(1) Let $M$ be a $C^k$ manifold of dimension $n$, where $k \geq 1$. Let $(U, \phi)$ be a chart for $M$ around a point $p \in M$, and let $\theta_{(U,\phi,p)} : \mathbb{R}^n \to T_p M$ be defined as in class:

$$\theta_{(U,\phi,p)}(\vec{u}) = [(U, \phi, \vec{u})].$$

Verify that $\theta_{(U,\phi,p)}$ is a bijection.

(2) (universal line bundle) Recall that $P_n(\mathbb{R}) = \{ \ell \subset \mathbb{R}^{n+1} | \ell \text{ is a 1 dimensional linear subspace of } \mathbb{R}^{n+1} \}$.

Define $E = \{ (\ell, v) \in P_n(\mathbb{R}) \times \mathbb{R}^{n+1} | v \in \ell \} \subset P_n(\mathbb{R}) \times \mathbb{R}^{n+1}$.

Let $p_1 : P_n(\mathbb{R}) \times \mathbb{R}^{n+1} \to P_n(\mathbb{R})$ be the projection to the first factor, and let $\pi : E \to P_n(\mathbb{R})$ be the restriction of $p_1$ to $E$.

Prove that $\pi : E \to P_n(\mathbb{R})$ is a $C^\infty$ vector bundle of rank 1 over $P_n(\mathbb{R})$.

(3) Show that if $\delta$ is a derivation on $C^0_0(\mathbb{R}^n)$ then $\delta = 0$.

(4) Let $M$ be a smooth submanifold of a smooth manifold $N$, and let $X, Y$ be smooth vector fields on $M$. Let $p \in M$ and let $U$ be an open neighborhood of $p$ in $N$.

(a) Suppose that $\tilde{X}, \tilde{Y} \in C^\infty(U, TU)$ are smooth vector fields on $U$ such that for all $q \in U \cap M$

$$\tilde{X}(q) = X(q) \in T_q M, \quad \tilde{Y}(q) = Y(q) \in T_q M.$$ 

Show that $[\tilde{X}, \tilde{Y}](q) \in T_q M$ for all $q \in U \cap M$.

(b) Let $f$ be a smooth function on $M$, and let $\tilde{f}$ be a smooth function on $U$ such that $\tilde{f}(q) = f(q)$ for all $q \in U \cap M$.

Let $g = [X,Y]f \in C^\infty(M)$ and let $\tilde{g} = [\tilde{X}, \tilde{Y}]\tilde{f} \in C^\infty(U)$.

Show that $\tilde{g}(q) = g(q)$ for all $q \in U \cap M$. 