(1) Let \((\vec{a}, \vec{b}) \in TS^n = \{(\vec{x}, \vec{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} | |\vec{x}| = 1, \vec{x} \cdot \vec{y} = 0\}\). Assume that \(\vec{b} \neq \vec{0}\). Prove that 
\[
\gamma : \mathbb{R} \to S^n, \quad t \mapsto \cos(|\vec{b}|t)\vec{a} + \sin(|\vec{b}|t)\frac{\vec{b}}{|\vec{b}|}
\]
is the unique geodesic on \((S^n, g_{can})\) such that \(\gamma(0) = \vec{a}\) and \(\gamma'(0) = \vec{b}\). (Hint: You may use Problem 3 of Assignment 8.)

(2) do Carmo Chapter 3 Exercise 1 (page 77-78).

(3) Let \(X\) be a left invariant vector field on a Lie group \(G\), and let \(\gamma : \mathbb{R} \to G\) be an integral curve of \(X\). Prove that \(\gamma\) is a geodesic with respect to any bi-invariant metric on \(G\). (Hint: see do Carmo Chapter 3 Exercise 3, page 80-81)

(4) Let \(F : M \to N\) be a smooth map between smooth manifolds, and define \(F_* : \mathcal{X}(M) \to C^\infty(M, F^*TN)\) by 
\[
(F_*X)(p) = dF_p(X(p)) \in T_{F(p)}N = (F^*TN)_p
\]
where \(X \in \mathcal{X}(M)\) and \(p \in M\). Let \(h\) be a Riemannian metric on \(N\), let \(\nabla\) be the Levi-Civita connection on \((N, h)\), and let \(D = F^*\nabla\) be the pull back connection on \(F^*TN\). Prove the following statements:

(a) (symmetric) For all \(X, Y \in \mathcal{X}(M)\),
\[
D_X(F_*Y) - D_Y(F_*X) = F_*([X, Y]).
\]

(b) (compatible with the metric) For all \(X \in \mathcal{X}(M)\) and \(V, W \in C^\infty(M, F^*TN)\),
\[
X\langle V, W \rangle = \langle D_XV, W \rangle + \langle V, D_XW \rangle.
\]