Assignment 15
Due on Monday, February 16, 2015

Notation: \([dC]\) = do Carmo, *Riemannian Geometry*

\[SL(2, \mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \}.\]

\[SU(1, 1) = \{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}.\]

\[S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 = 1 \}.\]

1. Let \((M, g)\) be a Riemannian manifold, and let \(\sigma : M \to M\) be an isometry. Let \(M^\sigma = \{x \in M | \sigma(x) = x\}\) be the set of fixed points of \(\sigma\). Suppose that \(M^\sigma\) is nonempty, and is a submanifold of \(M\). Prove that \(M^\sigma\) is a totally geodesic submanifold of \(M\).

   Indeed, the following stronger result holds: let \(K\) be any set of isometries of a Riemannian manifold \((M, g)\), and let \(M^K\) be the set of points of \(M\) which are left fixed by all elements of \(K\). Then each connected component of \(M^K\) is a closed totally geodesic submanifold of \(M\).)

2. Given any positive constant \(K > 0\), define a Riemannian metric \(g_K\) on \(\mathbb{R}^n\) by

   \[g = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1 + K \sum_{i=1}^n x_i^2)^2}.\]

   Prove that:
   
   (a) \((\mathbb{R}^n, g)\) has constant sectional curvature \(K\).

   (b) \((\mathbb{R}^n, g)\) is not complete.

3. Let \((x, y)\) be coordinates of \(\mathbb{R}^2\), and let \(z = x + iy\).

   (a) Let \(H^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\}\) be the upper half plane, equipped with the Riemannian metric

   \[g = \frac{dx^2 + dy^2}{y^2} = -4dzd\bar{z}.\]

   Prove that the map \(z \mapsto \frac{az + b}{cz + d}\), where \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})\), is an isometry of \((H^2, g)\).

   (b) Let \(D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}\), equipped with the Riemannian metric

   \[h = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} = \frac{4dzd\bar{z}}{(1 - |z|^2)^2}.\]

   Prove that the map \(z \mapsto \frac{az + b}{\bar{\beta}z + \bar{\alpha}}\), where \(\begin{pmatrix} a & b \\ \bar{\alpha} & \bar{\beta} \end{pmatrix} \in SU(1, 1)\), is an isometry of \((D^2, h)\).