1. Gromov-Witten Theory: an Overview

$\begin{cases} \text{symplectic approach} \\ \text{algebraic approach} \end{cases}$ (1990's)

Symplectic Approach

(X, ω, J) compact Kähler manifold

ω: symplectic structure

J: complex structure

Naively, Gromov-Witten (GW) invariants of X count parametrized holomorphic curves in X:

$$f: C \xrightarrow{\text{hol.}} X$$

C: compact Riemann surface

More generally

(X, ω, J)

ω: symplectic structure

J: almost complex structure, compatible with ω

$J: TX \to TX$, $J^2 = -Id_{TX}$

$\omega(Ju, Jv) = \omega(u, v)$, $\omega(u, Ju) > 0$ if $u \neq 0$

Naively, GW invariants of X count

J-holomorphic (pseudo-holomorphic) curves in X:

$f : (C, j) \to (X, J) \qquad df \circ j = J \circ df$

(C: compact Riemann surface)

M. Gromov, "Pseudo holomorphic curves in symplectic manifolds," Invent. 1985

D. McDuff, D. Salamon, "J-holomorphic curves and Symplectic topology"

Algebraic Approach

X smooth projective variety $/ \mathbb{C}$

($X \hookrightarrow \mathbb{P}^N \Rightarrow$ compact Kähler manifold)

Naively, GW invariants of X count parametrized algebraic curves in X:

$f : C \to X$

Fix $g \in \mathbb{Z}_{\geq 0}$ (genus), $\beta \in H_2(X; \mathbb{Z})$ (effective curve class), $Z_1, \dots, Z_n$ cycles in X (oriented closed submanifolds)

$\gamma_i \in PD([Z_i]) \in H^{d_i}(X; \mathbb{Z})$, where $H^{d_i}(X; \mathbb{Z}) \cong H_*(X; \mathbb{Z})$

$d_i =$ (real) codimension of Z_i in X

$\langle \gamma_1, \dots, \gamma_n \rangle^X_{g,n,\beta}$ "=" # of genus g, deg β curve in X passing through $Z_1, .., Z_n$

$\overline{M}_{g,n}(X,\beta)$ Kontsevich moduli space of

n-pointed, genus g, deg β stable maps

$f : (C, x_1 \dots x_n) \to X$ $\qquad f_*[C] = \beta$

C: genus g nodal curve

$x_1 \dots x_n$: distinct nonsingular points (ordered)

$h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = 1-g$

isomorphism:

$$\begin{tikzcd}
(C, x_1 \dots x_n) \arrow[d, "\varphi", "\wr"'] \arrow[dr, "f"] & \\
(C', x_1' \dots x_n') \arrow[r, "f'"'] & X
\end{tikzcd}$$

stability: $\mathrm{Aut}([f : (C, x_1 \dots x_n) \to X])$ is finite

$ev_i : \overline{M}_{g,n}(X,\beta) \to X$ evaluation at i-th marked pt

$[f : (C, x_1 \dots x_n) \to X] \mapsto f(x_i)$

Suppose that $\overline{M}_{g,n}(X,\beta)$ is a smooth projective variety of dim. d ($\Rightarrow$ compact complex manifold
seldom true
of dim d)

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,n} = \# \, ev_1^{-1}(Z_1) \cap \cdots \cap ev_n^{-1}(Z_n)$$

if $ev_1^{-1}(Z_1), \ldots, ev_n^{-1}(Z_n)$ intersect transversally at finitely many points

fundamental class

$$\langle \gamma_1, \ldots, \gamma_n \rangle^X_{g,n} = \langle \underbrace{ev_1^*(\gamma_1) \cup \cdots ev_n^*(\gamma_n)}_{\in H^{d_1+\cdots+d_n}(\overline{M}_{g,n}(X,\beta);\mathbb{Z})}, \underbrace{[\overline{M}_{g,n}(X,\beta)]}_{\in H_{2d}(\overline{M}_{g,n}(X,\beta);\mathbb{Z})} \rangle$$

$$= \langle \underbrace{\gamma_1 \times \cdots \times \gamma_n}_{\in H^{d_1+\cdots+d_n}(X^n;\mathbb{Z})}, \underbrace{(ev_1 \times \cdots \times ev_n)_* [\overline{M}_{g,n}(X,\beta)]}_{\in H_{2d}(X^n;\mathbb{Z})} \rangle$$

