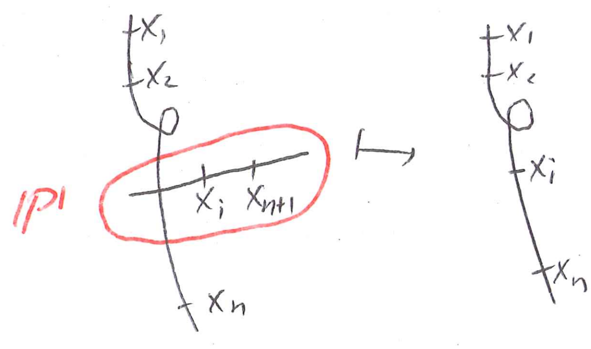
4.2 Universal Curve

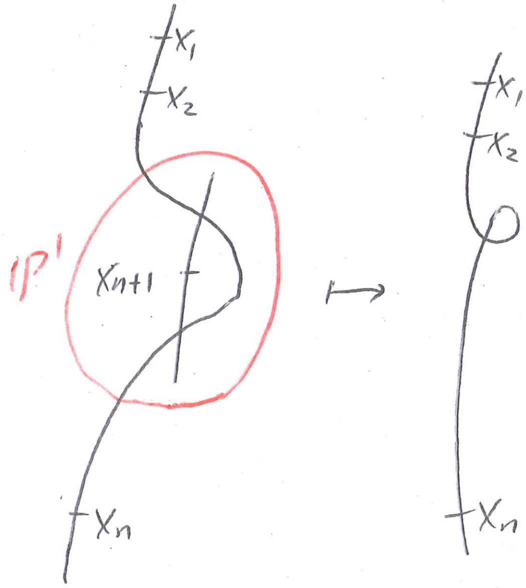
$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forget the $(n+1)$th marked point

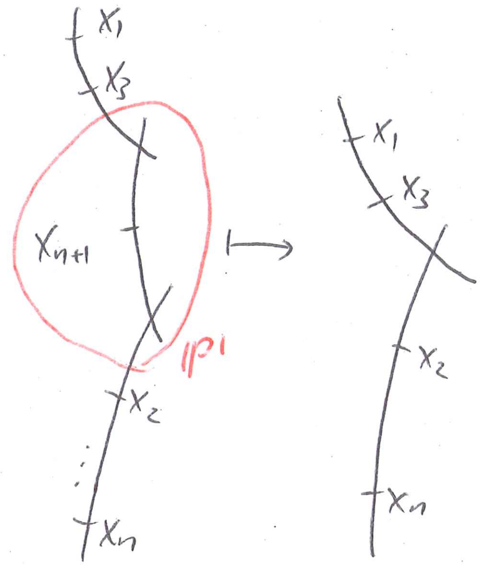
$2g-2+n>0$ (and contract an unstable irreducible component)

Set theoretic description





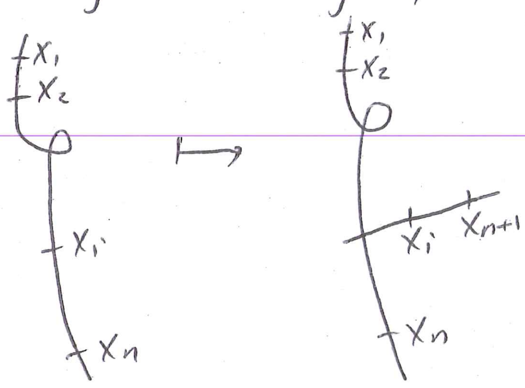




$\overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ can be identified

with the universal curve $\pi: \mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$

$s_i: \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n+1}$



$s_i(\overline{\mathcal{M}}_{g,n}) = D(0, \{i, n+1\} \mid g, \{1,\dots,n\}-\{i\})$

$$\begin{tikzcd}
\mathcal{U}_{g,n+1} \arrow[r, hook, "c"] \arrow[d] & \mathcal{U}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{U}_{g,n} \arrow[r] \arrow[d] & \mathcal{U}_{g,n} \arrow[d] \\
\overline{\mathcal{M}}_{g,n+1} \arrow[u, bend left, "s_i' \ (i=1,\dots,n+1)"] \arrow[r, phantom, "="] \arrow[rr, bend right, "\text{forget } x_{n+1}"'] & \mathcal{U}_{g,n} \arrow[u, bend left, "\pi^* s_i"] \arrow[u, bend right, "\Delta"'] \arrow[r, "\pi"] & \overline{\mathcal{M}}_{g,n} \arrow[u, bend right, "s_i \ (i=1,\dots,n)"']
\end{tikzcd}$$

