Assignment 24

This will be a problem on the take home final exam.

Let \( N \) be a 4-manifold equipped with a pseudo Riemannian metric \( \bar{g}_{\mu\nu} \) of signature \( -, +, +, + \) on \( N \). Let \( s_{\mu\nu} \) be a symmetric \((0,2)\) tensor on \( N \) such that \( \bar{g}_{\mu\nu}(t) := \bar{g}_{\mu\nu} + ts_{\mu\nu} \) is a pseudo Riemannian metric of signature \( -, +, +, + \) on \( N \) when \( t \in (-\epsilon, \epsilon) \), where \( \epsilon \) is some positive number. Let \( R \) and \( \dot{R}(t) \) denote the scalar curvatures defined by \( \bar{g}_{\mu\nu} = \bar{g}_{\mu\nu}(0) \) and \( \bar{g}_{\mu\nu}(t) \), respectively; let \( \dot{R}_{\mu\nu}, \ddot{R}_{\mu\nu}(t) \) denote the Ricci curvatures defined by \( \bar{g}_{\mu\nu} \) and \( \bar{g}_{\mu\nu}(t) \), respectively. We use Einstein’s summation convention: “\( \bar{g}^{\mu\nu}s_{\mu\nu} \)” means “\( \sum_{\mu,\nu=0}^{3} \bar{g}^{\mu\nu}s_{\mu\nu} \)”, etc.

(a) Prove that
\[
\frac{d}{dt}\bigg|_{t=0} \sqrt{-\det(\bar{g}(t))} = \frac{1}{2} \bar{g}^{\mu\nu}s_{\mu\nu}\sqrt{-\det(\bar{g})}.
\]

(b) Define a smooth function \( \text{Tr}(s) \) on \( N \) and a vector field \( X^\mu \) on \( N \) by
\[
\text{Tr}(s) := \bar{g}^{\mu\nu}s_{\mu\nu}, \quad X^\mu = (s^{\mu\nu} - \bar{g}^{\mu\nu}\text{Tr}(s))_{,\nu}
\]
where \( ,\nu \) denotes the covariant derivative with respect to \( \frac{\partial}{\partial x^\nu} \). Prove that
\[
\frac{d}{dt}\bigg|_{t=0} \dot{R}(t) = -\dot{R}^{\mu\nu}s_{\mu\nu} + \text{div}\, X.
\]

(c) Use (a) and (b) to prove that
\[
\frac{d}{dt}\bigg|_{t=0} \left( \dot{R}(t)\sqrt{-\det(\bar{g}(t))} \right) = \left( -\left( \dot{R}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu}\bar{R} \right)s_{\mu\nu} + \text{div}\, X \right)\sqrt{-\det(\bar{g})}.
\]