$= 0$ unless $d_1 + \cdots + d_n = 2d$

In general, $\overline{M}_{g,n}(X,\beta)$ is a

proper (compact, Hausdorff) DM stack (singular orbifold)

with a perfect obstruction theory
(virtual tangent bundle $T^{vir} = T^1 - T^2$)
(orbi) vector bundle

virtual (complex) dimension

$$d^{vir} = \operatorname{rank} T^1 - \operatorname{rank} T^2$$

$$= \underbrace{\langle c_1(TX), \beta\rangle + \dim X (1-g)}_{\text{deformation of the map } f} + \underbrace{3g-3+n}_{\text{deformation of the domain } (C, x_1, \dots, x_n)}$$

$h^0(C, f^*TX) - h^1(C, f^*TX)$
obstruction to deforming f

$$\overset{R\text{-}R}{=} \deg(f^*TX) + \operatorname{rank}(f^*TX)(1-g)$$

$$= \langle c_1(TX), \beta\rangle + \dim X (1-g)$$

$$d^{vir} = \langle c_1(TX), \beta\rangle + (\dim X - 3)(1-g) + n$$

Li-Tian

Behrend-Fantechi, Behrend

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in H_{2d^{vir}}(\overline{M}_{g,n}(X,\beta);\mathbb{Q})$$
$$A_{d^{vir}}(\overline{M}_{g,n}(X,\beta);\mathbb{Q})$$

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} := \langle ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n), [\overline{M}_{g,n}(X,\beta)]^{vir} \rangle$$
$$(= 0 \quad \text{unless} \quad d_1 + \cdots + d_n = 2d^{vir})$$

(primary) GW invariants of X.

2. Quantum Cohomology

$$\Lambda = \mathbb{Q}[[Q^\beta : \beta \text{ effective}]] \qquad Q^{\beta_1+\beta_2} = Q^{\beta_1} Q^{\beta_2}$$

Example $X = \mathbb{P}^n$ $\qquad H_2(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}\ell$

$\qquad \ell = [\mathbb{P}^1]$

$\beta = d\ell$ is effective $\Leftrightarrow d \in \mathbb{Z}_{\geq 0}$

$$\Lambda = \mathbb{Q}[[Q]] \qquad Q^\beta = Q^d, \ Q = Q^\ell$$

(small) quantum cohomology

$$QH^*(X) = (H^*(X;\Lambda), *) \xrightarrow{Q \to 0} (H^*(X;\mathbb{Q}), \cup)$$

quantum product $\qquad$ classical cohomology

$a, b \in H^*(X;\mathbb{Q})$

$(a,b) := \int_X a \cup b \in \mathbb{Q}$ Poincaré pairing
non-degenerate
supersymmetric

extend to $H^*(X;\Lambda)$

$a, b \in H^*(X;\mathbb{Q})$, define $a * b \in H^*(X;\Lambda)$ by

$$(a*b, c) = \underbrace{(a\cup b, c)}_{\text{classical}} + \underbrace{\sum_{\beta > 0} \langle a, b, c\rangle^X_{0,3,\beta} Q^\beta}_{\text{quantum}} \in \Lambda$$