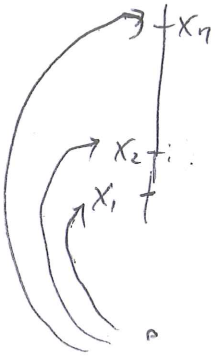
$$\pi \circ s_i = c \circ s_i'$$

$$\Delta = c \circ s_{n+1}'$$

Scheme theoretic construction:

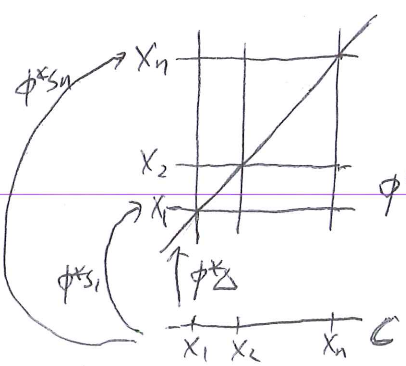
F. Knudsen, "The projectivity of the moduli space of stable curves II: the stacks $M_{g,n}$"

To understand the diagram on p. 32, we first consider $\xi = [(C, x_1, \ldots, x_n)] \in \mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ (i.e. C is smooth)

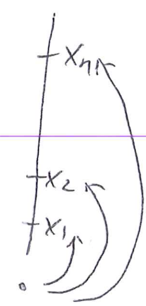


$$\begin{tikzcd} C \arrow[r, "\tilde{\Phi}"] \arrow[d] \arrow[dr, "\phi"] & \mathcal{U}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi"'] \\ \xi \arrow[r] & \overline{\mathcal{M}}_{g,n} \arrow[u, bend right, "s_i"'] \end{tikzcd} \quad i = 1, \ldots, n \qquad \phi(C) = \xi$$

$$\phi^* \mathcal{U}_{g,n} = \tilde{\phi}^* \pi^* \mathcal{U}_{g,n}$$



$$\begin{tikzcd} \phi^*\mathcal{U}_{g,n} = C \times C \arrow[r] \arrow[d] & C \arrow[d] \\ C \arrow[r] \arrow[u, bend left, "\phi^* s_i"] \arrow[u, bend right, "\phi^*\Delta"'] & \xi \end{tikzcd}$$

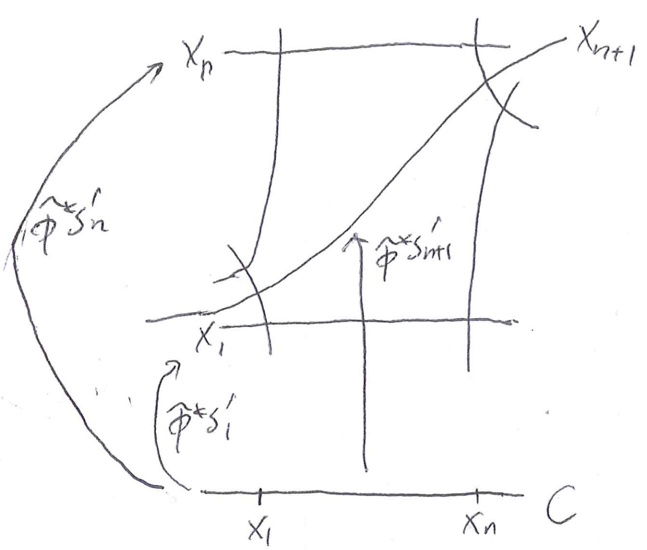


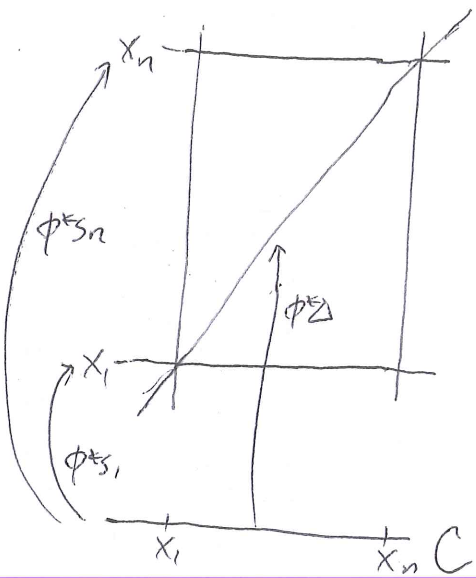
$\phi^* s_i(z) = (z, x_i) \quad i = 1, \ldots, n$

