Mathematics G4402. Modern Geometry  
Assignment 7

Due on Wednesday, November 4, 2015

[dC] = do Carmo, Riemannian Geometry

(1) Let $G$ be a Lie group, and let $\mathfrak{g} = T_eG$ be its Lie algebra, where $e$ is the identity element of $G$. Define $i : G \to G$ by $i(x) = x^{-1}$. Prove the following statements.

(a) If $X$ is a left invariant vector field on $G$ then $i_*X$ is a right invariant vector field on $G$.

(b) For any $\xi \in \mathfrak{g}$, the map $t \mapsto \exp(t\xi)$ is a group homomorphism from $\mathbb{R}$ to $G$.

(c) The differential of $i$ at the $e$ is $di_e = -\text{Id}_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$, where $\text{Id}_\mathfrak{g}$ is the identity map from $\mathfrak{g}$ to $\mathfrak{g}$.

(d) Given any $\xi \in \mathfrak{g}$, let $X^L_\xi$ (resp. $X^R_\xi$) be the unique left (resp. right) invariant vector field on $G$ such that $X^L_\xi(e) = \xi$ (resp. $X^R_\xi(e) = \xi$). Then for any $\xi, \eta \in \mathfrak{g}$,

$$[X^R_\xi, X^R_\eta](e) = -[X^L_\xi, X^L_\eta](e).$$

(2) Let $G_1$ and $G_2$ be Lie groups, and let $e_1 \in G_1$ and $e_2 \in G_2$ be the identity elements. Suppose that $f : G_1 \to G_2$ is a group homomorphism and a smooth map. Prove that $df_{e_1} : T_{e_1}G \to T_{e_2}H$ is a Lie algebra homomorphism.

(3) Let $a_{ij} : GL(n, \mathbb{R}) \to \mathbb{R}$ be the entries of the matrix, so that $a_{ij}$, $i, j = 1, \ldots, n$ are global coordinates on $GL(n, \mathbb{R})$. Let $\tilde{g}_n$ be the Riemannian metric on $GL(n, \mathbb{R})$ defined by $\tilde{g}_n = \sum_{i,j=1}^n da_{ij}^2$. Let $i : SO(n) \to GL(n, \mathbb{R})$ be the inclusion, which is a smooth embedding. Show that $g_n = i^* \tilde{g}_n$ is a bi-invariant Riemannian metric on $SO(n)$.

(4) Let $G$ be a compact connected Lie group ($\dim G = n$).

(a) Let $\omega$ be a left invariant $C^\infty$ $n$-form on $G$, that is, $L_x^*\omega = \omega$ for all $x \in G$. Prove that $\omega$ is right invariant. (Hint: see [dC] page 47.)

(b) Show that there exists a left-invariant $C^\infty$ $n$-form on $G$.

(c) Let $\langle \cdot, \cdot \rangle$ be a left invariant metric on $G$, and let $\omega$ be a left invariant $C^\infty$ $n$-form on $G$ such that $\int_G \omega > 0$. Define a new Riemannian metric $\langle \cdot, \cdot \rangle$ on $G$ by

$$\langle (u, v) \rangle_y = \int_G \langle (dR_x)_y (u), (dR_x)_y (v) \rangle_{yx} \omega,$$

$x, y \in G, u, v \in T_y G$.

Prove that the new Riemannian metric $\langle \cdot, \cdot \rangle$ is bi-invariant.