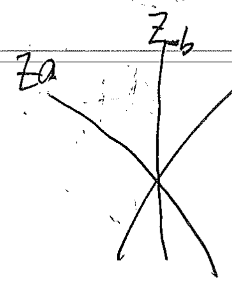
$(a\cup b, c) = \int_X a\cup b\cup c$

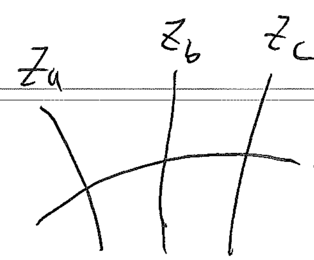
$\beta > 0$: nonzero effective class in $H_2(X;\mathbb{Z})$

$\downarrow Q \to 0$

0

$a * b \xrightarrow{Q\to 0} a\cup b$

classical


quantum


rational curve

<u>Example</u> $H^*(\mathbb{P}^r;\mathbb{Q}) = \mathbb{Q}[H]/\langle H^{r+1}\rangle = \bigoplus_{i=0}^{r} \mathbb{Q}H^i$

$$H^i \cup H^j = \begin{cases} H^{i+j} & i+j \le r \\ 0 & i+j \ge r+1 \end{cases}$$

$(H^i, H^j) = \delta_{i+j, r}$

$$H^i * H^j = \begin{cases} H^{i+j} & i+j \leq r \\ Q\, H^{i+j-r-1} & i+j \geq r+1 \end{cases}$$

$$H^r * H^r = Q H^{r-1}$$

$$\langle H^r, H^r, H \rangle^{\mathbb{P}^r}_{g=0,\, n=3,\, d=1} = 1$$

$$H^*(\mathbb{P}^r; \mathbb{Q}) = \mathbb{Q}[H] / \langle H^{r+1} \rangle$$

$$QH^*(\mathbb{P}^r) = \mathbb{Q}[\![Q]\!][H] / \langle H^{r+1} - Q \rangle$$

For $\mathbb{P}^r$ (more generally, Fano manifold)

may work over polynomial ring instead of power series ring: $QH^*(\mathbb{P}^r) = \mathbb{Q}[Q, H] / \langle H^{r+1} - Q \rangle$

$$\downarrow Q=0$$

$$H^*(\mathbb{P}^r; \mathbb{Q}) = \mathbb{Q}[H] / \langle H^{r+1} \rangle$$

3. Moduli Spaces of Stable Curves

References

Books:

- J. Harris, I. Morrison, Moduli of Curves (1998)
- E. Arbarello, M. Cornalba, P. Griffiths, Geometry of Algebraic Curves II (2010)

Papers:

- Deligne, Mumford "The irreducibility of the space of curves of given genus" (1969)
- "The projectivity of the moduli space of stable curves"

  I, $\underbrace{\text{II, III}}_{\text{Knutsen (1982-3)}}$

  Knutsen-Mumford (1976)

3.1 Set theoretic definition

3.2 Moduli functor

3.3 Fine moduli (stack)

3.4 Coarse moduli (scheme)

3.1 Set theoretic definition

g, n, nonnegative integers

$2g-2+n>0 \iff (g,n) \neq (0,0),(0,1),(0,2),(1,0)$

compact Riemann surfaces

$$M_{g,n} = \left\{ (C, x_1, \dots, x_n) \;\middle|\; \begin{array}{l} C \text{ connected nonsingular projective curve} / \mathbb{C} \\ \text{genus}(C) = g \\ x_1, \dots, x_n \in C \text{ distinct points} \end{array} \right\} \Big/ \cong$$

$\chi = (C - \{x_1, \dots, x_n\}) = 2-2g-n < 0$

$\boxed{g=0}$ $n \geq 3$

$$M_{0,n} = \{ (C, x_1, \dots, x_n) \mid C \cong \mathbb{P}^1,\ x_1, \dots, x_n \in C \text{ distinct} \} / \cong$$

$$= \{ (x_1, \dots, x_n) \in (\mathbb{P}^1)^n \mid x_1, \dots, x_n \text{ distinct} \} / PSL(2,\mathbb{C})$$

$$\cong \{ (x_4, \dots, x_n) \in (\mathbb{P}^1 - \{0,1,\infty\})^{n-3} ; x_4, \dots, x_n \text{ distinct}$$

$$= (\mathbb{P}^1 - \{0,1,\infty\})^{n-3} \setminus \underset{\text{big diagonal}}{\Delta} \quad \underset{\text{dense}}{\overset{\text{open}}{\subset}} (\mathbb{P}^1)^{n-3}$$