$\phi^* \Delta(z) = (z, z)$

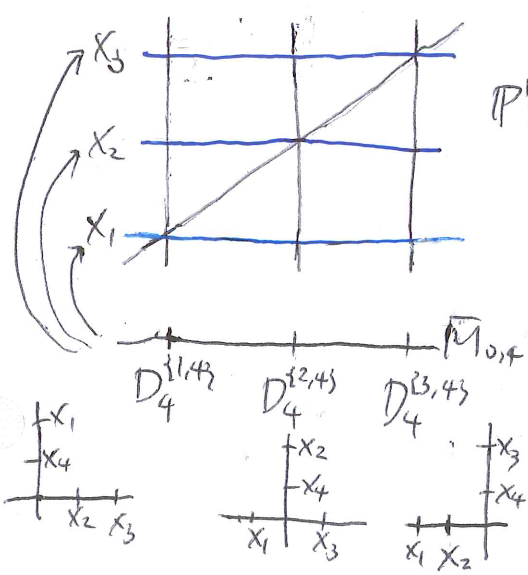
$$\begin{tikzcd}
\tilde{\Phi}^*\mathcal{U}_{g,n+1} \arrow[d, phantom, "\parallel"] & \phi^*\mathcal{U}_{g,n} \arrow[d, phantom, "\parallel"] \\
Bl_{\{(x_1,x_1),\dots,(x_n,x_n)\}}(C\times C) \arrow[r] \arrow[d] & C\times C \arrow[d] \\
C \arrow[u, bend left, "\tilde{\Phi}^* s_i'"] \arrow[r, phantom, "="] & C
\end{tikzcd}$$

$i=1,\dots,n+1$

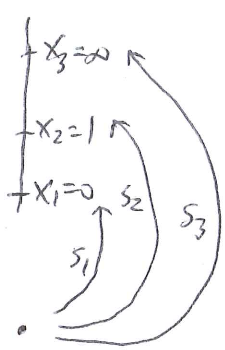




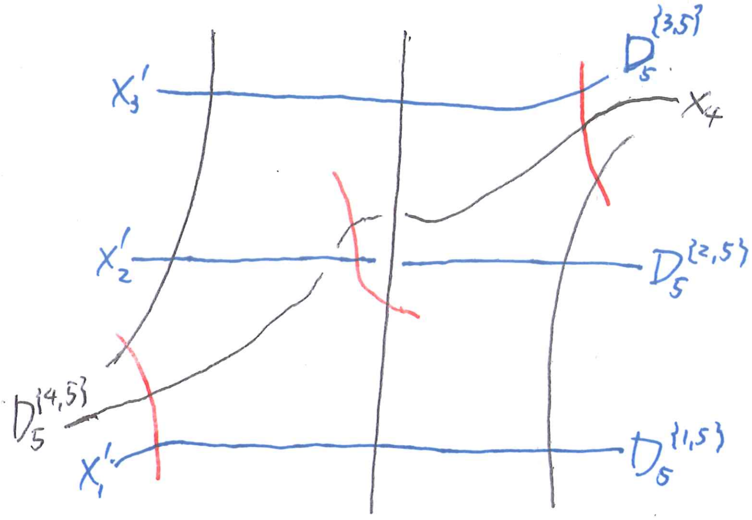
$\boxed{g=0}$

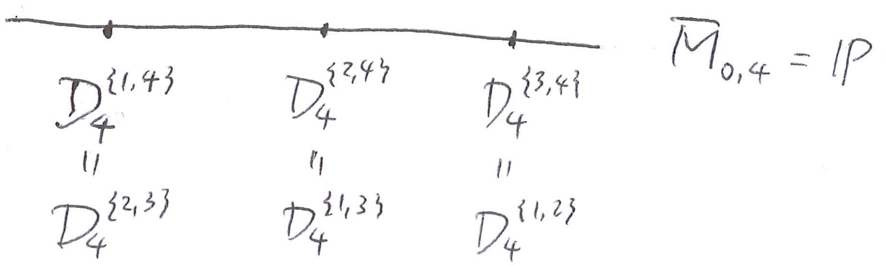


$$\begin{tikzcd}
\mathbb{P}^1\times\mathbb{P}^1 = U_{0,3}\times_{\overline{M}_{0,3}} U_{0,3} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_{0,3} = \mathbb{P}^1 \arrow[d] \\
\mathbb{P}^1 = U_{0,3} \arrow[r] & \overline{M}_{0,3} = \{pt\}
\end{tikzcd}$$



$$U_{0,4} = Bl_{\{(0,0),(1,1),(\infty,\infty)\}}(\mathbb{P}^1 \times \mathbb{P}^1)$$





Exercise

(a) Identify 10 divisors in the above figure with

$$D_5^{\{i,j\}} \qquad 1 \leq i < j \leq 5$$

(b) Show that there exists a subset

$$\{e_1, \ldots, e_5\} \subset \{D_5^{\{i,j\}} : 1 \leq i, j \leq 5\}$$

such that $\quad H_2(\overline{M}_{0,5}; \mathbb{Z}) = \bigoplus_{i=1}^{5} \mathbb{Z} e_i$

