An open interval in \( \mathbb{R} \) is of the form \((a, b)\), where \(-\infty \leq a < b \leq +\infty\).

1. Let \( M \) be a smooth submanifold of a smooth manifold \( N \), and let \( X, Y \) be smooth vector fields on \( M \). Let \( p \in M \) and let \( U \) be an open neighborhood of \( p \) in \( N \).
   (a) Suppose that \( \tilde{X}, \tilde{Y} \in C^\infty(U, T U) \) are smooth vector fields on \( U \) such that for all \( q \in U \cap M \)
   \[ \tilde{X}(q) = X(q) \in T_q M, \quad \tilde{Y}(q) = Y(q) \in T_q M. \]
   Show that \([\tilde{X}, \tilde{Y}](q) \in T_q M\) for all \( q \in U \cap M \).
   (b) Let \( f \) be a smooth function on \( M \), and let \( \tilde{f} \) be a smooth function on \( U \) such that \( \tilde{f}(q) = f(q) \) for all \( q \in U \cap M \).
   Let \( g = [X, Y]f \in C^\infty(M) \) and let \( \tilde{g} = [\tilde{X}, \tilde{Y}]\tilde{f} \in C^\infty(U) \).
   Show that \( \tilde{g}(q) = g(q) \) for all \( q \in U \cap M \).

2. Let \( X \) be a smooth vector field on a smooth manifold \( M \), and let \( \gamma : I \to M \) be a nonconstant integral curve of \( X \), where \( I \) is an open interval in \( \mathbb{R} \). Prove the following statements.
   (a) \( \gamma \) is an immersion.
   (b) If \( \gamma \) is not injective, then there exists a smooth embedding \( i : S^1 \to M \) such that \( i(S^1) = \gamma(I) \).

3. Let \( X \) be the vector field on \( \mathbb{R} \) defined by \( X(x) = x^2 \frac{\partial}{\partial z} \). Given \( x \in \mathbb{R} \), let \( \phi_x : I_x \to \mathbb{R} \) be the unique integral curve of \( X \) such that \( \phi_x(0) = x \), where \( I_x \) is an open interval containing 0, and \( \phi_x \) cannot be extended to a larger open interval containing \( I_x \). Find \( \phi_x \) and \( I_x \) for all \( x \in \mathbb{R} \).

4. Let \( X, Y, Z \) be the vector fields defined on \( \mathbb{R}^3 \) by
   \[ X = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \quad Y = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}, \quad Z = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \]
   (a) Show that the map \( (a, b, c) \mapsto aX + bY + cZ \) is an isomorphism from \( \mathbb{R}^3 \) onto a subspace of the space of smooth vector fields on \( \mathbb{R}^3 \), and that the bracket of vector fields on \( \mathbb{R}^3 \) corresponds to the cross product on \( \mathbb{R}^3 \).
   (b) Compute the flow of the vector field \( aX + bY + cZ \) where \( a, b, c \in \mathbb{R} \).