Due on Monday, October 6, 2014

Assignment 4

[1] Let $X, Y$ be smooth vector fields on a smooth manifold $M$. Show that
\[ L_X L_Y T - L_Y L_X T = L_{[X,Y]} T \]
for any smooth tensor $T$ on $M$. (See [GHL] page 43.)

(2) Let $X, Y$ be smooth vector fields on a smooth $n$-manifold $M$, and let $s \in \{0, 1, \ldots, n\}$. Prove the following identities.
(a) $L_X \omega = d(i_X \omega) + i_X (d\omega)$ for any $\omega \in \Omega^s(M)$. (See [GHL] page 43, 44.)
(b) $L_X (i_Y \omega) - i_Y (L_X \omega) = i_{[X,Y]} \omega$ for any $\omega \in \Omega^s(M)$.
(c) $L_f X \omega = df \wedge i_X \omega + f L_X \omega$ for any $f \in C^\infty(M)$ and any $\omega \in \Omega^s(M)$.

(3) Let $\omega$ be a smooth $s$-form on a smooth manifold $M$. Prove that for any smooth vector fields $X_0, X_1, \ldots, X_s$ on $M$
\[ d\omega(X_0, X_1, \ldots, X_s) = \sum_{i=0}^s (-1)^i X_i \left( \omega(X_0, \ldots, \hat{X}_i, \ldots, X_s) \right) \]
\[ + \sum_{0 \leq i < j \leq s} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_s) \]
(See [GHL] page 44.)

(4) Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $i : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion map, which is a smooth embedding. We define
\[ \tilde{\omega} = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy \in \Omega^2(\mathbb{R}^3), \]
\[ \tilde{\eta} = -y dx + x dy \in \Omega^1(\mathbb{R}^3), \]
and let $\omega = i^* \tilde{\omega} \in \Omega^2(S^2)$, $\eta = i^* \tilde{\eta} \in \Omega^1(S^2)$.
(a) Define two smooth vector fields on $\mathbb{R}^3$: \[ \tilde{X} = -zx \frac{\partial}{\partial x} - zy \frac{\partial}{\partial y} + (x^2 + y^2) \frac{\partial}{\partial z}, \quad \tilde{Y} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}. \]
Show that $\tilde{X}(p), \tilde{Y}(p) \in T_p S^2$ for all $p \in S^2$, so that we may define two smooth vector fields $\tilde{X}, \tilde{Y}$ on $S^2$ by
\[ X(p) = \tilde{X}(p), \quad Y(p) = \tilde{Y}(p), \quad p \in S^2. \]
(b) Compute $L_X \omega, L_Y \omega, L_X Y, L_X \eta, L_Y \eta$.
(c) $d\eta = \lambda \omega$ for some $\lambda \in C^\infty(S^2)$. Find $\lambda(x, y, z)$ for all $(x, y, z) \in S^2$.
(d) $i_X \omega \wedge i_Y \omega = \phi \omega$ for some $\phi \in C^\infty(S^2)$. Find $\phi(x, y, z)$ for all $(x, y, z) \in S^2$.
(e) Let $\pi : S^2 \to P_2(\mathbb{R}) = S^2/\{\pm 1\}$ be the projection, which is a surjective local diffeomorphism. Does there exist $\tilde{\omega} \in \Omega^2(P_2(\mathbb{R}))$ such that $\omega = \pi^* \tilde{\omega}$?