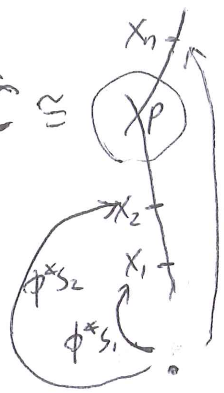
(c) Compute $\quad e_i \cdot e_j \quad 1 \leq i < j \leq 5$

We now consider $\xi = [(C, x_1, \dots, x_n)] \in \overline{\mathcal{M}}_{g,n}$

where C has nodal singularities

let $p \in C$ be a node

$\operatorname{Spec}\left(\frac{\mathbb{C}[\![x,y]\!]}{\langle xy \rangle}\right) = \hat{C} \cong$



$$\begin{array}{ccc} C & \xrightarrow{\tilde{\phi}} & \mathcal{U}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \\ \downarrow & \searrow^{\phi} & \downarrow \pi \uparrow s_i \quad i=1,\dots,n \\ \xi & \longrightarrow & \overline{\mathcal{M}}_{g,n} \end{array} \qquad \phi(C) = \xi$$

$$\begin{array}{ccc} \operatorname{Spec} \frac{\mathbb{C}[\![x,y,u,v]\!]}{\langle xy, uv \rangle} = \hat{C} \times \hat{C} & \longrightarrow & \hat{C} = \operatorname{Spec}\left(\frac{\mathbb{C}[\![x,y]\!]}{\langle xy \rangle}\right) \\ \downarrow & & \downarrow \\ \operatorname{Spec}\left(\frac{\mathbb{C}[\![u,v]\!]}{\langle uv \rangle}\right) = \hat{C} & \longrightarrow & pt = \operatorname{Spec} \mathbb{C} \end{array}$$

$\therefore$

$$\begin{array}{ccccc} \hat{\mathbb{C}}^2 \times_{\mathbb{C}} \hat{\mathbb{C}}^2 & \longrightarrow & \hat{\mathbb{C}}^2 & & (x,y) \\ \downarrow & & \downarrow & & \downarrow \\ \hat{\mathbb{C}}^2 & \longrightarrow & \hat{\mathbb{C}} & & xy \\ (u,v) & \longmapsto & uv & & \end{array}$$

$$\begin{array}{ccccc} & & x & \longleftarrow & x \\ & & y & \longleftarrow & y \\ u \quad v & & \frac{\mathbb{C}[\![x,y,u,v]\!]}{\langle xy - uv \rangle} & \longleftarrow & \mathbb{C}[\![x,y]\!] \quad xy \\ \uparrow \quad \uparrow & & \uparrow & & \uparrow \quad \uparrow \\ u \quad v & & \mathbb{C}[\![u,v]\!] & \longleftarrow & \mathbb{C}[\![t]\!] \quad t \\ & & uv & \longleftarrow & t \end{array}$$

Local model:

