3.2 Moduli Functor

Definition

An <u>n-pointed genus g stable curve</u> over a scheme B (over $\operatorname{Spec}\mathbb{C}$) is a flat projective morphism $\pi : C \to B$ with distinct sections

$s_i : B \to C \quad i = 1, \dots, n$

such that $\forall b \in B$

$(C_b, s_1(b), \dots, s_n(b))$ is an n-pointed genus g stable curve

$$\begin{tikzcd} C_b \subset C \arrow[r, hook, "\text{closed}"', "\text{immersion}"] \arrow[d, "\pi"] & \mathbb{P}^N \times B \arrow[dl, "pr_2"] \\ b \in B & \end{tikzcd}$$

$C_b \subset \mathbb{P}^N$

Hilbert polynomial

$$P_t = \underbrace{(H.C_b)}_{\deg} t + 1 - \underbrace{g_a(C_b)}_{\text{genus}}$$

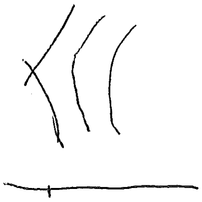
locally constant

Example

flat

$\{XY - tZ^2\} \subset \mathbb{P}^2 \times \mathbb{A}^1$

$$\begin{matrix} [X,Y,Z],\ t \\ \downarrow pr_2 \\ \mathbb{A}^1 \end{matrix}$$



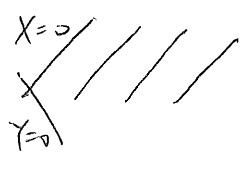
$t=0$

non-flat

$\{XY = 0 = tX\} \subset \mathbb{P}^2 \times \mathbb{A}^1$

$$\downarrow pr_2$$

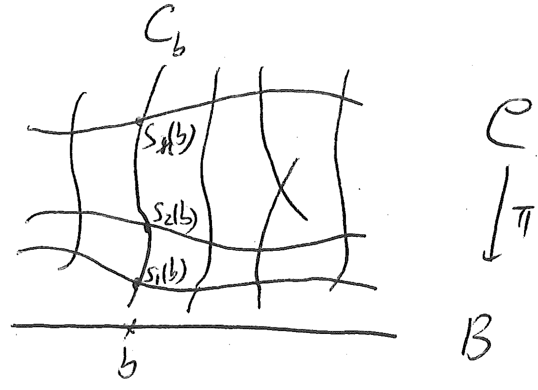
$\mathbb{A}^1$



$t=0$

$$\begin{array}{c} \mathcal{C} \\ \pi \big\downarrow \big\uparrow s_i \\ B \end{array}$$

$i = 1, \ldots, n$



(1) Define a contravariant functor

$$\overline{F}_{g,n} : Sch \longrightarrow Set$$

$Sch = Sch/\mathbb{C}$: category of schemes $/ \mathrm{Spec}\, \mathbb{C}$

Set : category of sets

$$B \mapsto \left\{ \begin{array}{c} \mathcal{C} \\ \pi \downarrow \uparrow s_i \\ B \end{array} \; i = 1, \ldots, n \quad \text{n-pointed genus g stable curve over B} \right\} \Big/ \cong$$

isomorphism

$$(*) \quad \begin{tikzcd} \mathcal{C}' \arrow[rr, "\cong \psi"] \arrow[dr, "\pi'"'] & & \mathcal{C} \arrow[dl, "\pi"] \\ & B \arrow[ul, bend left, "s_i'"] \arrow[ur, bend right, "s_i"'] & \end{tikzcd}$$