$\dim M_{0,n} = n-3$

$M_{0,n}$ quasi-projective

$M_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$

$\boxed{g=1}$ $\widetilde{M}_{1,1}$ moduli of framed elliptic curves

$\widetilde{M}_{1,1} = \{(E, p, \alpha, \beta) : E$ connected nonsingular complex projective curve, genus$(E)=1$, $p \in E$, α, β symplectic basis of $H_1(E;\mathbb{Z})\}$

$\downarrow$ forget α, β

$M_{1,1}$

$$\alpha \cdot \beta = -\beta \cdot \alpha = 1, \quad \alpha \cdot \alpha = \beta \cdot \beta = 0$$

$\widetilde{M}_{1,1} = \mathfrak{h} = \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}$

$\tau = \dfrac{\int_\beta \omega}{\int_\alpha \omega}$ ω any nonzero holomorphic 1-form ω on E (independent of choice)

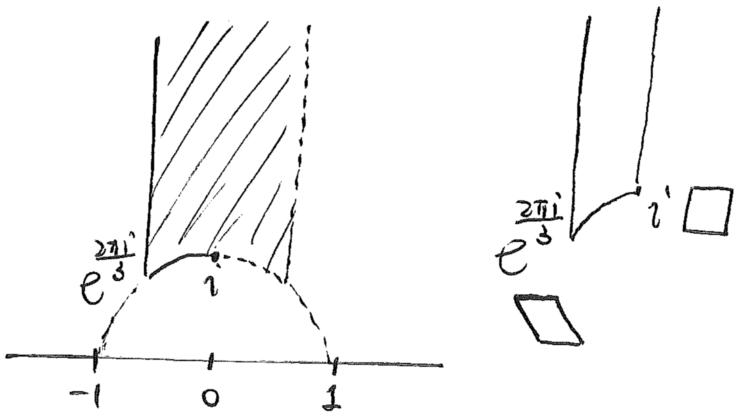
$\alpha \cdot \beta = 1 \Rightarrow \mathrm{Im}\left(\dfrac{\int_\beta \omega}{\int_\alpha \omega}\right) > 0$ $\qquad E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$

$SL(2,\mathbb{C})$ acts on $\mathfrak{h}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$ $\qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \tau = \tau, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \tau = \tau + 1$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \tau = -1/\tau$

fundamental domain $\{\tau \in \mathfrak{h} : -\frac{1}{2} \le \mathrm{Re}\,\tau < \frac{1}{2},$
$|\tau| \geq 1$ if $\mathrm{Re}\,\tau \leq 0$
$|\tau| > 1$ if $\mathrm{Re}\,\tau > 0\}$



$$M_{1,1} = \widetilde{M}_{1,1}/SL(2,\mathbb{Z}) = \mathfrak{h}/SL(2,\mathbb{Z})$$

$\mathcal{M}_{1,1} = [\mathfrak{h}/SL(2,\mathbb{C})]$ fine moduli stack

$$\downarrow$$

$$\mathcal{M}_{1,1}^{rig} = [\mathfrak{h}/PSL(2,\mathbb{Z})], \quad PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$$

$$\downarrow$$

$M_{1,1} = \mathbb{C}$ coarse moduli scheme (the j-line)

(modular) compactification $\overline{M}_{g,n}$ parametrizing isomorphism classes of n-pointed genus g stable curves

$\overline{M}_{0,4} = \mathbb{P}^1$

$\overline{M}_{0,n} \neq (\mathbb{P}^1)^{n-3}$ for $n \geq 5$

We will see $\overline{M}_{0,5} = Bl_{\{3 \text{ points}\}}(\mathbb{P}^1 \times \mathbb{P}^1)$