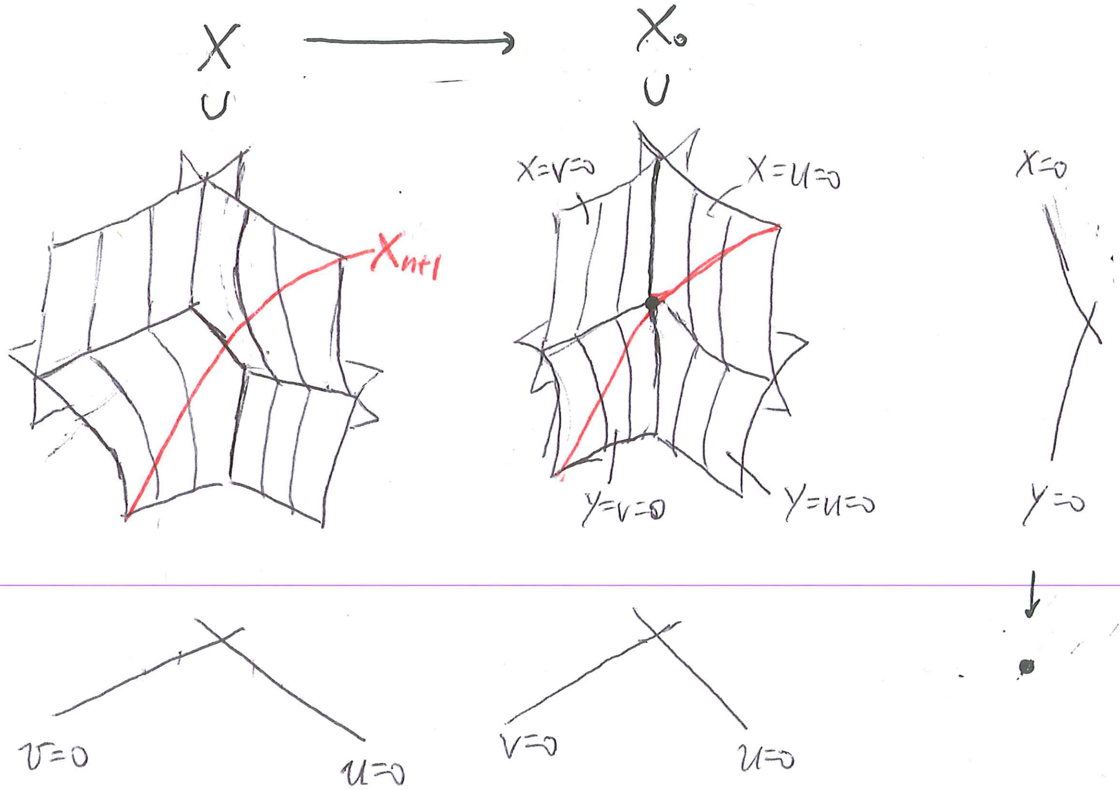
$$\begin{array}{ccc} X_0 \cong \{(x,y,z,u) \in \mathbb{C}^4 : xy - uv = 0\} & \longrightarrow & \mathbb{C}^2 \\ (x,y,u,v) & \longmapsto & (x,y) \in \\ \downarrow & & \downarrow \\ \mathbb{C}^2 \ni (u,v) & \longrightarrow & \mathbb{C} \end{array}$$

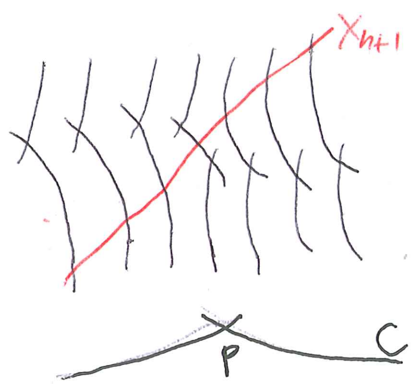
$$X = \{((x,y,u,v),[t_0,t_1]) \in \mathbb{C}^4 \times \mathbb{P}^1 : (x,v),(u,y) \in [t_0,t_1]\}$$

$$\begin{tikzcd}
\{\vec{0}\} \times \mathbb{P}^1 \arrow[d] & \subset & X \arrow[d] & \subset & \mathbb{C}^4 \times \mathbb{P}^1 \arrow[d, "pr_1"] \arrow[dr, "pr_2"] & \\
\vec{0} & \in & X_0 & \subset & \mathbb{C}^4 & \mathbb{P}^1
\end{tikzcd}$$

$\vec{0} = (0,0,0,0)$

$$\begin{tikzcd}
X \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[d, "pr_2|_X"] \\
\mathbb{P}^1
\end{tikzcd}$$





4.3 Cohomology of $\overline{M}_{g,n}$

Sean Keel, "Intersection theory of moduli space of stable n-pointed curves of genus 0"
Tran. AMS 1992

$$A^*(\overline{M}_{0,n};\mathbb{Z}) = \frac{\mathbb{Z}[D^S ; S\subset\{1,\dots,n\},\ |S|\geq 2,\ |S^c|\geq 2]}{\{\text{relations } (1),(2),(3)\}} = H^*(\overline{M}_{0,n};\mathbb{Z})$$

$S\subset\{1,\dots,n\}$

$$A^1(\overline{M}_{0,n};\mathbb{Z}) = H^2(\overline{M}_{0,n};\mathbb{Z}) = \mathrm{Pic}(\overline{M}_{0,n})$$
$$\wr\wr$$
$$A_{n-4}(\overline{M}_{0,n};\mathbb{Z}) = H_{2(n-4)}(\overline{M}_{0,n};\mathbb{Z})$$

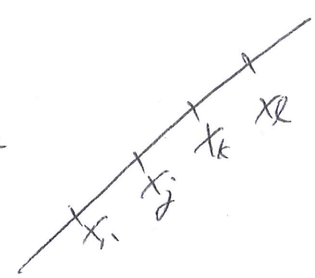
$D^S := D(0,S\,|\,0,S^c)$, $\quad S^c = \{1,\dots,n\}-S$