$s_i = \psi \circ s_i'$

$\pi \circ \psi = \pi'$

B_1, B_2 schemes, $\phi : B_1 \to B_2$

$\rotatebox{90}{$\in$}$

$\mathrm{Mor}_{Sch}(B_1, B_2)$

$$\overline{F}_{g,n}(B_2) \to \overline{F}_{g,n}(B_1)$$

$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ B_2 \arrow[u, bend right, "s_i"'] \end{tikzcd} \longmapsto \begin{tikzcd} C \times_{B_2} B_1 \arrow[d, "\phi^* \pi"'] \\ B_1 \arrow[u, bend right, "\phi^* s_i"'] \end{tikzcd} \qquad \begin{tikzcd} C \times_{B_2} B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "\pi"] \\ B_1 \arrow[r, "\Phi"] & B_2 \end{tikzcd}$$

well-defined up to canonical isomorphism

(2) Consider a slightly different contravariant functor:

$$\overline{\mathcal{F}}_{g,n} : Sch \to (\text{Groupoids})$$
$$B \mapsto \overline{\mathcal{F}}_{g,n}(B)$$

Object: n-pointed genus g stable curve over B

morphisms: isomorphisms $(*)$ on p. 15

$2g-2+n>0$

$$\overline{\mathcal{F}}_{g,n} \cong \overline{F}_{g,n} \iff g=0$$

3.3 Fine Moduli

Given a scheme M, define a contravariant functor $h_M : Sch \to Set$

$$B \mapsto Mor_{Sch}(B, M)$$

B_1, B_2 schemes $\phi : B_1 \to B_2$

$$\in Mor_{Sch}(B_1, B_2)$$

$$h_M(B_2) = Mor_{Sch}(B_2, M) \to h_M(B_1) = Mor_{Sch}(B_1, M)$$

$$(f : B_2 \to M) \mapsto (f \circ \phi : B_1 \to M)$$

Definition

A contravariant functor $h : Sch \to \mathbf{Set}$ is representable by a scheme M if there is an isomorphism $\Psi : h_M \to h$ of functors. In this case, we say M is a fine moduli of the functor h. (M is unique up to isomorphism.)

e.g. Hilbert scheme of subcurve of (arithmetic) genus g and degree d in $\mathbb{P}^r$

Suppose that there exists a fine moduli $\overline{M}_{g,n}$ of the functor $\overline{F}_{g,n}$. Then

$$\begin{array}{ccc} h_{\overline{M}_{g,n}}(\overline{M}_{g,n}) & \overset{\simeq}{\to} & \overline{F}_{g,n}(\overline{M}_{g,n}) \\ \| & & \\ \mathrm{Mor}_{Sch}(\overline{M}_{g,n}, \overline{M}_{g,n}) & & \end{array}$$

$$id_{\overline{M}_{g,n}} \mapsto \begin{array}{c} U_{g,n} \\ \tilde{\pi}\downarrow \uparrow \tilde{s}_i \\ \overline{M}_{g,n} \end{array} \quad \text{universal } n\text{-pointed genus } g \text{ stable curve}$$

Given any n-pointed genus g stable curve

$$\begin{array}{c} C \\ \pi\downarrow \uparrow s_i \\ B \end{array} \quad i=1,\dots,n \qquad \text{over any scheme } B,$$

there exists $\phi \in h_{\overline{M}_{g,n}}(B) = \mathrm{Mor}_{Sch}(B, \overline{M}_{g,n})$

such that $\begin{array}{c} C \\ \pi\downarrow \uparrow s_i \\ B \end{array}$ is isomorphic to $\begin{array}{c} U_{g,n} \times_{\overline{M}_{g,n}} B \\ \phi^*\tilde{\pi} \downarrow \uparrow \phi^*\tilde{s}_i \\ B \end{array}$

$$\begin{tikzcd} U_{g,n} \times_{\overline{M}_{g,n}} B \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & U_{g,n} \arrow[d, "\tilde{\pi}"'] \\ B \arrow[r, "\phi"] \arrow[u, bend left, "s_i"] & \overline{M}_{g,n} \arrow[u, bend right, "\tilde{s}_i"'] \end{tikzcd}$$

<u>Fact</u> $(2g-2+n>0)$

(1) $\overline{\mathcal{F}}_{0,n} \cong \mathcal{F}_{0,n}$ is represented by a smooth projective scheme $\overline{M}_{0,n}$

