1. Let $G \times M \to M$ be a properly discontinuous action of a group $G$ on a smooth manifold $M$. Let $\varphi_g, g \in G$, and the smooth manifold $M/G$, be defined as in [dC] page 22 and 23, 4.8 Example.

(a) Prove that $M/G$ is orientable if and only if there exists an orientation on $M$ that is preserved by all $\varphi_g, g \in G$.

(b) Prove that $P^n(\mathbb{R})$ is orientable if and only if $n$ is odd.

2. Let $G_1$ and $G_2$ be Lie groups, and let $e_1 \in G_1$ and $e_2 \in G_2$ be the identity elements. Suppose that $f : G_1 \to G_2$ is a group homomorphism and a smooth map. Prove that $df_{e_1} : T_{e_1}G \to T_{e_2}H$ is a Lie algebra homomorphism.

3. Let $a_{ij} : GL(n, \mathbb{R}) \to \mathbb{R}$ be the entries of the matrix, so that $a_{ij}, i, j = 1, \ldots, n$ are global coordinates on $GL(n, \mathbb{R})$. Let $\tilde{g}_n$ be the Riemannian metric on $GL(n, \mathbb{R})$ defined by $\tilde{g}_n = \sum_{i,j=1}^n da_{ij}^2$. Let $i : SO(n) \to GL(n, \mathbb{R})$ be the inclusion, which is a smooth embedding. Show that $g_n = i^*\tilde{g}_n$ is a bi-invariant Riemannian metric on $SO(n)$.

4. Let $G$ be a compact connected Lie group ($\dim G = n$).

(a) Let $\omega$ be a left invariant $C^\infty$ $n$-form on $G$, that is, $L^*_x\omega = \omega$ for all $x \in G$. Prove that $\omega$ is right invariant. (Hint: see [dC] page 47.)

(b) Show that there exists a left-invariant $C^\infty$ $n$-form on $G$.

(c) Let $\langle \ , \ \rangle$ be a left invariant metric on $G$, and let $\omega$ be a left invariant $C^\infty$ $n$-form on $G$ such that $\int_G \omega > 0$. Define a new Riemannian metric $\langle \langle \ , \ \rangle \rangle$ on $G$ by

$$\langle \langle u, v \rangle \rangle_y = \int_G \langle (dR_x)_y(u), (dR_x)_y(v) \rangle_{yx} \omega,$$

$x, y \in G, \ u, v \in T_yG$.

Prove that the new Riemannian metric $\langle \langle \ , \ \rangle \rangle$ is bi-invariant.