(1) $D^S = D^{S^c}$

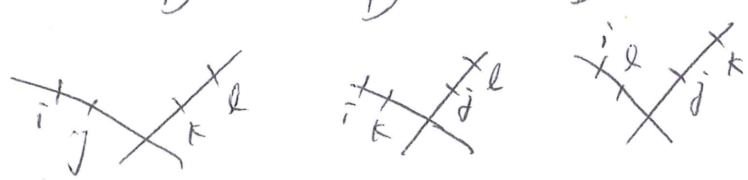
(2) i,j,k,l distint $\quad \{i,j,k,l\}\subset\{1,2,\dots,n\}$

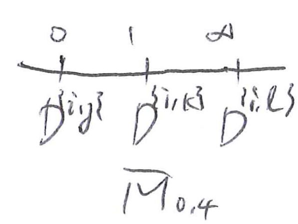
$$\sum_{\substack{i,j\in S\\ k,l\notin S}} D^S = \sum_{\substack{i,k\in S\\ j,l\notin S}} D^S = \sum_{\substack{i,l\in S\\ j,k\notin S}} D^S$$

$$\overline{M}_{0,n} \xrightarrow{\pi_{i,j,k,l}} \overline{M}_{0,4}$$

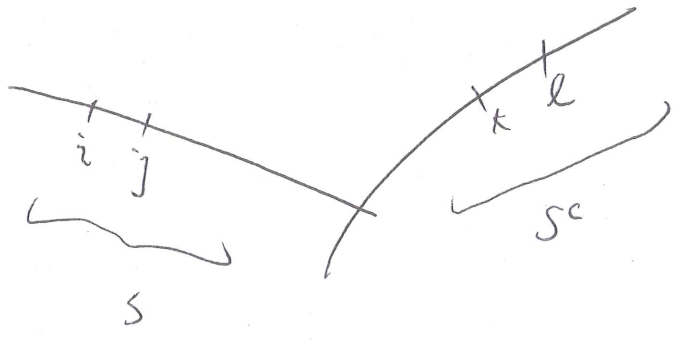


$$D^{\{i,j\}} \sim D^{\{i,k\}} \sim D^{\{i,l\}}$$





$$\pi_{i,j,k,l}^{*} D^{i,j} = \sum_{\substack{i,j\in S\\ k,l\notin S}} D^{S} \qquad \text{etc.}$$



(3) $D^S D^T = 0$ unless one of the following holds

$S\subset T,\ T\subset S,\ S\subset T^c,\ T^c\subset S$

ψ-classes

$$\begin{tikzcd} \mathcal{U}_{g,n} \arrow[d, "\pi"] \\ \overline{\mathcal{M}}_{g,n} \arrow[u, bend left, "s_i"] \end{tikzcd}$$

relative dualizing sheaf

$$\mathcal{L}_i = s_i^{*}\,\omega_\pi \in Pic(\overline{\mathcal{M}}_{g,n}) = Pic(H_{\nu,g,n}, PGL(r+1))$$

$$\overline{\mathcal{M}}_{g,n} = [H_{\nu,g,n}/PGL(r+1)]$$

$r = \nu(2g-2+n)-g$

$Pic(H_{\nu,g,n}, PGL(r+1))$ isomorphism classes of $PGL(r+1)$-equivariant line bundles over $H_{\nu,g,n}$

$$\begin{array}{ccc} \mathcal{L}_i & \ni & T^{*}_{x_i}C \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & [C, x_1, \dots, x_n] \end{array}$$

$$\psi_i = c_1(\mathcal{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n})$$
$$A^1(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$$
$$H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$$

Hodge bundle and λ-classes $(g \geq 1)$

$\mathbb{E} = \pi_* \omega_\pi$

$$\begin{array}{ccc} \mathbb{E} & \ni & H^0(C, \omega_C) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & [(C, x_1 \ldots x_n)] \end{array}$$

$\mathbb{E}$ rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$

$\leftrightarrow$ $PGL(r+1)$-equivariant vector bundle over $H_{r,g,n}$

$g \geq 1$ $\quad \lambda_j = c_j(\mathbb{E}) \in A^j(\overline{\mathcal{M}}_{g,n})$

$A^j(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$

$H^{2j}(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$

$$\begin{array}{ccc} \mathbb{E} & \ni & H^{1,0}(C) \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,n} & \ni & [C, x_1 \ldots x_n] \end{array} \qquad \begin{array}{ccc} \mathbb{E} \oplus \mathbb{E}^\vee \cong \mathbb{E} \oplus \overline{\mathbb{E}} & \ni & H^1(C; \mathbb{C}) \supset H^1(C; \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,n} & \ni & [C, x_1 \ldots x_n] \end{array}$$

$\mathbb{E} \oplus \mathbb{E}^\vee$ flat $\Rightarrow 1 = c(\mathbb{E} \oplus \mathbb{E}^\vee) = c(\mathbb{E}) c(\mathbb{E}^\vee)$

