Mathematics G4402. Modern Geometry  
Fall 2014  
Assignment 8

Due Monday, November 10, 2014

(1) Let \( \mathcal{H} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > 0\} \) be the upper half plane, and define a Riemannian metric on \( \mathcal{H} \) by
\[
g = \frac{dy_1^2 + dy_2^2}{y_2^2}.
\]
(a) Compute the Christoffel symbols \( \Gamma^k_{ij}, \ i, j, k \in \{1, 2\} \), for the Levi-Civita connection \( \nabla \) on \( (\mathcal{H}, g) \).
(b) Define \( \gamma : \mathbb{R} \to \mathcal{H} \) by \( \gamma(t) = (t, 1) \). Then \( \gamma \) is a smooth curve in \( \mathcal{H} \). Let
\[
V(t) = a(t) \frac{\partial}{\partial y_1} + b(t) \frac{\partial}{\partial y_2}
\]
be the unique parallel (w.r.t. \( \nabla \)) vector field along \( \gamma \) such that \( V(0) = \frac{\partial}{\partial y_2} \). Find \( a(t), b(t) \) for \( t \in \mathbb{R} \).

(2) Let \( F : (M, g) \to (N, h) \) be an isometric immersion. For any \( p \in M \), let \( \pi_p \) be the orthogonal projection from \( T_{F(p)}N \) to the image of \( dF_p : T_p M \to T_{F(p)}N \). Let \( X, Y \) be \( C^\infty \) vector fields on \( M \) which are \( F \)-related to \( C^\infty \) vector fields \( \tilde{X}, \tilde{Y} \) on \( N \), respectively. Let \( \nabla \) and \( \tilde{\nabla} \) be the Levi-Civita connections on \( (M, g) \) and on \( (N, h) \), respectively. Prove that for any \( p \in M \),
\[
dF_p((\nabla_X Y)(p)) = \pi_p((\tilde{\nabla}_{\tilde{X}} \tilde{Y})(F(p)))
\]

(3) Let \( M \) be a submanifold of \( \mathbb{R}^N \), and let \( g \) be the Riemannian metric induced from the Euclidean metric on \( \mathbb{R}^N \). Let \( \gamma : (a, b) \to M \) be a smooth curve in \( M \). Then \( \gamma(t) = (x_1(t), \ldots, x_N(t)) \) where \( x_i \) are smooth functions on \( I \). Define
\[
\frac{d^2 \gamma}{dt^2} = \sum_{i=1}^N \frac{d^2 x_k}{dt^2} \frac{\partial}{\partial x_k}.
\]
Use Problem 2 to prove that
\[
\frac{D}{dt} \frac{d\gamma}{dt} = \pi_{\gamma(t)}\left( \frac{d^2 \gamma}{dt^2} \right)
\]
where \( \frac{D}{dt} \) is the covariant derivative of the Levi-Civita connection on \( (M, g) \), and \( \pi_{\gamma(t)} \) is the orthogonal projection from \( T_{\gamma(t)}\mathbb{R}^N \) to \( T_{\gamma(t)}M \).