(2) $g>0$: $\overline{\mathcal{F}}_{g,n}$ is not represented by a scheme

(3) $\overline{\mathcal{F}}_{g,n}$ is represented by a proper smooth Deligne-Mumford (DM) stack

(2) Suppose that $\overline{\mathcal{F}}_{g,n}$ is represented by a scheme $\overline{M}_{g,n}$

e.g. $g \geq 2, n=0$ $\exists$ genus g stable curve C such that

$\mathrm{Aut}(C) \supsetneq \{id\}$ Pick $\phi \in \mathrm{Aut}(C)$, $\phi \neq id_C$

$$\begin{array}{l} \mathcal{C} = C \times \mathbb{P}^1 / (x,0) \sim (\phi(x), 1) \\ \pi \downarrow \\ B = \mathbb{P}^1 / 0 \sim 1 \leftarrow \mathbb{P}^1 \end{array}$$

$\phi : B \to \overline{M}_{g,0}$ constant map to $[C] \in \overline{M}_{g,0}$

$U_{g,0} \times_{\overline{M}_{g,0}} B = C \times B$, which is not isomorphic to $\begin{array}{c} \mathcal{C} \\ \downarrow \pi \\ B \end{array}$ $\rightarrow\leftarrow$

Remark

Similar construction for $\overline{F}_{g,n}$ or more general moduli problems with nontrivial Aut.

(3) If $(C, x_1, \dots, x_n)$ is a n-pointed genus g stable curve that $\omega_C(\underset{\substack{\| \\ x_1+\cdots+x_n}}{D})^{\otimes\nu}$ is very ample if $\nu \geq 3$

$\Rightarrow$ ν-log canonical embedding

$$C \overset{\deg d}{\hookrightarrow} \mathbb{P}^r \qquad d = \nu(2g-2+n)$$

$$r+1 = d+1-g$$

Hilbert polynomial

$$p(t) = dt+1-g = \nu t(2g-2+n)+1-g$$

$H_{\nu,g,n}$ Hilbert scheme of ν-log-canonically embedded n-pointed genus g curve

smooth quasi-projective variety

$$\dim H_{\nu,g,n} = (r+1)^2-1+3g-3+n$$

$$\overline{\mathcal{M}}_{g,n} \cong [H_{\nu,g,n}/PGL(r+1)] \quad \text{quotient stack}$$

3.4 Coarse Moduli

$\overline{M}_{g,n} = H_{r,g,n}/PGL(r+1)$ GIT quotient
projective scheme

is a coarse moduli space of $\overline{F}_{g,n} : Sch \to Set$,
in the following sense.

Definition

A scheme M is a coarse moduli of a contravariant functor $h : Sch \to Set$ if there is a natural transformation of functors

$\Phi : h \to h_M$

$$h(B) \xrightarrow{\Phi(B)} h_M(B) = Mor_{Sch}(B, M)$$

$$B_1 \xrightarrow{\phi} B_2$$

$$\begin{array}{ccc} h(B_2) & \xrightarrow{h(\phi)} & h(B_1) \\ {\scriptstyle \Phi(B_2)}\downarrow & & \downarrow{\scriptstyle \Phi(B_1)} \\ Mor_{Sch}(B_2, M) & \xrightarrow{h_M(\phi)} & Mor_{Sch}(B_1, M) \\ f & \longmapsto & f \circ \phi \end{array}$$

(a) $h(\operatorname{Spec}\mathbb{C}) \xrightarrow{\Phi(\operatorname{Spec}\mathbb{C})} h_M(\operatorname{Spec}\mathbb{C}) = \operatorname{Mor}_{Sch}(\operatorname{Spec}\mathbb{C}, M) = M(\mathbb{C})$

is a set bijection

(b) For any scheme M' and any natural transformation $\Phi' : h \to h_{M'}$ of functors, there is a unique morphism $p : M \to M'$ of schemes such that the induced natural transformation

$$p : h_M \longrightarrow h_{M'}$$

$$\operatorname{Mor}_{Sch}(B, M) = h_M(B) \xrightarrow{p(B)} h_{M'}(B) = \operatorname{Mor}_{Sch}(B, M')$$
$$f \mapsto p \circ f$$

satisfies $\Phi' = p \circ \Phi$

$$\begin{tikzcd} h \arrow[r, "\Phi"] \arrow[dr, "\Phi'"'] & h_M \arrow[d, "p"] \arrow[dl, phantom, "?"] \\ & h_{M'} \end{tikzcd}$$