This identity holds over $\overline{\mathcal{M}}_{g,n}$ (Mumford's relation)

$$1 = c(\mathbb{E}) c(\mathbb{E}^\vee) = (1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \lambda_2 + \cdots + (-1)^g \lambda_g)$$

$2\lambda_2 - \lambda_1^2 = 0 \quad \ldots \quad \lambda_g^2 = 0$

David Mumford, "Towards an enumerative geometry of the moduli space of curves" Progress in Math 1983

Hodge integrals

$$\int_{[\overline{M}_{g,n}]} \underbrace{\psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}}_{\in\, H^{2(\sum_{i=1}^n d_i + \sum_{j=1}^g j k_j)}(\overline{M}_{g,n};\mathbb{Q})} \in \mathbb{Q}$$

$[\overline{M}_{g,n}] \in H_{2(3g-3+n)}(\overline{M}_{g,n};\mathbb{Q})$

It is zero unless $\sum_{i=1}^n d_i + \sum_{j=1}^g j k_j = 3g-3+n$

Mumford: GRR

Hodge integrals $\{ \int_{[\overline{M}_{g,n}]} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \}$

$\big\downarrow$ reduced to

descendant integrals $\{ \int_{[\overline{M}_{g,n}]} \psi_1^{d_1} \cdots \psi_n^{d_n} \}$

E. Arbarello, M. Cornalba

"The Picard group of the moduli space of curves"

Topology 1987

Theorem

(1) $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ is a free abelian group of finite rank.

$$\begin{array}{ll} \overline{\mathcal{M}}_{g,n} & \text{fine moduli stack} \\ \downarrow p & \\ \overline{M}_{g,n} & \text{coarse moduli scheme} \end{array}$$

$$0 \to \mathrm{Pic}(\overline{M}_{g,n}) \xrightarrow{p^*} \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$$

subgroup of finite index

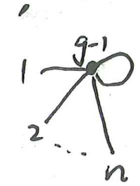
(isomorphism when $g=0$)

$$\mathrm{Pic}(\overline{M}_{g,n})\otimes\mathbb{Q} = \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})\otimes\mathbb{Q}$$

(2) $g \geq 3,\ n \geq 0$ $\qquad \delta_{a,A} = c_1(\mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(D(a,A|g-a,A^c)))$

$\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ is generated by

$\lambda, \psi_1, \ldots, \psi_n,\ \delta_{irr},\ \{\delta_{a,A} : \begin{array}{l} 0 \leq a \leq g \\ 2a-2+|A| \geq 0 \\ 2(g-a)-2+|A^c| \geq 0 \end{array}\}$



with relation $\delta_{a,A} = \delta_{g-a,A^c}$

(3) $g = 1, 2$

$\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ is generated by

$$\lambda_1,\ \psi_1, \dots, \psi_n,\ \delta_{irr},\ \left\{ \delta_{a,A} : \begin{array}{l} 0 \le a \le g \\ 2a-2+|A| \ge 0 \\ 2(g-a)-2+|A^c| \ge 0 \end{array} \right\}$$

with relation $\delta_{a,A} = \delta_{g-a,A^c}$ and other relations

$\boxed{g=1}$

$$\begin{array}{ccc} \mathcal{U}_{1,1} & & \mathbb{E} \longrightarrow \mathcal{L}_1 \\ s_1 \uparrow \downarrow \pi & & \omega \in H^0(C, \omega_C) \longmapsto \omega(x_1) \in T^*_{x_1}C \\ \overline{\mathcal{M}}_{1,1} & & \mathbb{E} \cong \mathcal{L}_1 \ \Rightarrow\ \lambda_1 = \psi_1 \end{array}$$

$$\begin{array}{c} \overline{\mathcal{M}}_{1,1} \\ p \downarrow \ \deg \tfrac{1}{2} \\ \overline{M}_{1,1} = \mathbb{P}^1 \end{array}$$

$H^2(\overline{\mathcal{M}}_{1,1}; \mathbb{Q}) = H^2(\overline{M}_{1,1}; \mathbb{Q}) = \mathbb{Q}H$

Poincaré dual of a point

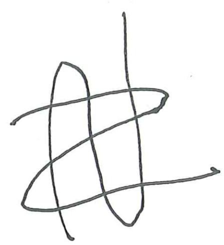
$\delta = \delta_{irr} = \frac{1}{2}H$

$\psi_1 = c\delta$ for some $c \in \mathbb{Q}$

<u>Proposition</u> $\psi_1 = \frac{1}{12}\delta \quad \Rightarrow \int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{12}\int_{\overline{M}_{1,1}} \delta$

$$= \frac{1}{12}\int_{\mathbb{P}^1}\left(\frac{1}{2}H\right) = \frac{1}{24}$$

<u>Proof</u> We consider a generic pencil of plane cubics:



