Assignment 9

Due Wednesday, November 18, 2015

- $I$ is an open interval in $\mathbb{R}$, i.e., $I = (a, b)$ for some $-\infty \leq a < b \leq +\infty$.
- $[dC]$ = do Carmo, *Riemannian Geometry*

1. Let $M$ be a submanifold of $\mathbb{R}^N$, and let $g$ be the Riemannian metric induced from the Euclidean metric on $\mathbb{R}^N$. Let $\gamma : I \to M$ be a smooth curve in $M$. Then $\gamma(t) = (x_1(t), \ldots, x_N(t))$ where $x_i$ are smooth functions on $I$. Define

$$\frac{d^2 \gamma}{dt^2} = \sum_{i=1}^{N} \frac{d^2 x_k}{dt^2} \frac{\partial}{\partial x_k}.$$ 

Prove that

$$\frac{D}{dt} \frac{d\gamma}{dt} = \pi_\gamma(t) \left( \frac{d^2 \gamma}{dt^2} \right)$$

where $\frac{D}{dt}$ is the covariant derivative of the Levi-Civita connection on $(M, g)$, and $\pi_\gamma(t)$ is the orthogonal projection from $T_\gamma(t)\mathbb{R}^N$ to $T_\gamma(t)M$. (Hint: You may use Problem (4) of Assignment 8.)

2. Let $(\vec{a}, \vec{b}) \in T S^n = \{(\vec{x}, \vec{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |\vec{x}| = 1, \vec{x} \cdot \vec{y} = 0\}$.

Assume that $\vec{b} \neq \vec{0}$. Prove that

$$\gamma : \mathbb{R} \to S^n, \quad t \mapsto \cos(|\vec{b}|t)\vec{a} + \sin(|\vec{b}|t)\frac{\vec{b}}{|\vec{b}|}$$

is the unique geodesic on $(S^n, g_{can})$ such that $\gamma(0) = \vec{a}$ and $\gamma'(0) = \vec{b}$. (Hint: You may use Problem (1) above.)

3. $[dC]$ Chapter 3 Exercise 1 a) and b).

4. Let $X$ be a left invariant vector field on a Lie group $G$, and let $\gamma : I \to G$ be an integral curve of $X$. Prove that $\gamma$ is a geodesic with respect to any bi-invariant metric on $G$. (Hint: see $[dC]$ page 80, Exercise 3)