Due on Wednesday, October 21, 2015

(1) Define a smooth function $Q$ on $\mathbb{R}^{n+1}$ by
$$Q(x_0, x_1, \ldots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2.$$ Define a smooth $(0, 2)$ symmetric tensor $q$ on $\mathbb{R}^{n+1}$ by
$$q = -dx_0^2 + dx_1^2 + \cdots + dx_n^2.$$

(a) (Hyperbolic space) Note that $-1$ is a regular value of the smooth function $Q$, so $H^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid Q(x_0, x_1, \ldots, x_n) = -1, x_0 > 0\}$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. Let $i : H^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map, and define $g = i^*q \in C^\infty(H^n, S^2T^*H^n)$. Show that $g$ is positive definite, so it is a Riemannian metric on $H^n$.

(b) (Poincaré disk) Show that $(x_0, x_1, \ldots, x_n) \mapsto \frac{1}{x_0 + 1}(x_1, \ldots, x_n)$ defines a diffeomorphism $f$ from $H^n$ onto the unit disk
$$D^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1^2 + \cdots + y_n^2 < 1\}.$$ Show that $(f^{-1})^*g = \rho \sum_{i=1}^n dy_i^2$ for some smooth, positive function $\rho$ on $D^n$, and find $\rho(y_1, \ldots, y_n)$.

(c) (Poincaré upper half space) Let $\mathcal{H}^n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n > 0\}$ be the $n$-dimensional upper half space. Define a Riemannian metric $h$ on $\mathcal{H}^n$ by
$$h = \frac{dy_1^2 + \cdots + dy_n^2}{y_n^2}.$$ Prove that $(\mathcal{H}^n, h)$ is isometric to $(D^n, (f^{-1})^*g)$. (Hint: see page 56-57 of S. Gallot, D. Hulin, and J. Lafontanie’s *Riemannian Geometry*, Third Edition.)

(2) Let $T^2$ be embedded in $\mathbb{R}^3$ as image of $\mathbb{R}^2$ by the map $\Phi$ defined by
$$\Phi(\theta, \phi) = ((a + b \cos \theta) \cos \phi, (a + b \cos \theta) \sin \phi, b \sin \theta)$$
where $a > b > 0$. Let $g$ be the Riemannian metric induced on $T^2$ by the Euclidean metric of $\mathbb{R}^3$.

(a) Write $g$ in the form $g = Ed\theta^2 + F(d\theta d\phi + d\phi d\theta) + Gd\phi^2$. (b) Compute the volume of $(T^2, g)$.

(3) Show that any isometry of the Euclidean space $\mathbb{R}^n$ must take straight lines to straight lines. Show that the only isometries of $\mathbb{R}^n$ are those of the form $x \mapsto Ax + b$ for constant $A \in O(n)$, $b \in \mathbb{R}^n$.

(4) Let $G \times M \to M$ be a properly discontinuous action of a group $G$ on a smooth manifold $M$. Let $\varphi_g, g \in G$, and the smooth manifold $M/G$, be defined as in 4.8 Example on page 22 and 23 of do Carmo’s *Riemannian Geometry*.

(a) Prove that $M/G$ is orientable if and only if there exists an orientation on $M$ that is preserved by all $\varphi_g, g \in G$.

(b) Prove that $P^n(\mathbb{R})$ is orientable if and only if $n$ is odd.