$C_1 = \{F_1(X,Y,Z)=0\}$

$C_0 = \{F_0(X,Y,Z)=0\}$

$$F_0, F_1 \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \cong \mathbb{C}^{10}$$

$$C_0 \cap C_1 = \{p_1, \dots, p_9\}$$

$$\mathcal{C} = \{([X,Y,Z],[t_0,t_1]) \in \mathbb{P}^2 \times \mathbb{P}^1 : t_0 F_0(X,Y,Z) + t_1 F_1(X,Y,Z) = 0\}$$

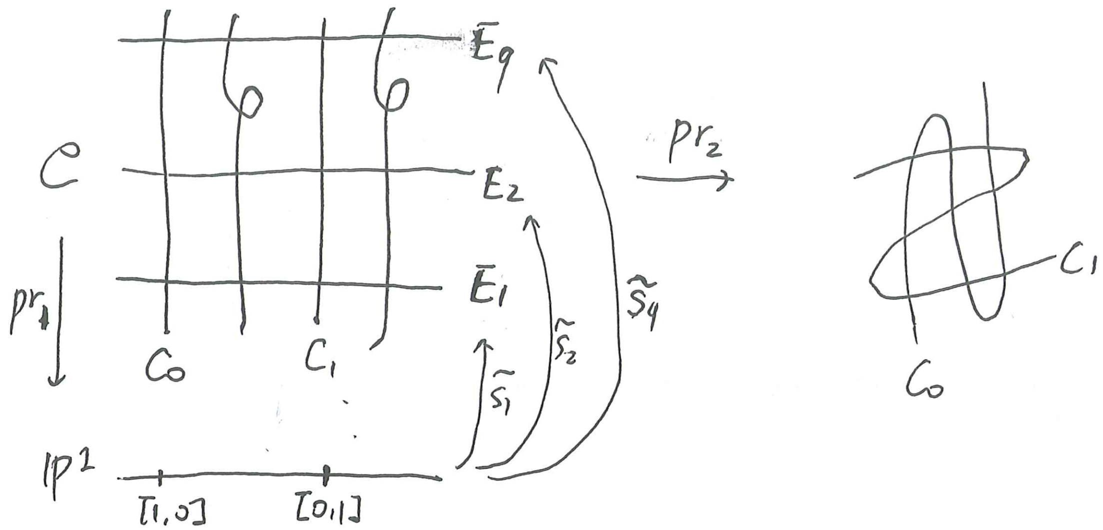
$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{P}^2 & & \mathbb{P}^1 \end{tikzcd}$$

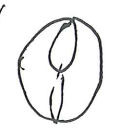
$$\mathcal{C} = Bl_{\{p_1,\dots,p_9\}}(\mathbb{P}^2)$$

Hodge diamond of $\mathcal{C}$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad \Rightarrow \chi(\mathcal{C}) = 12$$

Let $E_i := pr_1^{-1}(p_i) \cong \mathbb{P}^1$ be exceptional divisors
$i = 1, \dots, 9$



$\chi($  $) = 0$, $\chi($  $) = 1$

$\Rightarrow \mathcal{C} \xrightarrow{pr_1} \mathbb{P}^1$ 12 singular fibers

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "pr_2"] & U_{1,1} \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\phi"] \arrow[u, bend left, "\tilde{s}_1"] & \overline{\mathcal{M}}_{1,1} \arrow[u, bend right, "s_1"'] \end{tikzcd}$$

$\overline{\mathcal{M}}_{1,1} \supset D_{irr} = [pt/\mathbb{Z}_2]$

$\phi^{-1}(D_{irr}) =$ 12 points in $\mathbb{P}^1$

$\phi^* \delta = 12 H$ ⋯ (1)

$$\phi^* \mathcal{L}_1 = \hat{s}_1^* (N^*_{E_1/e}) = \hat{s}_1^* \mathcal{O}_e(-E_1)$$

$$\int_{\mathbb{P}^1} \phi^* \psi_1 = -E_1 \cdot E_1 = 1 \Rightarrow \phi^* \psi_1 = H \quad \text{---}(2)$$

By (1) and (2), $\therefore \delta = 12\psi_1$ $\square$

$$\mathcal{L}_1^{\otimes 12} = \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(D_{irr})$$

$$\mathcal{L}_1^{\otimes 12}\Big|_{\mathcal{M}_{1,1}} = \mathcal{O}_{\mathcal{M}_{1,1}}$$

$Pic(\mathcal{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z}$ generated by $\mathcal{L}_1|_{\mathcal{M}_{1,1}}$