4 Geometry and Topology of $\overline{M}_{g,n}$

4.1 Dual graphs and Stratification

$(C, x_1 \ldots x_n)$

C nodal curve
$x_1, \ldots x_n$ distinct points in $C - C_{sing}$

The dual graph of $(C, x_1 \ldots x_n)$ is
a labelled graph Γ

To each connected component C_v of $\tilde{C} \xrightarrow[\text{normalization}]{\nu} C$
we associate a vertex $v \in V(\Gamma)$

$\vec{g}: V(\Gamma) \to \mathbb{Z}_{\geq 0}$ $\quad \vec{g}(v) = g_v := \text{genus}(C_v)$

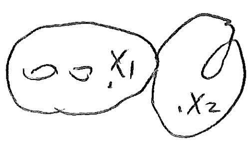
To each node y_e of C we associate
an edge $e \in E(\Gamma)$

$v \in e \iff y_e \in \nu(C_v)$

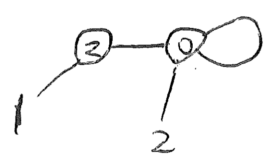
To each marked point x_i we associate
a leg $l_i \in L(\Gamma)$

Example

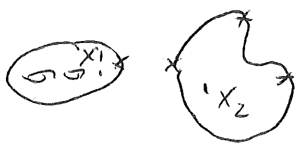
C



Γ



$\tilde{C}$



Given a dual graph of a pointed nodal curve

$$g(\Gamma) = \sum_{v \in V(\Gamma)} g(v) + b_1(\Gamma)$$

$$n(\Gamma) = \# L(\Gamma)$$

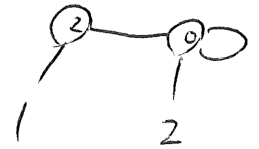
If Γ is the dual graph of $(C, x_1 \dots x_n)$ then

$g(\Gamma) = g_a(C)$, $\quad n(\Gamma) = n$.

We say Γ is <u>stable</u> if $(C, x_1 - x_n)$ is stable

$$G_{g,n} = \{ \Gamma \text{ stable dual graph} : g(\Gamma) = g, \ n(\Gamma) = n \}$$

$\Gamma \in G_{g,n} \longleftrightarrow \mathcal{M}_\Gamma$ stratum of $\overline{\mathcal{M}}_{g,n}$

$\Gamma =$ 

$\mathcal{M}_\Gamma = [(\mathcal{M}_{2,2} \times \mathcal{M}_{0,4})/\mathbb{Z}_2]$

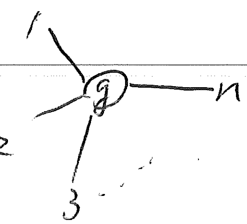
$$\overline{\mathcal{M}}_{g,n} = \bigsqcup_{\Gamma \in G_{g,n}} \mathcal{M}_\Gamma$$

$$\mathcal{M}_\Gamma = \left[\left(\prod_{v \in V(\Gamma)} \mathcal{M}_{g_v, n_v}\right) \Big/ \mathrm{Aut}(\Gamma)\right]$$

$G_{g,n}$ is a partially ordered set

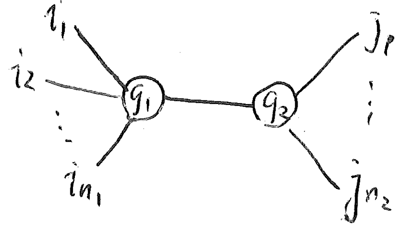
$$\Gamma_1 \le \Gamma_2 \iff \mathcal{M}_{\Gamma_1} \subset \overline{\mathcal{M}}_{\Gamma_2}$$

top stratum



codim $\mathcal{M}_\Gamma = |E(\Gamma)|$

Codim 1 strata

① $\Gamma =$ 

$g_1 + g_2 = g$

$$M_\Gamma = \left[(M_{g_1, n_1+1} \times M_{g_2, n_2+1}) / Aut(\Gamma) \right]$$