$\overline{M}_{1,1} = \mathbb{C} \sqcup \{\infty\}$

__Definition__

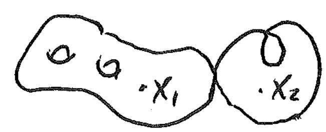
$(C, x_1, \dots, x_n)$ is an __n-pointed genus g prestable curve__ if

(1) C is a reduced, connected, projective curve with at most nodal singularities

(2) $x_1, \dots, x_n$ distinct points in $C - C_{sing}$
↳ set of nodes

(3) $g_a(C) = h^1(C, \mathcal{O}_C) = g$
↑
arithmetic genus

e.g. $g=3$, $n=2$



It is __stable__ if

(4) $\mathrm{Aut}(C, x_1, \dots, x_n)$ is finite

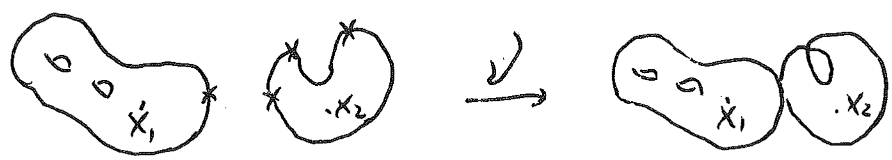
<u>Remark</u> $(C, x_1, \ldots, x_n)$ n-pointed genus g prestable curve. TFAE (The following are equivalent):

(a) $\mathrm{Aut}(C, x_1, \ldots, x_n)$ is finite

(b) $\tilde{C} \xrightarrow{\nu} C$ normalization

Special pts $\nu^{-1}(C_{sing} \sqcup \{x_1, \ldots, x_n\})$

e.g.



$\tilde{C}$ $\quad\quad\quad\quad$ C

$C' \subset \tilde{C}$ connected component

$g(C') = 0 \Rightarrow C'$ contains at least 3 special points

$g(C') = 1 \Rightarrow C'$ contains at least 1 special point

(c) Σ connected component of $C - (C_{sing} \cup \{x_1, \ldots, x_n\})$

then $\chi(\Sigma) < 0 \iff \exists$ complete $\underbrace{\text{hyperbolic metric}}_{K=-1}$ on Σ

(d) $\omega_C(x_1 + \cdots + x_n)$ is ample

$\omega_C(x_1 + \cdots + x_n)$ invertible sheaf on C

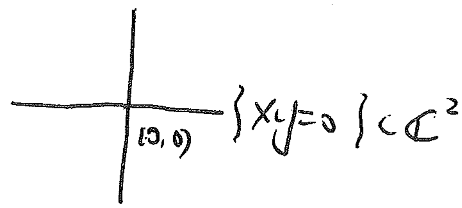
① $p \in C - \{x_1, \ldots, x_n\} - C_{sing}$

$(x) \in \mathrm{Spec}\, \mathbb{C}[x]$

dx

② $p \in C_{sing}$

$\langle x, y \rangle \in \mathrm{Spec}\left(\mathbb{C}[x,y]/\langle x y \rangle\right)$



$$\frac{dx}{x} = -\frac{dy}{y}$$

③ $\{x_1, \ldots, x_n\}$

$\langle x \rangle \in \mathrm{Spec}\, \mathbb{C}[x]$

$$\frac{dx}{x}$$

<u>Remark</u> $(C, x_1, \ldots, x_n)$ n-pointed genus g <u>stable</u> curve

$\Rightarrow$ $\omega_C(x_1 + \cdots + x_n)^{\otimes \nu}$ is very ample if $\nu \geq 3$

<u>Remark</u> $(C, x_1, \ldots, x_n)$ n-pointed genus $\underset{\sim}{0}$ stable curve

$\Rightarrow$ $\mathrm{Aut}(C, x_1, \ldots, x_n) = \{id_C\}$