Assignment 23

Due Monday, April 26, 2010

- For Problem (2) and (3), we use the notation in [PT] Section 2, Section 3.
- In [PT], \( \nabla \) is the Levi-Civita connection on the spacetime \( N^4 \), and \( \nabla \) is the Levi-Civita connection on the spacelike hypersurface \( M^3 \).
- Greek letters \( \alpha, \beta, \ldots \in \{0, 1, 2, 3\} \). English letters \( i, j, k, \cdots \in \{1, 2, 3\} \).

1. Let \( (M, g) \) be a Riemannian \( n \)-manifold with a volume form \( \omega_n \in \Omega^n(M) \) compatible with the Riemannian metric \( g \). Let \( S \) be an \((n-1)\) dimensional submanifold of \( M \) with a volume form \( \omega_{n-1} \in \Omega^{n-1}(S) \) compatible with the Riemannian metric \( \phi^*g \), where \( \phi: S \to M \) is the inclusion. Then \( M \) and \( S \) are oriented, and there exists a unique unit normal vector field \( \nu \) along \( S \) such that \( \omega_{n-1} = \iota_\nu \omega_n \). Prove the following statements.
   - (a) \( d(i_X \omega_n) = (\text{div} X) \omega_n \).
   - (b) \( \phi^*(i_X \omega_n) = \langle X, \nu \rangle \omega_{n-1} \).
   (Hint: see [GHL] Section 4.A.2.)

2. Let \( \cdot \) denote the Clifford multiplication, so that \( e^\alpha \cdot e^\beta = -e^\beta \cdot e^\alpha \). if \( \alpha \neq \beta \), and \( -e_0 \cdot e_0 = e_i \cdot e_i = -\text{Id}_S \), where \( \text{Id}_S: S \to S \) is the identity map.
   - (a) For a fixed point \( p \in M \), find the eigenvalues (with multiplicities) linear map \( T_0 + T_0 e_0 \cdot e_i \cdot : S \to S \).
   - (b) Prove that \( -\frac{1}{8} R_{\alpha \beta i j} e^i \cdot e^j \cdot e^\alpha \cdot e^\beta = \frac{1}{4} (R + 2 R_{00} + 2 R_{0i} e^0 \cdot e^i ) \).
   (Hint: it might be useful to consider the following cases, where \( i, j, k \) are distinct: \( R_{ikjk}, R_{ijki}, \text{etc.}; R_{ijij}, R_{ijji}; R_{01jk}, R_{01jk}, R_{00ij}, \text{etc.} \).

3. Let \( \{e_1, e_2, e_3\} \) be a local orthonormal frame of \( TM \), and let \( \{e^1, e^2, e^3\} \) be local orthonormal coframe of \( T^*M \). For any \( \phi \in \Gamma(S) \), we have
   \[
   \nabla^i \phi = \bar{\nabla}^i \phi - \frac{1}{2} R_{ij} e^0 \cdot e^i \cdot \phi,
   \]
   where \( \nabla_i := \nabla_{e_i} \) and \( \bar{\nabla}_i := \bar{\nabla}_{e_i} \).
   - (a) Let \( D = e^i \cdot \nabla_i \) be the hypersurface Dirac operator. Use (1) to prove that \( D \phi = e^i \cdot \nabla_i \phi - \frac{1}{2} (\text{tr} h) e^0 \cdot e^i \cdot \phi \).
   - (b) Given \( \phi, \psi \in \Gamma(S) \), define a complexified vector field \( X \) on \( M \) by \( \alpha(X) = \langle \phi, \alpha \cdot \psi \rangle \) for any \( \alpha \in \Omega^1(M) \). Prove that \( \langle \phi, D \psi \rangle - \langle D \phi, \psi \rangle = \text{div} X \).
   (Hint: the following facts might be useful:
   (i) \( \bar{\nabla} \) is compatible with \( \langle \phi, \psi \rangle = \langle \nabla_i \phi, \psi \rangle + \langle \phi, \nabla_i \psi \rangle \).
   (ii) \( e_0 \cdot \phi \) is Hermitian and \( e_1 \cdot, e_2 \cdot, e_3 \cdot \) are skew-Hermitian with respect to \( \langle \phi, \psi \rangle \).)