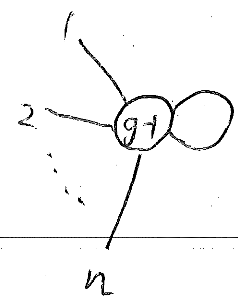
trivial unless $n_1 = n_2 = 0$, $g_1 = g_2 = \frac{g}{2}$

$I = \{i_1, \ldots, i_{n_1}\}$ $\quad I \cup J = \{1, \ldots, n\}$

$J = \{j_1, \ldots, j_{n_2}\}$ $\quad I \cap J = \phi$

$$3g_1 - 3 + n_1 + 1 + 3g_2 - 3 + n_2 + 1$$
$$= 3(g_1 + g_2) - 4 + (n_1 + n_2)$$
$$= (3g - 3 + n) - 1$$

closure $\quad \overline{M}_{g_1, n_1+1} \times \overline{M}_{g_2, n_2+1} \longrightarrow D(g_1, I \mid g_2, J) \underset{\text{divisor}}{\subset} \overline{M}_{g,n}$

② $\Gamma =$ 

$$M_\Gamma = [M_{g-1, n+2} / \underset{\mathbb{Z}/2\mathbb{Z}}{\overset{\|}{Aut(\Gamma)}}]$$

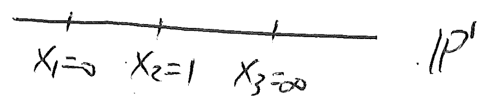
$$3(g-1) - 3 + n + 2 = 3g - 4 + n = (3g - 3 + n) - 1$$

$\boxed{g = 0}$ $\quad$ Given $I \subset \{1, \ldots, n\}$

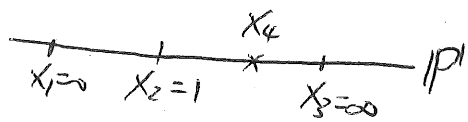
$$D_n^I := D(0, I \mid 0, \{1, \ldots, n\} - I) = D_n^{\{1, \ldots, n\} - I}$$

divisor in $\overline{M}_{0,n} = \overline{\mathcal{M}}_{0,n}$

$$M_{0,3} = \{ (\mathbb{P}^1, 0, 1, \infty) \} = \overline{M}_{0,3}$$

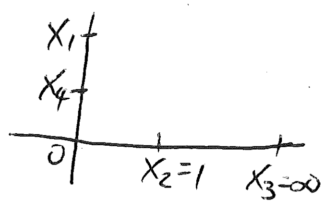


$$M_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$$



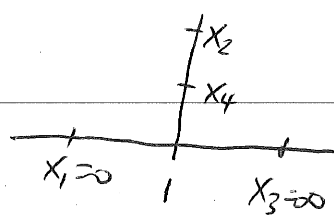
$$\overline{M}_{0,4} = \mathbb{P}^1$$

$0 \in \mathbb{P}^1 \longleftrightarrow$



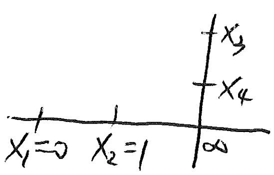
$\longleftrightarrow D_4^{\{1,4\}} \subset \overline{M}_{0,4}$

$1 \in \mathbb{P}^1 \longleftrightarrow$



$\longleftrightarrow D_4^{\{2,4\}} \subset \overline{M}_{0,4}$

$\infty \in \mathbb{P}^1 \longleftrightarrow$



$\longleftrightarrow D_4^{\{3,4\}} \subset \overline{M}_{0,4}$